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Global existence and exponential decay of strong solutions for the three-dimensional Boussinesq equations

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Abstract

In this paper, we consider the global existence of strong solutions to the three-dimensional Boussinesq equations on the smooth bounded domain Ω . Based on the blow-up criterion and uniform estimates, we prove that the strong solution exists globally in time if the initial L^2 -norm of velocity and temperature are small. Moreover, an exponential decay rate of the strong solution is obtained.

Keywords: Boussinesq equations; Blow-up criterion; Global strong solutions; Exponential decay rate

1 Introduction

In this paper, we consider the following three-dimensional incompressible Boussinesq equations in the Eulerian coordinates:

$$\begin{cases} u_t - \mu \Delta u + u \cdot \nabla u + \nabla P = \theta e_3, \\ \theta_t - \kappa \Delta \theta + u \cdot \nabla \theta = 0, \\ \nabla \cdot u = 0, \\ u(x, 0) = u_0(x), \quad \theta(x, 0) = \theta_0(x), \end{cases} \quad (1.1)$$

where $u = (u_1, u_2, u_3)(x, t)$, $\theta = \theta(x, t)$, $P(x, t)$ are unknown functions denoting fluid velocity vector field, absolute temperature, and scalar pressure, $t \geq 0$ is time, $x \in \Omega$ is spatial coordinate. μ is the kinematic viscosity, κ is the thermal diffusivity, and $e_3 = (0, 0, 1)$ is the unit vector in the x_3 direction. The given functions u_0 and θ_0 are the initial velocity and initial temperature, respectively.

Boussinesq system (1.1) has been widely used in atmospheric sciences and oceanic fluids [9], and there is a huge amount of literature on the well-posedness theory of strong and weak solutions for the three-dimensional Boussinesq equations. In 2004, Sawada and Taniuchi [12] established the local existence and uniqueness of strong solutions in the whole space. In 2008, Danchin and Paicu [6] obtained global small solutions in Lorentz spaces. In 2012, Brandolese and Schonbek [2] showed the polynomial decay rate of weak

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and strong solutions, global smooth solution and its stability were given by Liu and Li [8] in 2014. In 2016, Qin et al. [11] established the global classical solutions for axisymmetric equations with anisotropic initial data. In 2017, Ye [15] proved global existence of smooth solution to a modified Boussinesq model without thermal diffusion. In 2018, Wen and Ye [14] established the regularity and uniqueness of strong solution for the damped Boussinesq equations with zero thermal diffusion. Global well-posedness with fractional partial dissipation can be found in a recent work [16]. For local and global theories of solutions in two-dimensional space, we refer to [1, 3–5, 10, 17] and the references therein.

Inspired by the results of full compressible Navier–Stokes equations [13] and nonhomogeneous incompressible magnetohydrodynamic equations [7], in this paper we consider global existence and exponential decay of strong solutions to system (1.1) with the following initial-boundary conditions:

$$(u, \theta)(x, t)|_{t=0} = (u_0(x), \theta_0(x)), \tag{1.2}$$

$$(u, \theta)(x, t)|_{\partial\Omega} = 0. \tag{1.3}$$

Now, we state our main results as follows.

Theorem 1.1 (Blow-up criterion) *Suppose that the initial data satisfy $(u_0, \theta_0) \in H_0^1 \cap H^2$ and $\operatorname{div} u_0 = 0$ in Ω . Let (u, θ) be the strong solution to the initial-boundary value problem (1.1)–(1.3) on $\Omega \times (0, T]$ satisfying*

$$\begin{cases} (u, \theta) \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2), \\ (u_t, \theta_t) \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1). \end{cases} \tag{1.4}$$

If $T^* < +\infty$ is the maximal time of existence for the strong solution (u, θ) , then

$$\lim_{T \rightarrow T^*} \sup_{[0, T]} \|\nabla u\|_{L^\infty(0, T; L^2)} = +\infty. \tag{1.5}$$

