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# Hermite–Hadamard type inequalities for co-ordinated convex and quasi-convex functions and their applications

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## Abstract

In the article, we present several Hermite–Hadamard type inequalities for the co-ordinated convex and quasi-convex functions and give an application to the product of the moment of two continuous and independent random variables. Our results are generalizations of some earlier results. Additionally, an illustrative example on the probability distribution is given to support our results.

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## 1 Introduction

Let  $\mathcal{I} \subseteq \mathbb{R}$  be a nonempty interval. Then a real-valued function  $\Phi : \mathcal{I} \rightarrow \mathbb{R}$  is said to be convex (concave) if the inequality

$$\Phi[\varrho\mu + (1 - \varrho)v] \leq (\geq) \varrho\Phi(\mu) + (1 - \varrho)\Phi(v)$$

holds for all  $\mu, v \in \mathcal{I}$  and  $\varrho \in [0, 1]$ .

It is a fact that the convex (concave) function is one of the most basic and important functions in the theory of geometric function, it has widely applications in pure and applied mathematics, physics, mechanics, statistics and economics, and meteorology [1–30]. Recently, the generalizations, extensions, variants and refinements for the convexity (concavity) have attracted the interest of several researchers [31–40]. In particular, many remarkable inequalities and properties in many branches of mathematics can be found in the literature [41–70] using the convexity (concavity) theory. The well-known Hermite–Hadamard inequality states that the double inequality

$$\Phi\left(\frac{\lambda_1 + \lambda_2}{2}\right) \leq (\geq) \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \Phi(\mu) dx \leq (\geq) \frac{\Phi(\lambda_1) + \Phi(\lambda_2)}{2} \quad (1.1)$$

holds for all  $\lambda_1, \lambda_2 \in \mathcal{I}$  with  $\lambda_1 \neq \lambda_2$  if  $\Phi : \mathcal{I} \rightarrow \mathbb{R}$  is a convex (concave) function.

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In the past hundred years, inequality (1.1) has inspired many researchers to estimate the bounds for

$$\left| \frac{\Phi(\lambda_1) + \Phi(\lambda_2)}{2} - \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \Phi(\mu) d\mu \right|$$

and

$$\left| \Phi\left(\frac{\lambda_1 + \lambda_2}{2}\right) - \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \Phi(\mu) d\mu \right|,$$

and all the obtained results are called Hermite–Hadamard type inequalities.

It is well known that the multivariable functions also have the concept of convexity (concavity). An example, we recall the definition of convexity (concavity) for the bivariate functions.

Let  $\lambda_1, \lambda_2, \xi_1, \xi_2 \in \mathbb{R}$  such that  $\lambda_1 < \lambda_2$  and  $\xi_1 < \xi_2$ . Then a bivariate real-valued function  $\Phi : [\lambda_1, \lambda_2] \times [\xi_1, \xi_2] \rightarrow \mathbb{R}$  is said to be convex (concave) if the inequality

$$\Phi(\varrho\mu + (1 - \varrho)\rho, \varrho\nu + (1 - \varrho)\omega) \leq (\geq) \varrho\Phi(\mu, \nu) + (1 - \varrho)\Phi(\rho, \omega)$$

holds for all  $(\mu, \nu), (\rho, \omega) \in [\lambda_1, \lambda_2] \times [\xi_1, \xi_2]$  and  $\varrho \in [0, 1]$ .

In order to establish the relation involving the convexity between the bivariate and univariate functions, Dragomir [71] introduced the definition of the bivariate co-ordinated convex function as follows.

**Definition 1.1** (See [71]) Let  $\lambda_1, \lambda_2, \xi_1, \xi_2 \in \mathbb{R}$  such that  $\lambda_1 < \lambda_2$  and  $\xi_1 < \xi_2$ . Then a bivariate real-valued function  $\Phi : [\lambda_1, \lambda_2] \times [\xi_1, \xi_2] \rightarrow \mathbb{R}$  is said to be convex on the co-ordinates if both of the partial mappings  $\Phi_\nu : [\lambda_1, \lambda_2] \rightarrow \mathbb{R}$  and  $\Phi_\mu : [\xi_1, \xi_2] \rightarrow \mathbb{R}$  defined by

$$\Phi_\nu(\delta) = \Phi(\delta, \nu)$$

and

$$\Phi_\mu(\theta) = \Phi(\mu, \theta)$$

are convex for all  $\mu \in [\lambda_1, \lambda_2]$  and  $\nu \in [\xi_1, \xi_2]$ .

Latif and Alomari [72] proved that the bivariate real-valued function  $\Phi : [\lambda_1, \lambda_2] \times [\xi_1, \xi_2] \rightarrow \mathbb{R}$  is convex on the co-ordinates if and only if

$$\begin{aligned} &\Phi(\varrho\mu + (1 - \varrho)\nu, \varsigma\rho + (1 - \varsigma)\omega) \\ &\leq \varrho\varsigma\Phi(\mu, \rho) + \varrho(1 - \varsigma)\Phi(\mu, \omega) + \varsigma(1 - \varrho)\Phi(\nu, \rho) + (1 - \varrho)(1 - \varsigma)\Phi(\nu, \omega) \end{aligned}$$

for all  $\varrho, \varsigma \in [0, 1] \times [0, 1]$  and  $(\mu, \rho), (\nu, \omega) \in [\lambda_1, \lambda_2] \times [\xi_1, \xi_2]$ .

Dragomir [71] proved that every convex mapping  $\Phi : [\lambda_1, \lambda_2] \times [\xi_1, \xi_2] \rightarrow \mathbb{R}$  is convex on the co-ordinates and the reverse is not true, and established the Hermite–Hadamard type inequality for the co-ordinated convex function on the rectangle of the plane  $\mathbb{R}^2$ .

**Theorem 1.2** (See [71]) *Let  $\lambda_1, \lambda_2, \xi_1, \xi_2 \in \mathbb{R}$  such that  $\lambda_1 < \lambda_2$  and  $\xi_1 < \xi_2$ , and  $\Phi : [\lambda_1, \lambda_2] \times [\xi_1, \xi_2] \rightarrow \mathbb{R}$  be a co-ordinated convex function. Then one has*

$$\begin{aligned}
 & \Phi\left(\frac{\lambda_1 + \lambda_2}{2}, \frac{\xi_1 + \xi_2}{2}\right) \\
 & \leq \frac{1}{2} \left[ \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \Phi\left(\mu, \frac{\xi_1 + \xi_2}{2}\right) d\mu + \frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} \Phi\left(\frac{\lambda_1 + \lambda_2}{2}, \nu\right) d\nu \right] \\
 & \leq \frac{1}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi(\mu, \nu) d\nu d\mu \\
 & \leq \frac{1}{4} \left[ \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \Phi(\mu, \xi_1) d\mu + \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \Phi(\mu, \xi_2) d\mu \right. \\
 & \quad \left. + \frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} \Phi(\lambda_1, \nu) d\nu + \frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} \Phi(\lambda_2, \nu) d\nu \right] \\
 & \leq \frac{\Phi(\lambda_1, \xi_1) + \Phi(\lambda_1, \xi_2) + \Phi(\lambda_2, \xi_1) + \Phi(\lambda_2, \xi_2)}{4} \tag{1.2}
 \end{aligned}$$

with the best possible constant 1/4.

