Shi et al. *Journal of Inequalities and Applications* (2020) 2020:65
https://doi.org/10.1186/s13660-020-02334-8

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Schur-convexity for compositions of complete symmetric function dual

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Abstract

The Schur-convexity for certain compound functions involving the dual of the complete symmetric function is studied. As an application, the Schur-convexity of some special symmetric functions is discussed and some inequalities are established.

MSC: 05E05; 26B25

Keywords: Schur-convexity; Schur-geometric convexity; Schur-harmonic convexity; Completely symmetric function; Dual form

1 Introduction

Throughout the article, $\mathbb{R}$ denotes the set of real numbers, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ denotes n-tuple (n-dimensional real vectors), the set of vectors can be written as

$$\mathbb{R}^n = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}, i = 1, 2, \ldots, n\},$$
$$\mathbb{R}^n_+ = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_n) : x_i > 0, i = 1, 2, \ldots, n\},$$
$$\mathbb{R}^n_- = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_n) : x_i < 0, i = 1, 2, \ldots, n\}.$$

In particular, the notations $\mathbb{R}$ and $\mathbb{R}_+$ denote $\mathbb{R}^1$ and $\mathbb{R}^1_+$, respectively.

In recent years, the Schur-convexity, Schur-geometric, and Schur-harmonic convexities of various symmetric functions have been a hot topic of inequality research [1–30].

The following complete symmetric function is an important class of symmetric functions.

For $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, the complete symmetric function $c_n(\boldsymbol{x}, r)$ is defined as

$$c_n(\boldsymbol{x}, r) = \sum_{i_1+i_2+\cdots+i_n=r} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}, \tag{1}$$

where $c_0(\boldsymbol{x}, r) = 1$, $r \in \{1, 2, \ldots, n\}$, $i_1, i_2, \ldots, i_n$ are nonnegative integers.

It has been investigated by many mathematicians, and there are many interesting results in the literature.

Guan [4] discussed the Schur-convexity of $c_n(\boldsymbol{x}, r)$ and proved the following.

Proposition 1 *$c_n(\boldsymbol{x}, r)$ is increasing and Schur-convex on $\mathbb{R}_+^n$.*

Subsequently, Chu et al. [1] proved the following.

Proposition 2 *$c_n(\boldsymbol{x}, r)$ is Schur-geometrically convex and Schur-harmonically convex on $\mathbb{R}_+^n$.*

In 2016, Shi et al. [18] further considered the Schur-convexity of $c_n(\boldsymbol{x}, r)$ on $\mathbb{R}_-^n$, which proved the following proposition.

Proposition 3 *If r is an even integer (or odd integer, respectively), then $c_n(\boldsymbol{x}, r)$ is decreasing and Schur-convex (or increasing and Schur-concave, respectively) on $\mathbb{R}_-^n$.*

The dual form of the complete symmetric function $c_n(\boldsymbol{x}, r)$ is defined as

$$c_n^*(\boldsymbol{x}, r) = \prod_{i_1+i_2+\cdots+i_n=r} \sum_{j=1}^{n} i_j x_j, \tag{2}$$

where $c_0^*(\boldsymbol{x}, r) = 1$, $r \in \{1, 2, \ldots, n\}$, $i_1, i_2, \ldots, i_n$ are nonnegative integers.

Zhang and Shi [17] proved the following two propositions.

Proposition 4 *For $r = 1, 2, \ldots, n$, $c_n^*(\boldsymbol{x}, r)$ is increasing and Schur-concave on $\mathbb{R}_+^n$.*

Proposition 5 *For $r = 1, 2, \ldots, n$, $c_n^*(\boldsymbol{x}, r)$ is Schur-geometrically convex and Schur-harmonically convex on $\mathbb{R}_+^n$.*

Notice that

$$c_n^*(-\boldsymbol{x}, r) = (-1)^r c_n^*(\boldsymbol{x}, r),$$

it is not difficult to prove the following proposition.

Proposition 6 *If r is an even integer (or odd integer, respectively), then $c_n^*(\boldsymbol{x}, r)$ is decreasing and Schur-concave (or increasing and Schur-convex, respectively) on $\mathbb{R}_-^n$.*

In this paper we will study the Schur-convexity, Schur-geometric and Schur-harmonic convexities of the following composite function of $c_n^*(\boldsymbol{x}, r)$:

$$c_n^*\bigl(f(\boldsymbol{x}), r\bigr) = c_n^*\bigl(f(x_1), f(x_2), \ldots, f(x_n), r\bigr) = \prod_{i_1+i_2+\cdots+i_n=r} \sum_{j=1}^{n} i_j\bigl(f(x_j)\bigr), \tag{3}$$

where f is a positive function which satisfies certain conditions.