Theorem 1.2 (Global strong solution) *For any given $K_i > 0$ ($i = 1, 2$), suppose that the initial data satisfy $(u_0, \theta_0) \in H_0^1 \cap H^2$, $\operatorname{div} u_0 = 0$ in Ω , and*

$$\|\nabla u_0\|_{L^2} \leq \sqrt{K_1}, \quad \|\theta_0\|_{L^2} \leq \sqrt{K_2}. \tag{1.6}$$

Then the initial-boundary value problem (1.1)–(1.3) admits a unique global strong solution (u, θ) on $\Omega \times [0, T]$ for any $T > 0$, provided that there exists a constant $\varepsilon_0 > 0$ such that

$$\|u_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2 \leq \varepsilon_0, \tag{1.7}$$

where ε_0 depends on K_1, μ, κ , and some other known constants but is independent of T .

Remark 1.1 When $\theta = 0$, system (1.1) reduces to Navier–Stokes equations. Then Theorem 1.2 implies that Navier–Stokes equations admit a unique global strong solution on $\Omega \times [0, T]$ for any $T > 0$, provided that there exists a constant $\varepsilon_0 > 0$ such that

$$\|u_0\|_{L^2}^2 \leq \varepsilon_0, \tag{1.8}$$

where ε_0 depends on K_1, μ and some other known constants but is independent of T .

Theorem 1.3 (Asymptotic behavior) *Under the conditions of Theorem 1.2, it holds that*

$$\int_{\Omega} (\theta^2 + |\nabla u|^2) \leq C_1 \exp\{-C_2 t\} \tag{1.9}$$

for any $t \in [0, +\infty)$, provided that there exists a constant $\varepsilon_0 > 0$ such that

$$\|u_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2 \leq \varepsilon_0, \tag{1.10}$$

where positive constants C_1, C_2, ε_0 depend on K_1, K_2, μ, κ , and some other known constants but are independent of t .

Remark 1.2 In Theorem 1.3, we obtain exponential decay rate (1.9) in a bounded domain, which refines the polynomial decay in [2] and [8].

For the fixed viscosity and heat conduction, we need the “smallness” of initial velocity and temperature. But the velocity and temperature can be large. From the proof of Theorem 1.2 and Theorem 1.3, we have the following corollary.

Corollary 1.4 *For any given $K_i > 0$ ($i = 1, 2$), suppose that the initial data satisfy $(u_0, \theta_0) \in H_0^1 \cap H^2, \operatorname{div} u_0 = 0$ in Ω , and*

$$\|\nabla u_0\|_{L^2} \leq \sqrt{K_1}, \quad \|\theta_0\|_{L^2} \leq \sqrt{K_2}. \tag{1.11}$$

Then the initial-boundary value problem (1.1)–(1.3) admits a unique global strong solution (u, θ) on $\Omega \times [0, T]$ for any $T > 0$, provided that there exists a constant $\mu^ > 0$ such that*

$$\mu \geq \mu^*.$$

Furthermore, when μ and κ are large enough, the global strong solution satisfies the following exponential decay rate:

$$\int_{\Omega} (\theta^2 + |\nabla u|^2) \leq C_1 \exp\{-C_2 t\} \tag{1.12}$$

for any $t \in [0, +\infty)$, where positive constants C_1, C_2, μ^* depend on K_1, K_2, μ, κ , and some other known constants but are independent of t .

Notations In the following paragraph, positive generic constants are denoted by C , which may change in different places.

The rest of the paper is organized as follows. In Sect. 2, under the assumption of (1.5) is false, we prove that the maximal time of existence for the strong solution is $T^* = +\infty$, where the standard energy estimate is used and it holds uniform in time. In Sect. 3, we show that $\|\nabla u\|_{L^\infty(0,T;L^2)}$ will never blow up in finite time, which combines the blow-up criterion in Theorem 1.1, global existence of strong solution is proved in Theorem 1.2 provided the initial data of velocity and temperature are suitably small under the L^2 -norm. Finally, in Sect. 4, exponential decay rate of the strong solution is obtained.

2 Blow-up criterion

Now, we state some uniform a priori estimates to prove Theorem 1.1.