In [73], Latif et al. derived the variants of the Hermite–Hadamard type inequality (1.2), which are the weighted generalizations of (1.2).

**Theorem 1.3** (See [73]) *Let  $\Delta \subseteq \mathbb{R}^2$ ,  $\Delta^\circ$  be the interior of  $\Delta$ ,  $\lambda_1 < \lambda_2$  and  $\xi_1 < \xi_2$  such that  $[\lambda_1, \lambda_2] \times [\xi_1, \xi_2] \in \Delta^\circ$ ,  $\Phi : \Delta \rightarrow \mathbb{R}$  be a twice differentiable mapping on  $\Delta^\circ$ ,  $p : [\lambda_1, \lambda_2] \times [\xi_1, \xi_2] \rightarrow [0, \infty)$  be a continuous function symmetric with respect to  $(\lambda_1 + \lambda_2)/2$  and  $(\xi_1 + \xi_2)/2$ ,  $\Phi_{\rho\zeta} \in L([\lambda_1, \lambda_2] \times [\xi_1, \xi_2])$  and  $|\Phi_{\rho\zeta}|$  be a co-ordinated convex function on  $[\lambda_1, \lambda_2] \times [\xi_1, \xi_2]$ . Then*

$$\begin{aligned}
 & \left| \Phi\left(\frac{\lambda_1 + \lambda_2}{2}, \frac{\xi_1 + \xi_2}{2}\right) \int_{\xi_1}^{\xi_2} \int_{\lambda_1}^{\lambda_2} p(\mu, \nu) d\mu d\nu + \int_{\xi_1}^{\xi_2} \int_{\lambda_1}^{\lambda_2} \Phi(\mu, \nu) p(\mu, \nu) d\mu d\nu \right. \\
 & \quad \left. - \int_{\xi_1}^{\xi_2} \int_{\lambda_1}^{\lambda_2} \Phi\left(\mu, \frac{\xi_1 + \xi_2}{2}\right) p(\mu, \nu) d\mu d\nu - \int_{\xi_1}^{\xi_2} \int_{\lambda_1}^{\lambda_2} \Phi\left(\frac{\lambda_1 + \lambda_2}{2}, \nu\right) p(\mu, \nu) d\mu d\nu \right| \\
 & \leq \frac{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)}{4} \left[ |\Phi_{\rho\zeta}(\lambda_1, \xi_1)| + |\Phi_{\rho\zeta}(\lambda_1, \xi_2)| + |\Phi_{\rho\zeta}(\lambda_2, \xi_1)| + |\Phi_{\rho\zeta}(\lambda_2, \xi_2)| \right] \\
 & \quad \times \int_0^1 \int_0^1 \left( \int_{\xi_1}^{L_2(\zeta)} \int_{\lambda_1}^{L_1(\rho)} p(\mu, \nu) d\mu d\nu \right) d\rho d\zeta, \tag{1.3}
 \end{aligned}$$

where  $L_1(\rho) = \frac{1-\rho}{2}\lambda_1 + \frac{1+\rho}{2}\lambda_2$  and  $L_2(\zeta) = \frac{1-\zeta}{2}\xi_1 + \frac{1+\zeta}{2}\xi_2$ .

**Theorem 1.4** (See [73]) *Let  $q > 1$ ,  $\Delta \subseteq \mathbb{R}^2$ ,  $\Delta^\circ$  be the interior of  $\Delta$ ,  $\lambda_1 < \lambda_2$  and  $\xi_1 < \xi_2$  such that  $[\lambda_1, \lambda_2] \times [\xi_1, \xi_2] \in \Delta^\circ$ ,  $\Phi : \Delta \rightarrow \mathbb{R}$  be a twice differentiable mapping on  $\Delta^\circ$ ,  $p : [\lambda_1, \lambda_2] \times [\xi_1, \xi_2] \rightarrow [0, \infty)$  be a continuous function symmetric with respect to  $(\lambda_1 + \lambda_2)/2$  and  $(\xi_1 + \xi_2)/2$ ,  $\Phi_{\rho\zeta} \in L([\lambda_1, \lambda_2] \times [\xi_1, \xi_2])$  and  $|\Phi_{\rho\zeta}|^q$  be a co-ordinated convex function on*

$[\lambda_1, \lambda_2] \times [\xi_1, \xi_2]$ . Then we have

$$\begin{aligned} & \left| \Phi\left(\frac{\lambda_1 + \lambda_2}{2}, \frac{\xi_1 + \xi_2}{2}\right) \int_{\xi_1}^{\xi_2} \int_{\lambda_1}^{\lambda_2} p(\mu, \nu) d\mu d\nu + \int_{\xi_1}^{\xi_2} \int_{\lambda_1}^{\lambda_2} \Phi(\mu, \nu) p(\mu, \nu) d\mu d\nu \right. \\ & \quad \left. - \int_{\xi_1}^{\xi_2} \int_{\lambda_1}^{\lambda_2} \Phi\left(\mu, \frac{\xi_1 + \xi_2}{2}\right) p(\mu, \nu) d\mu d\nu - \int_{\xi_1}^{\xi_2} \int_{\lambda_1}^{\lambda_2} \Phi\left(\frac{\lambda_1 + \lambda_2}{2}, \nu\right) p(\mu, \nu) d\mu d\nu \right| \\ & \leq (\lambda_2 - \lambda_1)(\xi_2 - \xi_1) \left[ \frac{|\Phi_{\varrho\varsigma}(\lambda_1, \xi_1)| + |\Phi_{\varrho\varsigma}(\lambda_1, \xi_2)| + |\Phi_{\varrho\varsigma}(\lambda_2, \xi_1)| + |\Phi_{\varrho\varsigma}(\lambda_2, \xi_2)|}{4} \right]^{\frac{1}{q}} \\ & \quad \times \int_0^1 \int_0^1 \left( \int_{\xi_1}^{L_2(\varsigma)} \int_{\lambda_1}^{L_1(\varrho)} p(\mu, \nu) d\mu d\nu \right) d\varrho d\varsigma, \end{aligned} \tag{1.4}$$

where  $L_1(\varrho)$  and  $L_2(\varsigma)$  are defined as in Theorem 1.3.

Özdemir et al. [74] generalized the co-ordinated convex function to the co-ordinated quasi-convex function.

**Definition 1.5** (See [74]) A real-valued function  $\Phi : [\lambda_1, \lambda_2] \times [\xi_1, \xi_2] \subset \mathbb{R}^2 \rightarrow \mathbb{R}$  is said to be quasi-convex if the inequality

$$\Phi(\varrho\mu + (1 - \varrho)\rho, \varrho\nu + (1 - \varrho)\omega) \leq \max\{\Phi(\mu, \nu), \Phi(\rho, \omega)\}$$

holds for all  $(\mu, \nu), (\rho, \omega) \in [\lambda_1, \lambda_2] \times [\xi_1, \xi_2]$  and  $\varrho \in [0, 1]$ .