Our main results are as follows.

Theorem 1 *Let $I \subset \mathbb{R}$ be a symmetric convex set with nonempty interior, and let $f : I \to \mathbb{R}_+$ be continuous on I and differentiable in the interior of I.*

(a) *If f is a log-convex function on I, then for any $r = 1, 2, \ldots, n$, $c_n^*(f(\boldsymbol{x}), r)$ is a Schur-convex function on I^n;*

(b) *If f is a concave function on I, then for any $r = 1, 2, \dots, n$, $c_n^*(f(\boldsymbol{x}), r)$ is a Schur-concave function on I^n.*

Theorem 2 *Let $I \subset \mathbb{R}_+$ be a symmetric convex set with nonempty interior and let $f : I \to \mathbb{R}_+$ be continuous on I and differentiable in the interior of I.*

(a) *If f is an increasing and log-convex function on I, then for any $r = 1, 2, \dots, n$, $c_n^*(f(\boldsymbol{x}), r)$ is a Schur-geometrically convex function on I^n.*

(b) *If f is a descending and concave function on I, then for any $r = 1, 2, \dots, n$, $c_n^*(f(\boldsymbol{x}), r)$ is a Schur-geometrically concave function on I^n.*

Theorem 3 *Let $I \subset \mathbb{R}_+$ be a symmetric convex set with nonempty interior, and let $f : I \to \mathbb{R}_+$ be continuous on I and differentiable in the interior of I.*

(a) *If f is an increasing and log-convex function on I, then for any $r = 1, 2, \dots, n$, $c_n^*(f(\boldsymbol{x}), r)$ is a Schur-harmonically convex function on I^n.*

(b) *If f is a descending and concave function on I, then for any $r = 1, 2, \dots, n$, $c_n^*(f(\boldsymbol{x}), r)$ is a Schur-harmonically concave function on I^n.*

2 Definitions and lemmas

For convenience, we introduce some definitions as follows.

Definition 1 ([31, 32]) Let $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$.

(a) $\boldsymbol{x} \ge \boldsymbol{y}$ means $x_i \ge y_i$ for all $i = 1, 2, \dots, n$.

(b) Let $\Omega \subset \mathbb{R}^n$, φ: $\Omega \to \mathbb{R}$ is said to be increasing if $\boldsymbol{x} \ge \boldsymbol{y}$ implies $\varphi(\boldsymbol{x}) \ge \varphi(\boldsymbol{y})$. φ is said to be decreasing if and only if $-\varphi$ is increasing.

Definition 2 ([31, 32]) Let $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$.

(a) $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x} \prec \boldsymbol{y}$) if $\sum_{i=1}^k x_{[i]} \le \sum_{i=1}^k y_{[i]}$ for $k = 1, 2, \dots, n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, where $x_{[1]} \ge x_{[2]} \ge \cdots \ge x_{[n]}$ and $y_{[1]} \ge y_{[2]} \ge \cdots \ge y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order.

(b) Let $\Omega \subset \mathbb{R}^n$, φ: $\Omega \to \mathbb{R}$ is said to be a Schur-convex function on Ω if $\boldsymbol{x} \prec \boldsymbol{y}$ on Ω implies $\varphi(\boldsymbol{x}) \le \varphi(\boldsymbol{y})$. φ is said to be a Schur-concave function on Ω if and only if $-\varphi$ is Schur-convex function on Ω.

Definition 3 ([31, 32]) Let $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$.

(a) $\Omega \subset \mathbb{R}^n$ is said to be a convex set if $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, $0 \le \alpha \le 1$, implies $\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} = (\alpha x_1 + (1-\alpha)y_1, \alpha x_2 + (1-\alpha)y_2, \dots, \alpha x_n + (1-\alpha)y_n) \in \Omega$.

(b) Let $\Omega \subset \mathbb{R}^n$ be a convex set. A function φ: $\Omega \to \mathbb{R}$ is said to be a convex function on Ω if

$$\varphi\bigl(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}\bigr) \le \alpha\varphi(\boldsymbol{x}) + (1-\alpha)\varphi(\boldsymbol{y})$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, and all $\alpha \in [0, 1]$. φ is said to be a concave function on Ω if and only if $-\varphi$ is a convex function on Ω.