Lemma 2.1 *Under the conditions of Theorem 1.1, it holds that*

$$\sup_{0 \leq t \leq T} \|\theta(t)\|_{L^2}^2 + \int_0^T \kappa \|\nabla\theta(s)\|_{L^2}^2 ds = \|\theta_0\|_{L^2}^2 \leq K_2, \tag{2.1}$$

$$\sup_{0 \leq t \leq T} \|u(t)\|_{L^2}^2 + \int_0^T \mu \|\nabla u(s)\|_{L^2}^2 ds \leq \|u_0\|_{L^2}^2 + \frac{4}{\mu\kappa} \|\theta_0\|_{L^2}^2 \leq K_1 + \frac{4K_2}{\mu\kappa}. \tag{2.2}$$

Proof Multiplying temperature equation (1.1)₂ by θ integration by part and using divergence free property, we get

$$\frac{1}{2} \frac{d}{dt} \|\theta(t)\|_{L^2}^2 + \kappa \|\nabla\theta(s)\|_{L^2}^2 = 0, \tag{2.3}$$

integrating (2.3) over $(0, t)$, (2.1) is proved.

Multiplying momentum equation (1.1)₁ by u and using the Poincaré inequality, we obtain

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2}^2 + \mu \|\nabla u\|_{L^2}^2 \leq \|u\|_{L^2} \|\theta\|_{L^2} \leq \frac{\mu}{2} \|\nabla u\|_{L^2}^2 + \frac{2}{\mu} \|\nabla\theta\|_{L^2}^2.$$

Then we have

$$\frac{d}{dt} \|u(t)\|_{L^2}^2 + \mu \|\nabla u\|_{L^2}^2 \leq \frac{4}{\mu} \|\nabla\theta\|_{L^2}^2. \tag{2.4}$$

Integrating the above inequality over time $(0, t)$ and combining (2.1), one can get (2.2). \square

Lemma 2.2 *Under the conditions of Theorem 1.1, and suppose (1.5) is false. Then it holds that*

$$\sup_{0 \leq t \leq T} (\|\nabla u(t)\|_{L^2}^2 + \|\nabla\theta(t)\|_{L^2}^2) + \int_0^T (\|u_t(s)\|_{L^2}^2 + \|\nabla^2\theta(s)\|_{L^2}^2) ds \leq C, \tag{2.5}$$

where constant C depends on the initial data, μ, κ , and some other known constants but is independent of T .

Proof Multiplying momentum equation (1.1)₁ by u_t and integrating by parts, we have

$$\begin{aligned} \frac{\mu}{2} \frac{d}{dt} \|\nabla u(t)\|_{L^2}^2 + \|u_t\|_{L^2}^2 &= - \int (u \cdot \nabla u) \cdot u_t dx + \int \theta e_3 \cdot u_t dx \\ &\leq \|u\|_{L^6} \|\nabla u\|_{L^3} \|u_t\|_{L^2} + \|\theta\|_{L^2} \|u_t\|_{L^2} \\ &\leq C (\|\nabla u\|_{L^2}^{\frac{3}{2}} \|u\|_{H^2}^{\frac{1}{2}} \|u_t\|_{L^2} + \|\theta\|_{L^2} \|u_t\|_{L^2}). \end{aligned} \tag{2.6}$$

On the other hand, (1.1)₁ can be rewritten as

$$-\mu \Delta u + \nabla P = -u_t - u \cdot \nabla u + \theta e_3.$$

By the H^2 -theory of Stokes system, we derive that

$$\mu \|u\|_{H^2} \leq C \| -u_t - u \cdot \nabla u + \theta e_3 \|_{L^2} \leq C (\|u_t\|_{L^2} + \|\nabla u\|_{L^2}^{\frac{3}{2}} \|u\|_{H^2}^{\frac{1}{2}} + \|\theta\|_{L^2}),$$

and thus

$$\|u\|_{H^2} \leq \frac{C}{\mu} (\|u_t\|_{L^2} + \|\nabla u\|_{L^2}^3 + \|\theta\|_{L^2}). \tag{2.7}$$