**Definition 1.6** (See [74]) A real-valued function  $\Phi : [\lambda_1, \lambda_2] \times [\xi_1, \xi_2] \rightarrow \mathbb{R}$  is said to be quasi-convex on the co-ordinates if both of the partial mappings  $\Phi_\nu : [\lambda_1, \lambda_2] \rightarrow \mathbb{R}$  and  $\Phi_\mu : [\xi_1, \xi_2] \rightarrow \mathbb{R}$  defined by

$$\Phi_\nu(\delta) = \Phi(\delta, \nu), \quad \Phi_\mu(\theta) = \Phi(\mu, \theta)$$

are quasi-convex for all  $\mu \in [\lambda_1, \lambda_2]$  and  $\nu \in [\xi_1, \xi_2]$ .

In [75], Latif et al. provided an equivalent definition for the co-ordinated quasi-convex function as follows.

**Definition 1.7** (See [75]) A real-valued function  $\Phi : [\lambda_1, \lambda_2] \times [\xi_1, \xi_2] \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  is said to be quasi-convex on the co-ordinates if the inequality

$$\Phi(\varrho\mu + (1 - \varrho)\rho, \varsigma\nu + (1 - \varsigma)\omega) \leq \max\{\Phi(\mu, \nu), \Phi(\mu, \omega), \Phi(\rho, \nu), \Phi(\rho, \omega)\}$$

holds for all  $(\mu, \nu), (\rho, \omega) \in [\lambda_1, \lambda_2] \times [\xi_1, \xi_2]$  and  $(\varsigma, \varrho) \in [0, 1] \times [0, 1]$ .

The class of co-ordinated quasi-convex functions on  $[\lambda_1, \lambda_2] \times [\xi_1, \xi_2]$  is denoted  $QC([\lambda_1, \lambda_2] \times [\xi_1, \xi_2])$ . Özdemir [74] proved that every quasi-convex function is also a co-ordinated quasi-convex function, but the converse does not hold true. The Hermite–Hadamard type inequality (1.2) was generalized to the co-ordinated quasi-convex function by Latif et al. [75].

**Theorem 1.8** (See [75]) *Let  $\Delta \subseteq \mathbb{R}^2$ ,  $\Delta^\circ$  be the interior of  $\Delta$ ,  $\lambda_1 < \lambda_2$  and  $\xi_1 < \xi_2$  such that  $[\lambda_1, \lambda_2] \times [\xi_1, \xi_2] \in \Delta^\circ$ ,  $\Phi : \Delta \rightarrow \mathbb{R}$  be a differentiable mapping on  $\Delta^\circ$ ,  $\Phi_{\rho\varsigma} \in L([\lambda_1, \lambda_2] \times [\xi_1, \xi_2])$  and  $|\Phi_{\rho\varsigma}|$  be a co-ordinated quasi-convex function on  $[\lambda_1, \lambda_2] \times [\xi_1, \xi_2]$ . Then*

$$\begin{aligned}
 & \left| \frac{1}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi(\mu, \nu) d\mu d\nu \right. \\
 & \quad + \frac{\Phi(\lambda_1, \xi_1) + \Phi(\lambda_1, \xi_2) + \Phi(\lambda_2, \xi_1) + \Phi(\lambda_2, \xi_2)}{4} \\
 & \quad - \frac{1}{2} \left[ \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} [\Phi(\mu, \xi_1) + \Phi(\mu, \xi_2)] d\mu \right. \\
 & \quad \left. + \frac{1}{\xi_2 - \xi_1} \int_{\lambda_1}^{\lambda_2} [\Phi(\lambda_1, \nu) + \Phi(\lambda_2, \nu)] d\nu \right] \Big| \\
 & \leq K \left[ \sup \left\{ \left| \Phi_{\rho\varsigma}(\lambda_1, \xi_1) \right|, \left| \Phi_{\rho\varsigma} \left( \lambda_1, \frac{\xi_1 + \xi_2}{2} \right) \right|, \right. \right. \\
 & \quad \left. \left| \Phi_{\rho\varsigma} \left( \frac{\lambda_1 + \lambda_2}{2}, \xi_1 \right) \right|, \left| \Phi_{\rho\varsigma} \left( \frac{\lambda_1 + \lambda_2}{2}, \frac{\xi_1 + \xi_2}{2} \right) \right| \right\} \\
 & \quad + \sup \left\{ \left| \Phi_{\rho\varsigma}(\lambda_1, \xi_2) \right|, \left| \Phi_{\rho\varsigma} \left( \lambda_1, \frac{\xi_1 + \xi_2}{2} \right) \right|, \right. \\
 & \quad \left. \left| \Phi_{\rho\varsigma} \left( \frac{\lambda_1 + \lambda_2}{2}, \xi_2 \right) \right|, \left| \Phi_{\rho\varsigma} \left( \frac{\lambda_1 + \lambda_2}{2}, \frac{\xi_1 + \xi_2}{2} \right) \right| \right\} \\
 & \quad + \sup \left\{ \left| \Phi_{\rho\varsigma}(\lambda_2, \xi_1) \right|, \left| \Phi_{\rho\varsigma} \left( \lambda_2, \frac{\xi_1 + \xi_2}{2} \right) \right|, \right. \\
 & \quad \left. \left| \Phi_{\rho\varsigma} \left( \frac{\lambda_1 + \lambda_2}{2}, \xi_1 \right) \right|, \left| \Phi_{\rho\varsigma} \left( \frac{\lambda_1 + \lambda_2}{2}, \frac{\xi_1 + \xi_2}{2} \right) \right| \right\} \\
 & \quad + \sup \left\{ \left| \Phi_{\rho\varsigma}(\lambda_2, \xi_2) \right|, \left| \Phi_{\rho\varsigma} \left( \lambda_2, \frac{\xi_1 + \xi_2}{2} \right) \right|, \right. \\
 & \quad \left. \left| \Phi_{\rho\varsigma} \left( \frac{\lambda_1 + \lambda_2}{2}, \xi_2 \right) \right|, \left| \Phi_{\rho\varsigma} \left( \frac{\lambda_1 + \lambda_2}{2}, \frac{\xi_1 + \xi_2}{2} \right) \right| \right\} \Big], \tag{1.5}
 \end{aligned}$$

where  $K = (\lambda_2 - \lambda_1)(\xi_2 - \xi_1)/64$ .

More recent results on Hermite–Hadamard type inequalities and their applications can be found in the literature [76–96].

Motivated by Theorems 1.2–1.4 and 1.8, it is natural to ask the question: what are the weighted versions of the Hermite–Hadamard type inequality for the co-ordinated convex and quasi-convex functions?

The main purpose of the article is to present several weighted versions of the Hermite–Hadamard type inequality for the co-ordinated convex and quasi-convex functions, and give an application to the moment of continuous random variables of bivariate distribution functions in the probability theory. Finally, we provide an example on the probability distribution to support our results.

**2 Some auxiliary results**

First of all, we introduce several symbols as follows.