Definition 4 ([31, 32])

(a) A set $\Omega \subset \mathbb{R}^n$ is called a symmetric set if $\boldsymbol{x} \in \Omega$ implies $\boldsymbol{x}P \in \Omega$ for every $n \times n$ permutation matrix P.

(b) A function $\varphi : \Omega \to \mathbb{R}$ is called symmetric if, for every permutation matrix P, $\varphi(\boldsymbol{x}P) = \varphi(\boldsymbol{x})$ for all $\boldsymbol{x} \in \Omega$.

Lemma 1 (Schur-convex function decision theorem [31, 32]) *Let $\Omega \subset \mathbb{R}^n$ be symmetric and have a nonempty interior convex set. Ω^0 is the interior of Ω. $\varphi : \Omega \to \mathbb{R}$ is continuous on Ω and differentiable in Ω^0. Then φ is the Schur-convex (or Schur-concave, respectively) function if and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial \varphi}{\partial x_1} - \frac{\partial \varphi}{\partial x_2}\right) \geq 0 \ (or \ \leq 0, respectively) \tag{4}$$

holds for any $\boldsymbol{x} \in \Omega^0$.

The first systematical study of the functions preserving the ordering of majorization was made by Issai Schur in 1923. In Schur's honor, such functions are said to be "Schur-convex". They can be used extensively in analytic inequalities, combinatorial optimization, quantum physics, information theory, and other related fields. See [31].

Definition 5 ([33]) Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_+^n$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}_+^n$.

(a) $\Omega \subset \mathbb{R}_+^n$ is called a geometrically convex set if $(x_1^\alpha y_1^\beta, x_2^\alpha y_2^\beta, \ldots, x_n^\alpha y_n^\beta) \in \Omega$ for all $\boldsymbol{x}$, $\boldsymbol{y} \in \Omega$ and α, $\beta \in [0,1]$ such that $\alpha + \beta = 1$.

(b) Let $\Omega \subset \mathbb{R}_+^n$. The function φ: $\Omega \to \mathbb{R}_+$ is said to be a Schur-geometrically convex function on Ω if $(\log x_1, \log x_2, \ldots, \log x_n) \prec (\log y_1, \log y_2, \ldots, \log y_n)$ on Ω implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. The function φ is said to be a Schur-geometrically concave function on Ω if and only if $-\varphi$ is a Schur-geometrically convex function on Ω.

The Schur-geometric convexity was proposed by Zhang [33] in 2004, and it was investigated by Chu et al. [34], Guan [35], Sun et al. [36], and so on. We also note that some authors use the term "Schur multiplicative convexity".

In 2009, Chu ([1, 2, 37]) introduced the notion of Schur-harmonically convex function, and some interesting inequalities were obtained.

Definition 6 ([37]) Let $\Omega \subset \mathbb{R}_+^n$ or $\Omega \subset \mathbb{R}_-^n$.

(a) A set Ω is said to be harmonically convex if $\frac{\boldsymbol{xy}}{\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}} \in \Omega$ for every $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ and $\lambda \in [0,1]$, where $\boldsymbol{xy} = \sum_{i=1}^n x_i y_i$ and $\frac{1}{\boldsymbol{x}} = (\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n})$.

(b) A function $\varphi : \Omega \to \mathbb{R}_+$ is said to be Schur-harmonically convex on Ω if $\frac{1}{\boldsymbol{x}} \prec \frac{1}{\boldsymbol{y}}$ implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. A function φ is said to be a Schur-harmonically concave function on Ω if and only if $-\varphi$ is a Schur-harmonically convex function.

Remark 1 We extend the definition and determination theorem of Schur-harmonically convex function established by Chu as follows:

(a) $\Omega \subset \mathbb{R}_+^n$ is extended to $\Omega \subset \mathbb{R}_+^n$ or $\Omega \subset \mathbb{R}_-^n$;

(b) The function $\varphi : \Omega \to \mathbb{R}$ must not be a positive function.