Inserting (2.7) into (2.6), we obtain

$$\begin{aligned} & \frac{\mu}{2} \frac{d}{dt} \|\nabla u(t)\|_{L^2}^2 + \|u_t\|_{L^2}^2 \\ & \leq \frac{C}{\mu} (\|\nabla u\|_{L^2}^{\frac{3}{2}} \|u_t\|_{L^2}^{\frac{3}{2}} + \|\nabla u\|_{L^2}^3 \|u_t\|_{L^2} + \|\nabla u\|_{L^2}^{\frac{3}{2}} \|\theta\|_{L^2}^{\frac{1}{2}} \|u_t\|_{L^2}) + C \|\theta\|_{L^2} \|u_t\|_{L^2} \\ & \leq \frac{1}{2} \|u_t\|_{L^2}^2 + C \left(\frac{1}{\mu^4} \|\nabla u\|_{L^2}^6 + \frac{1}{\mu^2} \|\nabla u\|_{L^2}^6 + \frac{1}{\mu^2} \|\nabla u\|_{L^2}^3 \|\theta\|_{L^2} + \|\theta\|_{L^2}^2 \right) \\ & \leq \frac{1}{2} \|u_t\|_{L^2}^2 + C \left(\frac{1}{\mu^4} \|\nabla u\|_{L^2}^2 + \frac{1}{\mu^2} \|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 \right). \end{aligned} \tag{2.8}$$

Then, integrating (2.8) over $(0, t)$, we have

$$\mu \|\nabla u(t)\|_{L^2}^2 + \int_0^t \|u_t(s)\|_{L^2}^2 ds \leq C \left(1 + \frac{1}{\mu^5} + \frac{1}{\mu^3} + \frac{1}{\kappa} \right), \tag{2.9}$$

where we have used Lemma 2.1 and constant C depending on K_1 and K_2 .

Next, applying the operator ∇ on both sides of temperature equation (1.1)₂ and multiplying by $\nabla \theta$, integrating the resulting equation over Ω , we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla \theta(t)\|_{L^2}^2 + \kappa \|\nabla^2 \theta\|_{L^2}^2 &= - \int \nabla(u \cdot \nabla \theta) \cdot \nabla \theta \, dx \\ &\leq \|\nabla u\|_{L^3} \|\nabla \theta\|_{L^6} \|\nabla \theta\|_{L^2} \\ &\leq C \|u\|_{H^2} \|\nabla^2 \theta\|_{L^2} \|\nabla \theta\|_{L^2} \\ &\leq \frac{\kappa}{2} \|\nabla^2 \theta\|_{L^2}^2 + \frac{C}{\kappa} \|u\|_{H^2}^2 \|\nabla \theta\|_{L^2}^2, \end{aligned} \tag{2.10}$$

integrating (2.10) over time $(0, t)$ and using Gronwall's inequality, which completes the proof of Lemma 2.2. □

Lemma 2.3 *Under the conditions of Theorem 1.1, and suppose (1.5) is false. Then it holds that*

$$\sup_{0 \leq t \leq T} (\|u_t(t)\|_{L^2}^2 + \|\theta_t(t)\|_{L^2}^2) + \int_0^T (\|\nabla u_t(s)\|_{L^2}^2 + \|\nabla \theta_t(s)\|_{L^2}^2) ds \leq C, \tag{2.11}$$

where constant C depends on the initial data, μ , κ , and some other known constants but is independent of T .

Proof Taking the operator ∂_t to (1.1)₁, multiplying the resulting equation by u_t and integrating by parts, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u_t(t)\|_{L^2}^2 + \mu \|\nabla u_t\|_{L^2}^2 &= - \int (u_t \cdot \nabla u) \cdot u_t \, dx + \int \theta_t e_3 \cdot u_t \, dx \\ &\leq \|\nabla u\|_{L^3} \|u_t\|_{L^6} \|u_t\|_{L^2} + \|\theta_t\|_{L^2} \|u_t\|_{L^2} \\ &\leq C(\|u\|_{H^2} \|\nabla u_t\|_{L^2} \|u_t\|_{L^2} + \|\theta_t\|_{L^2} \|u_t\|_{L^2}) \\ &\leq \frac{\mu}{2} \|\nabla u_t\|_{L^2}^2 + \frac{C}{\mu} (\|u\|_{H^2}^2 \|u_t\|_{L^2}^2 + \|\theta_t\|_{L^2}^2). \end{aligned} \tag{2.12}$$