Let  $\omega(\mu, v) : [\lambda_1, \lambda_2] \times [\xi_1, \xi_2] \rightarrow [0, \infty)$  be a continuous real-valued function such that  $\int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \omega(\mu, v) \, dv \, d\mu = 1$ . Then we denote the integral  $\int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \mu \omega(\mu, v) \, dv \, d\mu$  by  $\Upsilon_1$ , the integral  $\int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} v \omega(\mu, v) \, dv \, d\mu$  by  $\Upsilon_2$  and the integral  $\int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \mu v \omega(\mu, v) \, dv \, d\mu$  by  $\Upsilon_3$ , that is,

$$\begin{aligned} \Upsilon_1 &= \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \mu \omega(\mu, v) \, dv \, d\mu, & \Upsilon_2 &= \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \mu v \omega(\mu, v) \, dv \, d\mu, \\ \Upsilon_3 &= \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} v \omega(\mu, v) \, d\mu. \end{aligned}$$

We show an outcome in which the function  $\omega(\mu, v)$  is symmetric on the co-ordinates with respect to  $\frac{\lambda_1 + \lambda_2}{2}$  and  $\frac{\xi_1 + \xi_2}{2}$ .

**Lemma 2.1** *If  $\omega(\mu, v) : [\lambda_1, \lambda_2] \times [\xi_1, \xi_2] \rightarrow [0, \infty)$  is symmetric on the co-ordinates with respect to the midpoints  $\frac{\lambda_1 + \lambda_2}{2}$  and  $\frac{\xi_1 + \xi_2}{2}$ . Then*

$$\Upsilon_1 = \frac{\lambda_1 + \lambda_2}{2}, \quad \Upsilon_3 = \frac{\xi_1 + \xi_2}{2}, \quad \Upsilon_2 = \left(\frac{\lambda_1 + \lambda_2}{2}\right) \left(\frac{\xi_1 + \xi_2}{2}\right).$$

*Proof* It follows from the hypothesis that

$$\begin{aligned} \Upsilon_1 &= \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \mu \omega(\mu, v) \, dv \, d\mu = \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \mu w(\lambda_1 + \lambda_2 - \mu, v) \, dv \, d\mu \\ &= \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\lambda_1 + \lambda_2 - \mu) \omega(\mu, v) \, dv \, d\mu, \end{aligned}$$

which gives the desired result due to

$$\int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \omega(\mu, v) \, dv \, d\mu = 1.$$

Similarly, one can prove that

$$\Upsilon_3 = \frac{\xi_1 + \xi_2}{2}$$

and

$$\Upsilon_2 = \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \mu v \omega(\mu, v) \, dv \, d\mu = \left(\frac{\lambda_1 + \lambda_2}{2}\right) \left(\frac{\xi_1 + \xi_2}{2}\right). \quad \square$$

**Lemma 2.2** *Let  $\Omega \subseteq \mathbb{R}^2$ ,  $\Omega^\circ$  be the interior of  $\Omega$ ,  $\lambda_1 < \lambda_2$  and  $\xi_1 < \xi_2$  such that  $[\lambda_1, \lambda_2] \times [\xi_1, \xi_2] \subseteq \Omega^\circ$ ,  $\omega : [\lambda_1, \lambda_2] \times [\xi_1, \xi_2] \rightarrow [0, \infty)$  be a continuous mapping, and  $\Phi : \Omega \rightarrow \mathbb{R}$  be a twice partially differentiable mapping on  $\Omega^\circ$  such that  $\Phi_{\theta_5} \in L([\lambda_1, \lambda_2] \times [\xi_1, \xi_2])$ . Then*

$$\begin{aligned} &\frac{1}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \left[ \Phi(\lambda_1, \xi_1) \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\lambda_2 - \mu)(\xi_2 - v) \omega(\mu, v) \, dv \, d\mu \right. \\ &\quad \left. + \Phi(\lambda_1, \xi_2) \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\lambda_2 - \mu)(v - \xi_1) \omega(\mu, v) \, dv \, d\mu \right] \end{aligned}$$

$$\begin{aligned}
 & + \Phi(\lambda_2, \xi_1) \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\mu - \lambda_1)(\xi_2 - v)\omega(\mu, v) \, dv \, d\mu \\
 & + \Phi(\lambda_2, \xi_2) \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\mu - \lambda_1)(v - \xi_1)\omega(\mu, v) \, dv \, d\mu \Big] \\
 & + \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi(\mu, v)\omega(\mu, v) \, dv \, d\mu \\
 & - \frac{1}{\xi_2 - \xi_1} \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\lambda_2 - \mu)\Phi(\lambda_1, v)\omega(\mu, v) \, dv \, d\mu \\
 & - \frac{1}{\xi_2 - \xi_1} \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\mu - \lambda_1)\Phi(\lambda_2, v)\omega(\mu, v) \, dv \, d\mu \\
 & - \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\xi_2 - v)\Phi(\mu, \xi_1)\omega(\mu, v) \, dv \, d\mu \\
 & - \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (v - \xi_1)\Phi(\mu, \xi_2)\omega(\mu, v) \, dv \, d\mu \\
 = & \frac{(\Upsilon_1 - \lambda_1)(\Upsilon_3 - \xi_1)}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \int_0^1 \int_0^1 H(\omega, \lambda_1, \xi_1, \Upsilon_1, \Upsilon_3; r, \rho) \\
 & \times \Phi_{r\rho}((1-r)\lambda_1 + \Upsilon_1 r, (1-\rho)\xi_1 + \Upsilon_3\rho) \, d\rho \, dr \\
 & + \frac{(\lambda_2 - \Upsilon_1)(\Upsilon_3 - \xi_1)}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \int_0^1 \int_0^1 H(\omega, \Upsilon_1, \xi_1, \lambda_2, \Upsilon_3; r, \rho) \\
 & \times \Phi_{r\rho}((1-r)\Upsilon_1 + \lambda_2 r, (1-\rho)\xi_1 + \Upsilon_3\rho) \, d\rho \, dr \\
 & + \frac{(\Upsilon_1 - \lambda_1)(\xi_2 - \Upsilon_3)}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \int_0^1 \int_0^1 H(\omega, \lambda_1, \Upsilon_3, \Upsilon_1, \xi_2; r, \rho) \\
 & \times \Phi_{r\rho}((1-r)\lambda_1 + \Upsilon_1 r, (1-\rho)\Upsilon_3 + \xi_2\rho) \, d\rho \, dr \\
 & + \frac{(\lambda_2 - \Upsilon_1)(\xi_2 - \Upsilon_3)}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \int_0^1 \int_0^1 H(\omega, \Upsilon_1, \Upsilon_3, \lambda_2, \xi_2; r, \rho) \\
 & \times \Phi_{r\rho}((1-r)\Upsilon_1 + \lambda_2 r, (1-\rho)\Upsilon_3 + \xi_2\rho) \, d\rho \, dr, \tag{2.1}
 \end{aligned}$$

where

$$\begin{aligned}
 & H(w, \Upsilon_1, \gamma, \beta, \epsilon; r, \rho) \\
 = & \int_{(1-r)\alpha+\beta r}^{\lambda_2} \int_{(1-\rho)\gamma+\epsilon\rho}^{\xi_2} (\lambda_2 - \mu)(\xi_2 - v)\omega(\mu, v) \, dv \, d\mu \\
 & - \int_{(1-r)\alpha+\beta r}^{\lambda_2} \int_{\xi_1}^{(1-z)\gamma+\epsilon\rho} (\lambda_2 - \mu)(v - \xi_1)\omega(\mu, v) \, dv \, d\mu \\
 & - \int_{\lambda_1}^{(1-r)\alpha+\beta r} \int_{(1-\rho)\gamma+\epsilon\rho}^{\xi_2} (\mu - \lambda_1)(\xi_2 - v)\omega(\mu, v) \, dv \, d\mu \\
 & + \int_{\lambda_1}^{(1-r)\alpha+\beta r} \int_{\xi_1}^{(1-\rho)\gamma+\epsilon\rho} (\mu - \lambda_1)(v - \xi_1)\omega(\mu, v) \, dv \, d\mu
 \end{aligned}$$

and  $(\alpha, \gamma), (\beta, \epsilon) \in [\lambda_1, \lambda_2] \times [\xi_1, \xi_2]$ .