Lemma 2 ([31, 32]) *Let the set $\mathbb{A}, \mathbb{B} \subset \mathbb{R}$, $\varphi : \mathbb{B}^n \to \mathbb{R}$, $f : \mathbb{A} \to \mathbb{B}$ and $\psi(x_1, x_2, \ldots, x_n) = \varphi(f(x_1), f(x_2), \ldots, f(x_n)) : \mathbb{A}^n \to \mathbb{R}$.*

(a) *If f is convex and φ is increasing and Schur-convex, then ψ is Schur-convex;*

(b) *If f is concave, φ is increasing and Schur-concave, then ψ is Schur-concave.*

Lemma 3 *Let the set $\Omega \subset \mathbb{R}_+^n$. The function $\varphi : \Omega \to \mathbb{R}_+$ is differentiable.*
(a) *If φ is increasing and Schur-convex, then φ is Schur geometrically convex.*
(b) *If φ is decreasing and Schur-concave, then φ is Schur geometrically concave.*

Lemma 4 *Let the set $\Omega \subset \mathbb{R}_+^n$. The function $\varphi : \Omega \to \mathbb{R}_+$ is differentiable.*
(a) *If φ is increasing and Schur-convex, then φ is Schur-harmonically convex.*
(b) *If φ is decreasing and Schur-concave, then φ is Schur-harmonically concave.*

Lemma 5 ([31, 32]) *Let $(\boldsymbol{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Then*

$$\big(A(\boldsymbol{x}), A(\boldsymbol{x}), \dots, A(\boldsymbol{x})\big) \prec (\boldsymbol{x} = (x_1, x_2, \dots, x_n), \tag{5}$$

where $A(\boldsymbol{x}) = \frac{1}{n}\sum_i^n x_i$.

Lemma 6 ([22]) *Let*

$$q(t) = \frac{u^t - 1}{t}.$$

If $u > 1$, then $q(t)$ is a log-convex function on $\mathbb{R}_+$.

3 Proof of main results

Proof of Theorem 1 For the case of $r = 1$ and $r = 2$, it is easy to prove that $c_n^*(f(\boldsymbol{x}), r)$ is Schur-convex on I^n.

Now consider the case of $r \geq 3$. By the symmetry of $c_n^*(f(\boldsymbol{x}), r)$, without loss of generality, we can set $x_1 > x_2$.

$$\begin{aligned} c_n^*\big((\boldsymbol{x}), r\big) = {} & \prod_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1\neq 0, i_2=0}} \sum_{j=1}^n i_j f(x_j) \times \prod_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1=0, i_2\neq 0}} \sum_{j=1}^n i_j f(x_j) \\ & \times \prod_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1\neq 0, i_2\neq 0}} \sum_{j=1}^n i_j f(x_j) \times \prod_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1=0, i_2=0}} \sum_{j=1}^n i_j f(x_j). \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial c_n^*(f(\boldsymbol{x}), r)}{\partial x_1} = {} & c_n^*\big(f(\boldsymbol{x}), r\big) \\ & \times \Bigg(\sum_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1\neq 0, i_2=0}} \frac{i_1 f'(x_1)}{\sum_{j=1}^n i_j f(x_j)} + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1\neq 0, i_2\neq 0}} \frac{i_1 f'(x_1)}{\sum_{j=1}^n i_j f(x_j)} \Bigg) \\ = {} & c_n^*\big(f(\boldsymbol{x}), r\big) \Bigg(\sum_{\substack{k+k_3+\cdots+k_n=r \\ k\neq 0}} \frac{k f'(x_1)}{k f(x_1) + \sum_{j=3}^n i_j f(x_j)} \\ & + \sum_{\substack{k+m+i_3+\cdots+i_n=r \\ k\neq 0, m\neq 0}} \frac{k f'(x_1)}{k f(x_1) + m f(x_2) + \sum_{j=3}^n i_j f(x_j)} \Bigg). \end{aligned} \tag{6}$$