On the other hand, multiplying (1.1)₂ by θ_t and integrating the resulting equation over Ω , we obtain

$$\begin{aligned} \frac{\kappa}{2} \frac{d}{dt} \|\nabla \theta(t)\|_{L^2}^2 + \|\theta_t\|_{L^2}^2 &= - \int (u \cdot \nabla \theta) \theta_t \, dx \leq \|u\|_{L^\infty} \|\nabla \theta\|_{L^2} \|\theta_t\|_{L^2} \\ &\leq \frac{1}{2} \|\theta_t\|_{L^2}^2 + C \|u\|_{H^2}^2 \|\nabla \theta\|_{L^2}^2. \end{aligned} \tag{2.13}$$

Then, integrating (2.12), (2.13) over $(0, t)$ and using Gronwall’s inequality, we have

$$\mu \|u_t\|_{L^2}^2 + \kappa \|\nabla \theta\|_{L^2}^2 + \int_0^t (\mu^2 \|\nabla u_t(s)\|_{L^2}^2 + \|\theta_t(s)\|_{L^2}^2) \, ds \leq C. \tag{2.14}$$

Next, applying the operator ∂_t on both sides of temperature equation (1.1)₂ and multiplying by θ_t , we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\theta_t(t)\|_{L^2}^2 + \kappa \|\nabla \theta_t\|_{L^2}^2 &= - \int (u_t \cdot \nabla \theta + u \cdot \nabla \theta_t) \theta_t \, dx \\ &\leq \|u_t\|_{L^3} \|\nabla \theta\|_{L^6} \|\theta_t\|_{L^2} \\ &\leq \|u_t\|_{H^1}^2 + C \|\nabla^2 \theta\|_{L^2}^2 \|\theta_t\|_{L^2}^2, \end{aligned} \tag{2.15}$$

integrating (2.15) over time $(0, t)$, using Gronwall’s inequality, and taking (2.5), (2.14) into consideration, we obtain

$$\|\theta_t(t)\|_{L^2}^2 + \int_0^t \|\nabla \theta_t(s)\|_{L^2}^2 \, ds \leq C, \tag{2.16}$$

which completes the proof of Lemma 2.3. □

From Lemmas 2.1–2.3, we can see that Theorem 1.1 is proved.

3 Global strong solution

In this section, we prove the global existence and uniqueness of the strong solution. Assume that $T^* > 0$ is the maximal existence time of the strong solution. We prove $T^* = +\infty$ by using contradiction arguments. If $T^* < +\infty$, our aim is to prove that (1.5) is not true under the conditions of Theorem 1.2, which is the desired contradiction.

We define

$$M(T) = \sup_{0 \leq t \leq T} \|\nabla u\|_{L^2}^2.$$

Proposition 3.1 *Assume that the initial data satisfy the conditions in Theorem 1.2, and the local strong solution satisfies*

$$M(T) \leq 2K_1, \quad (x, t) \in \Omega \times [0, T], \tag{3.1}$$

then

$$M(T) \leq \frac{3}{2}K_1, \quad (x, t) \in \Omega \times [0, T], \tag{3.2}$$

provided that there exists ε_0 depending on K_1, μ, κ , and some other known constants but independent of T such that $\|u_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2 \leq \varepsilon_0$ is suitably small.

Lemma 3.2 *Under the condition of Proposition 3.1, there exists C^* depending on K_1, μ, κ , and some other known constants but independent of T , it holds that*

$$M(T) \leq \frac{3}{2}K_1, \tag{3.3}$$

provided that $\varepsilon_0 \leq C^*$.