*Proof* Let

$$\sigma(\delta) = \begin{cases} 0, & \delta < 0, \\ 1, & \delta > 0. \end{cases}$$

Then we clearly see that

$$\begin{aligned} & \Phi(\mu, v) - \Phi(\lambda_1, v) - \Phi(\mu, \xi_1) + \Phi(\lambda_1, \xi_1) \\ &= \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \sigma(\mu - \varrho) \sigma(v - \varsigma) \Phi_{\varrho\varsigma}(\varrho, \varsigma) \, d\varsigma \, d\varrho, \end{aligned} \tag{2.2}$$

$$\begin{aligned} & \Phi(\mu, v) - \Phi(\lambda_1, v) - \Phi(\mu, \xi_2) + \Phi(\lambda_1, \xi_2) \\ &= - \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \sigma(\mu - \varrho) \sigma(\varsigma - v) \Phi_{\varrho\varsigma}(\varrho, \varsigma) \, d\varsigma \, d\varrho, \end{aligned} \tag{2.3}$$

$$\begin{aligned} & \Phi(\mu, v) - \Phi(\lambda_2, v) - \Phi(\mu, \xi_1) + \Phi(\lambda_2, \xi_1) \\ &= - \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \sigma(\varrho - \mu) \sigma(v - \varsigma) \Phi_{\varrho\varsigma}(\varrho, \varsigma) \, d\varsigma \, d\varrho \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} & \Phi(\mu, v) - \Phi(\lambda_2, v) - \Phi(\mu, \xi_2) + \Phi(\lambda_2, \xi_2) \\ &= \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \sigma(\varrho - \mu) \sigma(\varsigma - v) \Phi_{\varrho\varsigma}(\varrho, \varsigma) \, d\varrho \, d\varsigma. \end{aligned} \tag{2.5}$$

It follows from (2.2) that

$$\begin{aligned} & \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\lambda_2 - \mu)(\xi_2 - v) \Phi(\mu, v) \omega(\mu, v) \, dv \, d\mu \\ & \quad - \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\lambda_2 - \mu)(\xi_2 - v) \Phi(\lambda_1, v) \omega(\mu, v) \, dv \, d\mu \\ & \quad - \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\lambda_2 - \mu)(\xi_2 - v) \Phi(\mu, \xi_1) \omega(\mu, v) \, dv \, d\mu \\ & \quad + \Phi(\lambda_1, \xi_1) \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\lambda_2 - \mu)(\xi_2 - v) \omega(\mu, v) \, dv \, d\mu \\ &= \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\lambda_2 - \mu)(\xi_2 - v) \left( \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \sigma(\mu - \varrho) \sigma(v - \varsigma) \Phi_{\varrho\varsigma}(\varrho, \varsigma) \, d\varsigma \, d\varrho \right) \\ & \quad \times \omega(\mu, v) \, dv \, d\mu \\ &= \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \left( \int_{\varrho}^{\lambda_2} \int_{\varsigma}^{\xi_2} (\lambda_2 - \mu)(\xi_2 - v) \omega(\mu, v) \, dv \, d\mu \right) \Phi_{\varrho\varsigma}(\varrho, \varsigma) \, d\varsigma \, d\varrho. \end{aligned} \tag{2.6}$$



Similarly, from (2.3)–(2.5) we have

$$\begin{aligned}
 & \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\lambda_2 - \mu)(v - \xi_1)\Phi(\mu, v)\omega(\mu, v) \, dv \, d\mu \\
 & \quad - \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\lambda_2 - \mu)(v - \xi_1)\Phi(\lambda_1, v)\omega(\mu, v) \, dv \, d\mu \\
 & \quad - \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\lambda_2 - \mu)(v - \xi_1)\Phi(\mu, \xi_1)\omega(\mu, v) \, dv \, d\mu \\
 & \quad + \Phi(\lambda_1, \xi_2) \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\lambda_2 - \mu)(v - \xi_1)\omega(\mu, v) \, dv \, d\mu \\
 & = - \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \left( \int_{\varrho}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\lambda_2 - \mu)(v - \xi_1)\omega(\mu, v) \, dv \, d\mu \right) \Phi_{\varrho\zeta}(\varrho, \zeta) \, d\zeta \, d\varrho, \tag{2.7}
 \end{aligned}$$

$$\begin{aligned}
 & \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\mu - \lambda_1)(\xi_2 - v)\Phi(\mu, v)\omega(\mu, v) \, dv \, d\mu \\
 & \quad - \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\mu - \lambda_1)(\xi_2 - v)\Phi(\lambda_2, v)\omega(\mu, v) \, dv \, d\mu \\
 & \quad - \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\mu - \lambda_1)(\xi_2 - v)\Phi(\mu, \xi_1)\omega(\mu, v) \, dv \, d\mu \\
 & \quad + \Phi(\lambda_2, \xi_1) \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\mu - \lambda_1)(\xi_2 - v)\omega(\mu, v) \, dv \, d\mu \\
 & = - \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \left( \int_{\lambda_1}^{\varrho} \int_{\zeta}^{\xi_2} (\mu - \lambda_1)(\xi_2 - v)\omega(\mu, v) \, dv \, d\mu \right) \Phi_{\varrho\zeta}(\varrho, \zeta) \, d\zeta \, d\varrho \tag{2.8}
 \end{aligned}$$

and

$$\begin{aligned}
 & \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\mu - \lambda_1)(v - \xi_1)\Phi(\mu, v)\omega(\mu, v) \, dv \, d\mu \\
 & \quad - \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\mu - \lambda_1)(v - \xi_1)\Phi(\lambda_2, v)\omega(\mu, v) \, dv \, d\mu \\
 & \quad - \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\mu - \lambda_1)(v - \xi_1)\Phi(\mu, \xi_2)\omega(\mu, v) \, dv \, d\mu \\
 & \quad + \Phi(\lambda_2, \xi_2) \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\mu - \lambda_1)(v - \xi_1)\omega(\mu, v) \, dv \, d\mu \\
 & = \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \left( \int_{\lambda_1}^{\varrho} \int_{\xi_1}^{\zeta} (\mu - \lambda_1)(v - \xi_1)\omega(\mu, v) \, dv \, d\mu \right) \Phi_{\varrho\zeta}(\varrho, \zeta) \, d\zeta \, d\varrho. \tag{2.9}
 \end{aligned}$$