By the same arguments,

$$\begin{aligned}\frac{\partial c_n^*(f(\boldsymbol{x}),r)}{\partial x_2} &= c_n^*\big(f(\boldsymbol{x}),r\big)\\ &= c_n^*\big(f(\boldsymbol{x}),r\big)\Bigg(\sum_{\substack{k+k_3+\cdots+k_n=r\\ k\neq 0}}\frac{kf'(x_2)}{kf(x_2)+\sum_{j=3}^n i_jf(x_j)}\\ &\quad+\sum_{\substack{k+m+i_3+\cdots+i_n=r\\ k\neq 0,m\neq 0}}\frac{kf'(x_2)}{kf(x_2)+mf(x_1)+\sum_{j=3}^n i_jf(x_j)}\Bigg),\\ \frac{\partial c_n^*(f(\boldsymbol{x}),r)}{\partial x_1}-\frac{\partial c_n^*(f(\boldsymbol{x}),r)}{\partial x_2} &= c_n^*\big(f(\boldsymbol{x}),r\big)(A_1+A_2),\end{aligned} \tag{7}$$

where

$$\begin{aligned}A_1 &= \sum_{\substack{k+k_3+\cdots+k_n=r\\ k\neq 0}}\left(\frac{kf'(x_1)}{kf(x_1)+\sum_{j=3}^n i_jf(x_j)}-\frac{kf'(x_2)}{kf(x_2)+\sum_{j=3}^n i_jf(x_j)}\right)\\ &= k\sum_{\substack{k+k_3+\cdots+k_n=r\\ k\neq 0}}\frac{k(f(x_2)f'(x_1)-f(x_1)f'(x_2))+(f'(x_1)-f'(x_2))\sum_{j=3}^n i_jf(x_j)}{(kf(x_1)+\sum_{j=3}^n i_jf(x_j))(kf(x_2)+\sum_{j=3}^n i_jf(x_j))}\end{aligned} \tag{8}$$

and

$$\begin{aligned}A_2 &= \sum_{\substack{k+m+i_3+\cdots+i_n=r\\ k\neq 0,m\neq 0}}\left(\frac{kf'(x_1)}{kf(x_1)+mf(x_2)+\sum_{j=3}^n i_jf(x_j)}-\frac{kf'(x_2)}{kf(x_2)+mf(x_1)+\sum_{j=3}^n i_jf(x_j)}\right)\\ &= k\sum_{\substack{k+m+i_3+\cdots+i_n=r\\ k\neq 0,m\neq 0}}\frac{\delta}{(kf(x_1)+mf(x_2)+\sum_{j=3}^n i_jf(x_j))(kf(x_2)+mf(x_1)+\sum_{j=3}^n i_jf(x_j))}\end{aligned}$$

where

$$\begin{aligned}\delta &= k\big(f(x_2)f'(x_1)-f(x_1)f'(x_2)\big)+m\big(f(x_1)f'(x_1)-f(x_2)f'(x_2)\big)\\ &\quad+\big(f'(x_1)-f'(x_2)\big)\sum_{j=3}^n i_jf(x_j).\end{aligned}$$

(a) Since the log-convex function must be convex function, so $f'(x_1)-f'(x_2)\geq 0$ and $f(x_2)f'(x_1)-f(x_1)f'(x_2)\geq 0$, and since $(f(x)f'(x))'=(f'(x))^2+f(x)f''(x)\geq 0$, so $f(x_1)f'(x_1)-f(x_2)f'(x_2)\geq 0$, and then $A_1\geq 0$ and $A_2\geq 0$. For $\boldsymbol{x}\in I^n$, we have

$$\frac{\partial c_n^*(f(\boldsymbol{x}),r)}{\partial x_1}-\frac{\partial c_n^*(f(\boldsymbol{x}),r)}{\partial x_2}\geq 0,$$

by Lemma 1, it follows that $c_n^*(f(\boldsymbol{x}),r)$ is Schur-convex on I^n.

(b) By Proposition 4, we know that $c_n^*(\boldsymbol{x},r)$ is increasing and Schur-concave on $\mathbb{R}_+^n$. Since f is concave, from (b) in Lemma 4 it follows that $c_n^*(f(\boldsymbol{x}),r)$ is Schur-concave on I^n.

The proof of Theorem 1 is completed. □

Proof of Theorem 2 Theorem 2 can be proved by Theorem 1 combined with Lemma 3.

The proof of Theorem 2 is completed. □

Proof of Theorem 3 Theorem 3 can be proved by Theorem 1 combined with Lemma 4.