Proof Recalling Lemma 2.2, from (2.8) we can see that

$$\begin{aligned} & \frac{\mu}{2} \frac{d}{dt} \|\nabla u(t)\|_{L^2}^2 + \|u_t\|_{L^2}^2 \\ & \leq \frac{1}{2} \|u_t\|_{L^2}^2 + C \left(\frac{1}{\mu^4} \|\nabla u\|_{L^2}^6 + \frac{1}{\mu^2} \|\nabla u\|_{L^2}^6 + \frac{1}{\mu^2} \|\nabla u\|_{L^2}^3 \|\theta\|_{L^2} + \|\theta\|_{L^2}^2 \right) \\ & \leq \frac{1}{2} \|u_t\|_{L^2}^2 + C \left(\frac{K_1^2}{\mu^4} \|\nabla u\|_{L^2}^2 + \frac{K_1^2}{\mu^2} \|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 \right). \end{aligned} \tag{3.4}$$

Then, integrating (3.4) over $(0, t)$, we obtain

$$\begin{aligned} & \|\nabla u(t)\|_{L^2}^2 + \frac{1}{\mu} \int_0^t \|u_t(s)\|_{L^2}^2 ds \\ & \leq \|\nabla u_0\|_{L^2}^2 + C \left(\frac{K_1^2}{\mu^5} + \frac{K_1^2}{\mu^3} \right) \int_0^t \|\nabla u(s)\|_{L^2}^2 ds + \frac{C}{\mu} \int_0^t \|\nabla \theta(s)\|_{L^2}^2 ds \\ & \leq \|\nabla u_0\|_{L^2}^2 + \frac{CK_1^2}{\mu^3} \left(\frac{1}{\mu^2} + 1 \right) \left(\frac{1}{\mu} \|u_0\|_{L^2}^2 + \frac{1}{\mu^2 \kappa} \|\theta_0\|_{L^2}^2 \right) + \frac{C}{\mu \kappa} \|\theta_0\|_{L^2}^2, \end{aligned} \tag{3.5}$$

where we have used Lemma 2.1. Then we have

$$\begin{aligned} & \|\nabla u(t)\|_{L^2}^2 \\ & \leq \|\nabla u_0\|_{L^2}^2 + \frac{CK_1^2}{\mu^4} \left(\frac{1}{\mu^2} + 1 \right) \|u_0\|_{L^2}^2 + \frac{CK_1^2}{\mu^5 \kappa} \left(\frac{1}{\mu^2} + 1 \right) \|\theta_0\|_{L^2}^2 + \frac{C}{\mu \kappa} \|\theta_0\|_{L^2}^2 \end{aligned}$$

$$\begin{aligned} &\leq \|\nabla u_0\|_{L^2}^2 + \frac{CK_1^2}{\mu^4} \left(\frac{1}{\mu^2} + 1\right) \|u_0\|_{L^2}^2 + \frac{C}{\mu\kappa} \left(\frac{K_1^2}{\mu^6} + \frac{K_1^2}{\mu^4} + 1\right) \|\theta_0\|_{L^2}^2 \\ &\leq K_1 + \frac{1}{2}K_1 = \frac{3}{2}K_1, \end{aligned}$$

provided $C^* \leq \frac{1}{2}K_1\delta^{-1}$, $\delta = \max\left\{\frac{CK_1^2}{\mu^4} \left(\frac{1}{\mu^2} + 1\right), \frac{C}{\mu\kappa} \left(\frac{K_1^2}{\mu^6} + \frac{K_1^2}{\mu^4} + 1\right)\right\}$. This completes the proof of Lemma 3.2. □

Hence, from Theorem 1.1 and Proposition 3.1, we can see that Theorem 1.2 is proved provided

$$\|u_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2 \leq \varepsilon_0 \leq C^*.$$

4 Decay estimates

Finally, in this section, based on the global in time strong solution, we have the following exponential decay rate.

Lemma 4.1 *Under the conditions of Theorem 1.2, we get that*

$$\int_{\Omega} (\theta^2 + |\nabla u|^2) \leq C_1 \exp\{-C_2 t\} \tag{4.1}$$

holds for any $t \in [0, +\infty)$, provided that

$$\varepsilon_0 \leq C^*$$

for some positive constants C_1, C_2 , and C^* depending on μ, κ, K_1, K_2 , and some other known constants but independent of t .