From (2.6)–(2.9), we get

$$\begin{aligned}
 & \frac{1}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \left[ \Phi(\lambda_1, \xi_1) \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\lambda_2 - \mu)(\xi_2 - v)\omega(\mu, v) \, dv \, d\mu \right. \\
 & \quad + \Phi(\lambda_1, \xi_2) \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\lambda_2 - \mu)(v - \xi_1)\omega(\mu, v) \, dv \, d\mu \\
 & \quad \left. + \Phi(\lambda_2, \xi_1) \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\mu - \lambda_1)(\xi_2 - v)\omega(\mu, v) \, dv \, d\mu \right]
 \end{aligned}$$

$$\begin{aligned}
 & + \Phi(\lambda_2, \xi_2) \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\mu - \lambda_1)(v - \xi_1)\omega(\mu, v) \, dv \, d\mu \Big] \\
 & - \frac{1}{\xi_2 - \xi_1} \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\xi_2 - v)\Phi(\mu, \xi_1)\omega(\mu, v) \, dv \, d\mu \\
 & - \frac{1}{\xi_2 - \xi_1} \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (v - \xi_1)\Phi(\mu, \xi_2)\omega(\mu, v) \, dv \, d\mu \\
 & - \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\lambda_2 - \mu)\Phi(\lambda_1, v)\omega(\mu, v) \, dv \, d\mu \\
 & - \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\mu - \lambda_1)\Phi(\lambda_1, v)\omega(\mu, v) \, dv \, d\mu \\
 & + \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi(\mu, v)\omega(\mu, v) \, dv \, d\mu \\
 = & \frac{1}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \left[ \int_{\varrho}^{\lambda_2} \int_{\varsigma}^{\xi_2} (\lambda_2 - \mu)(\xi_2 - v)\omega(\mu, v) \, dv \, d\mu \right. \\
 & - \int_{\varrho}^{\lambda_2} \int_{\xi_1}^{\varsigma} (\lambda_2 - \mu)(v - \xi_1)\omega(\mu, v) \, dv \, d\mu \\
 & - \int_{\lambda_1}^{\varrho} \int_{\varsigma}^{\xi_2} (\mu - \lambda_1)(\xi_2 - v)\omega(\mu, v) \, dv \, d\mu \\
 & \left. + \int_{\lambda_1}^{\varrho} \int_{\xi_1}^{\varsigma} (\mu - \lambda_1)(v - \xi_1)\omega(\mu, v) \, dv \, d\mu \right] \Phi_{\varrho\varsigma}(\varrho, \varsigma) \, d\varsigma \, d\varrho \tag{2.10}
 \end{aligned}$$

and

$$\begin{aligned}
 & \frac{1}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \left[ \int_{\varrho}^{\lambda_2} \int_{\varsigma}^{\xi_2} (\lambda_2 - \mu)(\xi_2 - v)\omega(\mu, v) \, dv \, d\mu \right. \\
 & - \int_{\varrho}^{\lambda_2} \int_{\xi_1}^{\varsigma} (\lambda_2 - \mu)(v - \xi_1)\omega(\mu, v) \, dv \, d\mu \\
 & - \int_{\lambda_1}^{\varrho} \int_{\varsigma}^{\xi_2} (\mu - \lambda_1)(\xi_2 - v)\omega(\mu, v) \, dv \, d\mu \\
 & \left. + \int_{\lambda_1}^{\varrho} \int_{\xi_1}^{\varsigma} (\mu - \lambda_1)(v - \xi_1)\omega(\mu, v) \, dv \, d\mu \right] \Phi_{\varrho\varsigma}(\varrho, \varsigma) \, d\varsigma \, d\varrho \\
 = & \frac{1}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \int_{\lambda_1}^{\gamma_1} \int_{\xi_1}^{\gamma_3} \left[ \int_{\varrho}^{\lambda_2} \int_{\varsigma}^{\xi_2} (\lambda_2 - \mu)(\xi_2 - v)\omega(\mu, v) \, dv \, d\mu \right. \\
 & - \int_{\lambda_1}^{\varrho} \int_{\varsigma}^{\xi_2} (\mu - \lambda_1)(\xi_2 - v)\omega(\mu, v) \, dv \, d\mu - \int_{\varrho}^{\lambda_2} \int_{\xi_1}^{\varsigma} (\lambda_2 - \mu)(v - \xi_1)\omega(\mu, v) \, dv \, d\mu \\
 & \left. + \int_{\lambda_1}^{\varrho} \int_{\xi_1}^{\varsigma} (\mu - \lambda_1)(v - \xi_1)\omega(\mu, v) \, dv \, d\mu \right] \Phi_{\varrho\varsigma}(\varrho, \varsigma) \, d\varsigma \, d\varrho \\
 & + \frac{1}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \int_{\gamma_1}^{\lambda_2} \int_{\xi_1}^{\gamma_3} \left[ \int_{\varrho}^{\lambda_2} \int_{\varsigma}^{\xi_2} (\lambda_2 - \mu)(\xi_2 - v)\omega(\mu, v) \, dv \, d\mu \right. \\
 & - \int_{\lambda_1}^{\varrho} \int_{\varsigma}^{\xi_2} (\mu - \lambda_1)(\xi_2 - v)\omega(\mu, v) \, dv \, d\mu - \int_{\varrho}^{\lambda_2} \int_{\xi_1}^{\varsigma} (\lambda_2 - \mu)(v - \xi_1)\omega(\mu, v) \, dv \, d\mu
 \end{aligned}$$

$$\begin{aligned}
 & + \int_{\lambda_1}^{\varrho} \int_{\xi_1}^{\varsigma} (\mu - \lambda_1)(v - \xi_1)\omega(\mu, v) \, dv \, d\mu \Big] \Phi_{\varrho\varsigma}(\varrho, \varsigma) \, d\varsigma \, d\varrho \\
 & + \frac{1}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \int_{\lambda_1}^{\gamma_1} \int_{\gamma_3}^{\xi_2} \left[ \int_{\varrho}^{\lambda_2} \int_{\varsigma}^{\xi_2} (\lambda_2 - \mu)(\xi_2 - v)\omega(\mu, v) \, dv \, d\mu \right. \\
 & - \int_{\lambda_1}^{\varrho} \int_{\varsigma}^{\xi_2} (\mu - \lambda_1)(\xi_2 - v)\omega(\mu, v) \, dv \, d\mu - \int_{\varrho}^{\lambda_2} \int_{\xi_1}^{\varsigma} (\lambda_2 - \mu)(v - \xi_1)\omega(\mu, v) \, dv \, d\mu \\
 & \left. + \int_{\lambda_1}^{\varrho} \int_{\xi_1}^{\varsigma} (\mu - \lambda_1)(v - \xi_1)\omega(\mu, v) \, dv \, d\mu \right] \Phi_{\varrho\varsigma}(\varrho, \varsigma) \, d\varsigma \, d\varrho \\
 & + \frac{1}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \int_{\gamma_1}^{\lambda_2} \int_{\gamma_3}^{\xi_2} \left[ \int_{\varrho}^{\lambda_2} \int_{\varsigma}^{\xi_2} (\lambda_2 - \mu)(\xi_2 - v)\omega(\mu, v) \, dv \, d\mu \right. \\
 & - \int_{\lambda_1}^{\varrho} \int_{\varsigma}^{\xi_2} (\mu - \lambda_1)(\xi_2 - v)\omega(\mu, v) \, dv \, d\mu - \int_{\varrho}^{\lambda_2} \int_{\xi_1}^{\varsigma} (\lambda_2 - \mu)(v - \xi_1)\omega(\mu, v) \, dv \, d\mu \\
 & \left. + \int_{\lambda_1}^{\varrho} \int_{\xi_1}^{\varsigma} (\mu - \lambda_1)(v - \xi_1)\omega(\mu, v) \, dv \, d\mu \right] \Phi_{\varrho\varsigma}(\varrho, \varsigma) \, d\varsigma \, d\varrho. \tag{2.11}
 \end{aligned}$$