The proof of Theorem 3 is completed. □

4 Applications

Let

$$c_n^*\left(\frac{1}{\boldsymbol{x}}, r\right) = \prod_{i_1+i_2+\cdots+i_n=r} \sum_{j=1}^{n} i_j\left(\frac{1}{x_j}\right). \tag{9}$$

Theorem 4 *The symmetric function $c_n^*(\frac{1}{\boldsymbol{x}}, r)$ is Schur-convex on $\mathbb{R}_+^n$. If r is an even integer (or odd integer, respectively), then $c_n^*(\frac{1}{\boldsymbol{x}}, r)$ is Schur-convex (or Schur-concave, respectively) on $\mathbb{R}_-^n$.*

Proof Let $f(x) = \frac{1}{x}$. Then $(\ln f(x))'' = \frac{1}{x^2}$, so $f(x)$ is log-convex on $\mathbb{R}_+$, by (a) in Theorem 1, it follows that $c_n^*(\frac{1}{\boldsymbol{x}}, r)$ is Schur-convex on $\mathbb{R}_+^n$.

For $\boldsymbol{x} \in \mathbb{R}_-^n$, $-\boldsymbol{x} \in \mathbb{R}_+^n$, so $c_n^*(\frac{1}{-\boldsymbol{x}}, r)$ is Schur-convex on $\mathbb{R}_-^n$. But

$$c_n^*\left(\frac{1}{-\boldsymbol{x}}, r\right) = (-1)^r c_n^*\left(\frac{1}{\boldsymbol{x}}, r\right).$$

This means that if r is an even integer, then

$$c_n^*\left(\frac{1}{\boldsymbol{x}}, r\right) = c_n^*\left(\frac{1}{-\boldsymbol{x}}, r\right)$$

is Schur-convex on $\mathbb{R}_-^n$.

If r is an odd integer, then

$$c_n^*\left(\frac{1}{\boldsymbol{x}}, r\right) = -c_n^*\left(\frac{1}{-\boldsymbol{x}}, r\right)$$

is Schur-concave on $\mathbb{R}_-^n$.

The proof of Theorem 4 is completed. □

By Theorem 4 and majorizing relation (7), it is not difficult to prove the following corollary.

Corollary 1 *If $\boldsymbol{x} \in \mathbb{R}_+^n$ or r is an even integer and $\boldsymbol{x} \in \mathbb{R}_-^n$, then we have*

$$\prod_{i_1+i_2+\cdots+i_n=r} \sum_{j=1}^{n} i_j\left(\frac{1}{x_j}\right) \geq \left(\frac{r}{A_n(\boldsymbol{x})}\right)^{\binom{n+r-1}{r}}, \tag{10}$$

where $A_n(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $\binom{n+r-1}{r} = \frac{(n+r-1)!}{r!((n+r-1)-r)!}$. If r is odd and $\boldsymbol{x} \in \mathbb{R}_-^n$, then inequality (10) is reversed.

Let

$$c_n^*\left(\frac{\mathbf{x}}{1-\mathbf{x}},r\right)=\prod_{i_1+i_2+\cdots+i_n=r}\sum_{j=1}^{n}i_j\left(\frac{x_j}{1-x_j}\right). \tag{11}$$

Theorem 5 *The symmetric function $c_n^*(\frac{\mathbf{x}}{1-\mathbf{x}},r)$ is Schur-convex, Schur-geometrically convex, and Schur-harmonically convex on $[\frac{1}{2},1]^n$.*

Proof Let $g(x)=\frac{x}{1-x}$. Then $(\ln g(x))''=\frac{2x-1}{x^2(1-x)^2}$, so $f(x)$ is log-convex on $[\frac{1}{2},1]$; by Theorem 1(a), it follows that $c_n^*(\frac{\mathbf{x}}{1-\mathbf{x}},r)$ is Schur-convex on $[\frac{1}{2},1]^n$. Noting that $g(x)$ is increasing on $[\frac{1}{2},1]$, by (a) in Theorem 2 and (a) in Theorem 3, it follows that $c_n^*(\frac{\mathbf{x}}{1-\mathbf{x}},r)$ is Schur-geometrically convex and Schur-harmonically convex on $[\frac{1}{2},1]^n$.

The proof of Theorem 5 is completed. □

From the majorizing relation (7), the following majorizing relation is established:

$$\big(\log G_n(\mathbf{x}),\log G_n(\mathbf{x}),\ldots,\log G_n(\mathbf{x})\big)\prec(\log x_1,\log x_2,\ldots,\log x_n).$$

By this majorizing relation and Theorem 5, it is not difficult to prove the following corollary.