Proof From (2.4) we can see that

$$\frac{d}{dt} \|u(t)\|_{L^2}^2 + \mu \|\nabla u\|_{L^2}^2 \leq \frac{4}{\mu} \|\nabla \theta\|_{L^2}^2. \tag{4.2}$$

Multiplying $\frac{2CK_1^2}{\mu^3} \left(\frac{1}{\mu^2} + 1\right)$ on both sides of inequality (4.2) and adding the resulting equation into (3.4), we obtain

$$\begin{aligned} &\frac{d}{dt} \left(\frac{2CK_1^2}{\mu^3} \left(\frac{1}{\mu^2} + 1\right) \|u(t)\|_{L^2}^2 + \frac{\mu}{2} \|\nabla u(t)\|_{L^2}^2 \right) \\ &\quad + \frac{2CK_1^2}{\mu^3} \left(\frac{1}{\mu^2} + 1\right) \|\nabla u(t)\|_{L^2}^2 + \frac{1}{2} \|u_t\|_{L^2}^2 \\ &\leq C \left(\frac{8K_1^2(1 + \mu^2)}{\mu^6} + 1 \right) \|\nabla \theta(t)\|_{L^2}^2. \end{aligned} \tag{4.3}$$

Next, multiplying $2C\left(\frac{8K_1^2(1+\mu^2)}{\mu^6\kappa} + \frac{1}{\kappa}\right)$ on both sides of inequality (2.3) and adding the resulting equation into (4.3), we have

$$\begin{aligned} & \frac{d}{dt} \left(\frac{2CK_1^2}{\mu^3} \left(\frac{1}{\mu^2} + 1 \right) \|u(t)\|_{L^2}^2 + \frac{\mu}{2} \|\nabla u(t)\|_{L^2}^2 + 2C \left(\frac{8K_1^2(1+\mu^2)}{\mu^6\kappa} + \frac{1}{\kappa} \right) \|\theta(t)\|_{L^2}^2 \right) \\ & + \frac{2CK_1^2}{\mu^3} \left(\frac{1}{\mu^2} + 1 \right) \|\nabla u(t)\|_{L^2}^2 + C \left(\frac{8K_1^2(1+\mu^2)}{\mu^6\kappa} + \frac{1}{\kappa} \right) \|\nabla \theta(t)\|_{L^2}^2 \leq 0. \end{aligned} \tag{4.4}$$

Taking

$$Y(t) = \frac{2CK_1^2}{\mu^3} \left(\frac{1}{\mu^2} + 1 \right) \|u(t)\|_{L^2}^2 + \frac{\mu}{2} \|\nabla u(t)\|_{L^2}^2 + 2C \left(\frac{8K_1^2(1+\mu^2)}{\mu^6\kappa} + \frac{1}{\kappa} \right) \|\theta(t)\|_{L^2}^2,$$

it can be estimated as

$$\begin{aligned} Y(t) & \leq \frac{2CK_1^2}{\mu^3} \left(\frac{1}{\mu^2} + 1 \right) \|\nabla u(t)\|_{L^2}^2 + \frac{\mu}{2} \|\nabla u(t)\|_{L^2}^2 + 2C \left(\frac{8K_1^2(1+\mu^2)}{\mu^6\kappa} + \frac{1}{\kappa} \right) \|\nabla \theta(t)\|_{L^2}^2 \\ & \leq \left(\frac{2CK_1^2(1+\mu^2)}{\mu^5} + \frac{\mu}{2} \right) \|\nabla u(t)\|_{L^2}^2 + 2C \left(\frac{8K_1^2(1+\mu^2)}{\mu^6\kappa} + \frac{1}{\kappa} \right) \|\nabla \theta(t)\|_{L^2}^2, \end{aligned} \tag{4.5}$$

where we have used the Poincaré inequality.

Then there exists a constant M depending on C, K_1, μ, κ such that

$$Y(t) \leq M \left(\frac{2CK_1^2}{\mu^3} \left(\frac{1}{\mu^2} + 1 \right) \|\nabla u(t)\|_{L^2}^2 + C \left(\frac{8K_1^2(1+\mu^2)}{\mu^6\kappa} + \frac{1}{\kappa} \right) \|\nabla \theta(t)\|_{L^2}^2 \right).$$

At last, from (4.4), we have

$$Y'(t) + \frac{1}{M} Y(t) \leq 0,$$

which deduces $Y(t) \leq Y(0) \exp\{-\frac{t}{M}\}$. This completes the proof of Theorem 1.3. □

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