Therefore, inequality (2.1) follows from (2.10) and (2.11). □

*Remark 2.3* Let  $\omega(\mu, v) = \frac{1}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)}$ . Then (2.1) reduces to

$$\begin{aligned}
 & \frac{\Phi(\lambda_1, \xi_1) + \Phi(\lambda_1, \xi_2) + \Phi(\lambda_2, \xi_1) + \Phi(\lambda_2, \xi_2)}{4} \\
 & + \frac{1}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi(\mu, v) \, dv \, d\mu \\
 & - \frac{1}{2(\lambda_2 - \lambda_1)} \int_{\lambda_1}^{\lambda_2} [\Phi(\mu, \xi_1) + \Phi(\mu, \xi_2)] \, d\mu \\
 & - \frac{1}{2(\xi_2 - \xi_1)} \int_{\lambda_1}^{\lambda_2} [\Phi(\lambda_1, v) + \Phi(\lambda_2, v)] \, d\mu \\
 & = \frac{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)}{16} \left[ \int_0^1 \int_0^1 \varrho\varsigma \Phi_{\varrho\varsigma} \left( \frac{1-\varrho}{2}\lambda_1 + \frac{1+\varrho}{2}\lambda_2, \frac{1-\varsigma}{2}\xi_1 + \frac{1+\varsigma}{2}\xi_2 \right) \, d\varsigma \, d\varrho \right. \\
 & + \int_0^1 \int_0^1 (-\varrho)\varsigma \Phi_{\varsigma\varrho} \left( \frac{1+\varrho}{2}\lambda_1 + \frac{1-\varrho}{2}\lambda_2, \frac{1-\varsigma}{2}\xi_1 + \frac{1+\varsigma}{2}\xi_2 \right) \, d\varsigma \, d\varrho \\
 & + \int_0^1 \int_0^1 \varrho(-\varsigma) \Phi_{\varrho\varsigma} \left( \frac{1-\varrho}{2}\lambda_1 + \frac{1+\varrho}{2}\lambda_2, \frac{1+\varsigma}{2}\xi_1 + \frac{1-\varsigma}{2}\xi_2 \right) \, d\varsigma \, d\varrho \\
 & \left. + \int_0^1 \int_0^1 (-\varsigma)(-\varrho) \Phi_{\varsigma\varrho} \left( \frac{1+\varrho}{2}\lambda_1 + \frac{1-\varrho}{2}\lambda_2, \frac{1+\varsigma}{2}\xi_1 + \frac{1-\varsigma}{2}\xi_2 \right) \, d\varsigma \, d\varrho \right]. \tag{2.12}
 \end{aligned}$$

The identity (2.12) was established in [75].

**Corollary 2.4** *If the function  $\omega(\mu, v)$  is symmetric with respect to  $\frac{\lambda_1 + \lambda_2}{2}$  and  $\frac{\xi_1 + \xi_2}{2}$  on the co-ordinates. Then Lemma 2.2 leads to*

$$\begin{aligned}
 & \frac{\Phi(\lambda_1, \xi_1) + \Phi(\lambda_1, \xi_2) + \Phi(\lambda_2, \xi_1) + \Phi(\lambda_2, \xi_2)}{4} + \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi(\mu, v)\omega(\mu, v) \, dv \, d\mu \\
 & - \frac{1}{\xi_2 - \xi_1} \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\xi_2 - v)\Phi(\mu, \xi_1)\omega(\mu, v) \, dv \, d\mu
 \end{aligned}$$

$$\begin{aligned}
 & - \frac{1}{\xi_2 - \xi_1} \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (v - \xi_1) \Phi(\mu, \xi_2) \omega(\mu, v) \, dv \, d\mu \\
 & - \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\lambda_2 - \mu) \Phi(\lambda_2, v) \omega(\mu, v) \, dv \, d\mu \\
 & - \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\mu - \lambda_1) \Phi(\lambda_2, v) \omega(\mu, v) \, dv \, d\mu \\
 & = \int_0^1 \int_0^1 H(\omega, \lambda_1, \xi_1, \lambda_2, \xi_2; \varrho, \varsigma) \Phi_{\varrho\varsigma}(\lambda_1\varrho + (1 - \varrho)\lambda_2, \xi_1\varsigma + (1 - \varsigma)\xi_2) \, d\varsigma \, d\varrho. \tag{2.13}
 \end{aligned}$$

**Lemma 2.5** *Let  $\mu \in [\lambda_1, \lambda_2]$ ,  $v \in [\xi_1, \xi_2]$ ,  $A : C([\lambda_1, \lambda_2] \times [\xi_1, \xi_2]) \rightarrow \mathbb{R}$  be a positive linear functional,  $e_i(\mu) = \mu^i$  and  $k_j(v) = v^j$  be the monomials ( $i, j \in \mathbb{N}$ ), and  $g$  be a co-ordinated convex function on  $[\lambda_1, \lambda_2] \times [\xi_1, \xi_2]$ . Then*

$$\begin{aligned}
 & A(g(e_1, k_1)) \\
 & \leq \frac{1}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} [A((\lambda_2 - e_1)(\xi_2 - k_1))g(\lambda_1, \xi_1) + A((\lambda_2 - e_1)(k_1 - \xi_1))g(\lambda_1, \xi_2) \\
 & \quad + A((e_1 - \lambda_1)(\xi_2 - k_1))g(\lambda_2, \xi_1) + A((e_1 - \lambda_1)(k_1 - \xi_1))g(\lambda_2, \xi_2)]. \tag{2.14}
 \end{aligned}$$

*Proof* It follows from the convexity of  $g$  on the co-ordinates that

$$\begin{aligned}
 g(e_1, k_1) & \leq \frac{(\lambda_2 - e_1)g(\lambda_1, k_1) + (e_1 - \lambda_1)g(\lambda_2, k_1)}{\lambda_2 - \lambda_1} \\
 & \leq \frac{1}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} [(\lambda_2 - e_1)(\xi_2 - k_1)g(\lambda_1, \xi_1) + (\lambda_2 - e_1)(k_1 - \xi_1)g(\lambda_1, \xi_2) \\
 & \quad + (e_1 - \lambda_1)(\xi_2 - k_1)g(\lambda_2, \xi_1) + (e_1 - \lambda_1)(k_1 - \xi_1)g(\lambda_2, \xi_2)]. \tag{2.15}
 \end{aligned}$$

Therefore, inequality (2.14) follows from (2.15) and the assumption of the functional  $A$ . □