Corollary 2 *If $\mathbf{x}\in[\frac{1}{2},1]^n$, then we have*

$$\prod_{i_1+i_2+\cdots+i_n=r}\sum_{j=1}^{n}i_j\left(\frac{x_j}{1-x_j}\right)\geq\left(\frac{rG_n(\mathbf{x})}{1-G_n(\mathbf{x})}\right)^{\binom{n+r-1}{r}}, \tag{12}$$

where $G_n(\mathbf{x})=\sqrt[n]{\prod_{i=1}^{n}x_i}$.

Let

$$c_n^*\left(\frac{1+\mathbf{x}}{1-\mathbf{x}},r\right)=\prod_{i_1+i_2+\cdots+i_n=r}\sum_{j=1}^{n}i_j\left(\frac{1+x_j}{1-x_j}\right). \tag{13}$$

Theorem 6

(a) *The symmetric function $c_n^*(\frac{1+\mathbf{x}}{1-\mathbf{x}},r)$ is Schur-convex, Schur-geometrically convex, and Schur-harmonically convex on $(0,1)^n$.*

(b) *If r is an even integer (or odd integer, respectively), then $c_n^*(\frac{1+\mathbf{x}}{1-\mathbf{x}},r)$ is Schur-convex (or Schur-concave, respectively) on $(1,+\infty)^n$.*

Proof (a) Let $h(x)=\frac{1+x}{1-x}$. Then $(\ln h(x))''=\frac{4x}{(1+x)^2(1-x)^2}$, so $f(x)$ is log-convex on $(0,1)$, by Theorem 1(a), it follows that $c_n^*(\frac{1+\mathbf{x}}{1-\mathbf{x}},r)$ is Schur-convex on $(0,1)^n$. Noting that $h(x)$ is increasing on $(0,1)^n$, by (a) in Theorem 2 and (a) in Theorem 3, it follows that $c_n^*(\frac{1+\mathbf{x}}{1-\mathbf{x}},r)$ is Schur-geometrically convex and Schur-harmonically convex on $(0,1)^n$.

(b) For $\mathbf{x}\in(1,+\infty)$, we consider

$$c_n^*\left(\frac{1+\mathbf{x}}{\mathbf{x}-1},r\right)=\prod_{i_1+i_2+\cdots+i_n=r}\sum_{j=1}^{n}i_j\left(\frac{1+x_j}{x_j-1}\right). \tag{14}$$

Let $h_1(x)=\frac{1+x}{x-1}$. Then $(\ln h_1(x))''=\frac{4x}{(1+x)^2(x-1)^2}$, so $f(x)$ is log-convex on $(1,+\infty)$, by (a) in Theorem 1, it follows that $c_n^*(\frac{1+\mathbf{x}}{\mathbf{x}-1},r)$ is Schur-convex on $(1,+\infty)^n$.

Noting that

$$c_n^*\left(\frac{1+\mathbf{x}}{1-\mathbf{x}},r\right)=(-1)^r c_n^*\left(\frac{1+\mathbf{x}}{\mathbf{x}-1},r\right),$$

combining the Schur-convexity of $c_n^*(\frac{1+\mathbf{x}}{\mathbf{x}-1},r)$, we can get (b) in Theorem 6.

The proof of Theorem 6 is completed. □

Let

$$c_n^*\left(\frac{1}{\mathbf{x}}-\mathbf{x},r\right)=\prod_{i_1+i_2+\cdots+i_n=r}\sum_{j=1}^{n} i_j\left(\frac{1}{x_j}-x_j\right). \tag{15}$$

Theorem 7

(a) *If r is an even integer (or odd integer, respectively), then $c_n^*(\frac{1}{\mathbf{x}}-\mathbf{x},r)$ is Schur-concave (or Schur-convex, respectively) on $\mathbb{R}_+^n$.*

(b) *The symmetric function $c_n^*(\frac{1}{\mathbf{x}}-\mathbf{x},r)$ is Schur-concave on $\mathbb{R}_-^n$.*

(c) *If r is an even integer, then $c_n^*(\frac{1}{\mathbf{x}}-\mathbf{x},r)$ is Schur-geometrically concave and Schur-harmonically concave on $(-\infty,1]^n$.*

Proof First consider

$$c_n^*\left(\mathbf{x}-\frac{1}{\mathbf{x}},r\right)=\prod_{i_1+i_2+\cdots+i_n=r}\sum_{j=1}^{n} i_j\left(x_j-\frac{1}{x_j}\right).$$

(a) Let $p(x)=x-\frac{1}{x}$. Then $p''(x)=-\frac{2}{x^3}$, so $f(x)$ is concave on $\mathbb{R}_+$, by Theorem 1(b), it follows that $c_n^*(\mathbf{x}-\frac{1}{\mathbf{x}},r)$ is Schur-concave on $\mathbb{R}_+^n$.

Noting that

$$c_n^*\left(\frac{1}{\mathbf{x}}-\mathbf{x},r\right)=(-1)^n c_n^*\left(\mathbf{x}-\frac{1}{\mathbf{x}},r\right),$$

combining the Schur-concavity of $c_n^*(\frac{1}{\mathbf{x}}-\mathbf{x},r)$, we can get (a) in Theorem 7.

(b) Noting that

$$c_n^*\left(\frac{1}{-\mathbf{x}}-(-\mathbf{x}),r\right)=(-1)^r c_n^*\left(\frac{1}{\mathbf{x}}-\mathbf{x},r\right),$$

combining (a) in Theorem 7, it is not difficult to verify that (b) in Theorem 7 holds.

(c) It is not difficult to verify that $p(x)=x-\frac{1}{x}$ is nonnegative and decreasing on $(-\infty,1]$, by Lemma 5 and Lemma 6, from (a) and (b) in Theorem 7, it follows that (c) in Theorem 7 holds.

The proof of Theorem 7 is completed. □

For $u>1$, let

$$c_n^*\left(\frac{u^{\mathbf{x}}-1}{\mathbf{x}},r\right)=\prod_{i_1+i_2+\cdots+i_n=r}\sum_{j=1}^{n} i_j\left(\frac{u^{x_j}-1}{x_j}\right). \tag{16}$$

Theorem 8 *The symmetric function $c_n^*(\frac{u^{\mathbf{x}}-1}{\mathbf{x}}, r)$ is Schur-convex, Schur-geometrically convex, and Schur-harmonically convex on $\mathbb{R}_+^n$ for $u > 1$.*

Proof Let $q(t) = \frac{u^t-1}{t}$. Then from Lemma 6 and (a) in Theorem 1, it follows that $c_n^*(\frac{u^{\mathbf{x}}-1}{\mathbf{x}}, r)$ is Schur-convex on $\mathbb{R}_+^n$ for $u > 1$.

Since

$$q'(t) = \frac{s(t)}{t^2},$$

where $s(t) = u^t(t\log u - 1) + 1$, $s'(t) = u^t \log u \log u^t > 0$, for $u > 1$ and $t > 0$, so $s(t) \geq s(0) = 0$, and then $q'(t) \geq 0$, that is, $q(t)$ is increasing on $\mathbb{R}_+^n$, by (a) in Theorem 2 and (a) in Theorem 3, it follows that $c_n^*(\frac{u^{\mathbf{x}}-1}{\mathbf{x}}, r)$ is Schur-geometrically convex and Schur-harmonically convex on $\mathbb{R}_+^n$.

The proof of Theorem 8 is completed. □

From the majorizing relation (7), the following majorizing relation is established:

$$\left(\frac{1}{H_n(\mathbf{x})}, \frac{1}{H_n(\mathbf{x})}, \ldots, \frac{1}{H_n(\mathbf{x})}\right) \prec \left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right).$$

By this majorizing relation and Theorem 8, it is not difficult to prove the following corollary.

Corollary 3 *If $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_+^n$ and $u > 1$, then*

$$\prod_{i_1+i_2+\cdots+i_n=r} \sum_{j=1}^{n} i_j \left(\frac{u^{x_j}-1}{x_j}\right) \geq \left(\frac{r(u^{H_n(\mathbf{x})}-1)}{H_n(\mathbf{x})}\right)^{\binom{n+r-1}{r}}, \tag{17}$$

where $H_n(\mathbf{x}) = \frac{n}{\sum_{i=1}^n x_i^{-1}}$.

Discovering and judging Schur convexity of various symmetric functions is an important subject in the study of the majorization theory. In recent years, many domestic scholars have made a lot of achievements in this field (see [24–30]).

Funding
The paper was supported by the General Project of Science and Technology Plan of Beijing Municipal Education Commission under Grant No. KM202011417012.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The author HS has made the conception of the manuscript and the writing of the first draft, the authors PW and JZ have made the revisions. All authors read and approved the final manuscript.

Publisher's Note

Received: 4 December 2019 Accepted: 4 March 2020 Published online: 11 March 2020

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