### 3 Main results

In order to provide compact demonstration, we use the notations for our coming results as follows:

$$\begin{aligned}
 & B_1(\lambda_1, \xi_1, \lambda_2, \xi_2) \\
 & = \int_{\lambda_1}^{\gamma_1} \int_{\xi_1}^{\gamma_3} (\gamma_1 - \mu)^2 (\gamma_3 - v)^2 \omega(\mu, v) \, dv \, d\mu \\
 & \quad - \int_{\lambda_1}^{\gamma_1} \int_{\gamma_3}^{\lambda_2} (\gamma_1 - \mu)^2 (\gamma_3 - v)^2 \omega(\mu, v) \, dv \, d\mu \\
 & \quad - \int_{\gamma_1}^{\lambda_2} \int_{\xi_1}^{\gamma_3} (\gamma_1 - \mu)^2 (\gamma_3 - v)^2 \omega(\mu, v) \, dv \, d\mu \\
 & \quad + \int_{\gamma_1}^{\lambda_2} \int_{\gamma_3}^{\xi_2} (\gamma_1 - \mu)^2 (\gamma_3 - v)^2 \omega(\mu, v) \, dv \, d\mu, \tag{3.1}
 \end{aligned}$$

$$B_2(\lambda_1, \xi_1, \lambda_2, \xi_2)$$

$$\begin{aligned}
 &= \int_{\lambda_1}^{\gamma_1} \int_{\xi_1}^{\gamma_3} (\gamma_1 - \mu)(\gamma_3 - v)^2 \omega(\mu, v) \, dv \, d\mu \\
 &\quad - \int_{\lambda_1}^{\gamma_1} \int_{\gamma_3}^{\xi_2} (\gamma_1 - \mu)(\gamma_3 - v)^2 \omega(\mu, v) \, dv \, d\mu \\
 &\quad - \int_{\gamma_1}^{\lambda_2} \int_{\xi_1}^{\gamma_3} (\gamma_1 - \mu)(\gamma_3 - v)^2 \omega(\mu, v) \, dv \, d\mu \\
 &\quad + \int_{\gamma_1}^{\lambda_2} \int_{\gamma_3}^{\xi_2} (\gamma_1 - \mu)(\gamma_3 - v)^2 \omega(\mu, v) \, dv \, d\mu, \tag{3.2}
 \end{aligned}$$

$$\begin{aligned}
 &B_3(\lambda_1, \xi_1, \lambda_2, \xi_2) \\
 &= \int_{\lambda_1}^{\gamma_1} \int_{\xi_1}^{\gamma_3} (\gamma_1 - \mu)^2 (\gamma_3 - v) \omega(\mu, v) \, dv \, d\mu \\
 &\quad - \int_{\lambda_1}^{\gamma_1} \int_{\gamma_3}^{\xi_2} (\gamma_1 - \mu)^2 (\gamma_3 - v) \omega(\mu, v) \, dv \, d\mu \\
 &\quad - \int_{\gamma_1}^{\lambda_2} \int_{\xi_1}^{\gamma_3} (\gamma_1 - \mu)^2 (\gamma_3 - v) \omega(\mu, v) \, dv \, d\mu \\
 &\quad + \int_{\gamma_1}^{\lambda_2} \int_{\gamma_3}^{\xi_2} (\gamma_1 - \mu)^2 (\gamma_3 - v) \omega(\mu, v) \, dv \, d\mu \tag{3.3}
 \end{aligned}$$

and

$$\begin{aligned}
 &B_4(\lambda_1, \xi_1, \lambda_2, \xi_2) \\
 &= \int_{\lambda_1}^{\gamma_1} \int_{\xi_1}^{\gamma_3} (\gamma_1 - \mu)(\gamma_3 - v) \omega(\mu, v) \, dv \, d\mu \\
 &\quad - \int_{\lambda_1}^{\gamma_1} \int_{\gamma_3}^{\xi_2} (\gamma_1 - \mu)(\gamma_3 - v) \omega(\mu, v) \, dv \, d\mu \\
 &\quad - \int_{\gamma_1}^{\lambda_2} \int_{\xi_1}^{\gamma_3} (\gamma_1 - \mu)(\gamma_3 - v) \omega(\mu, v) \, dv \, d\mu \\
 &\quad + \int_{\gamma_1}^{\lambda_2} \int_{\gamma_3}^{\xi_2} (\gamma_1 - \mu)(\gamma_3 - v) \omega(\mu, v) \, dv \, d\mu. \tag{3.4}
 \end{aligned}$$

**Theorem 3.1** *Let  $\Omega \subseteq \mathbb{R}^2$ ,  $\Omega^\circ$  be the interior of  $\Omega$ ,  $\lambda_1 < \lambda_2$  and  $\xi_1 < \xi_2$  such that  $[\lambda_1, \lambda_2] \times [\xi_1, \xi_2] \subseteq \Omega^\circ$ ,  $\Phi : \Omega \rightarrow \mathbb{R}$  be a twice partially differentiable mapping on  $\Omega^\circ$  such that  $\Phi_{\rho_5} \in L([\lambda_1, \lambda_2] \times [\xi_1, \xi_2])$  and  $|\Phi_{\rho_5}|$  is co-ordinated convex on  $[\lambda_1, \lambda_2] \times [\xi_1, \xi_2]$ , and  $\omega : [\lambda_1, \lambda_2] \times [\xi_1, \xi_2] \rightarrow [0, \infty)$  be a continuous mapping. Then one has*

$$\begin{aligned}
 &\left| \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi(\mu, v) \omega(\mu, v) \, dv \, d\mu - \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi(\gamma_1, v) \omega(\mu, v) \, dv \, d\mu \right. \\
 &\quad \left. - \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi(\mu, \gamma_3) \omega(\mu, v) \, dv \, d\mu + \Phi(\gamma_1, \gamma_3) \right| \\
 &\leq \frac{1}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \left[ |\Phi_{\rho_5}(\lambda_1, \xi_1)| A_1(\lambda_1, \xi_1, \lambda_2, \xi_2) + |\Phi_{\rho_5}(\lambda_1, \xi_2)| A_2(\lambda_1, \xi_1, \lambda_2, \xi_2) \right. \\
 &\quad \left. + |\Phi_{\rho_5}(\lambda_2, \xi_1)| A_3(\lambda_1, \xi_1, \lambda_2, \xi_2) + |\Phi_{\rho_5}(\lambda_2, \xi_2)| A_4(\lambda_1, \xi_1, \lambda_2, \xi_2) \right], \tag{3.5}
 \end{aligned}$$

where

$$\begin{aligned}
 &A_1(\lambda_1, \xi_1, \lambda_2, \xi_2) \\
 &= \frac{1}{4}B_1(\lambda_1, \xi_1, \lambda_2, \xi_2) + \frac{(\lambda_2 - \gamma_1)}{2}B_2(\lambda_1, \xi_1, \lambda_2, \xi_2) \\
 &\quad + \frac{(\xi_2 - \gamma_3)}{2}B_3(\lambda_1, \xi_1, \lambda_2, \xi_2) + (\lambda_2 - \gamma_1)(\xi_2 - \gamma_3)B_4(\lambda_1, \xi_1, \lambda_2, \xi_2), \\
 &A_2(\lambda_1, \xi_1, \lambda_2, \xi_2) \\
 &= \frac{1}{4}B_1(\lambda_1, \xi_1, \lambda_2, \xi_2) + \frac{(\lambda_2 - \gamma_1)}{2}B_2(\lambda_1, \xi_1, \lambda_2, \xi_2) \\
 &\quad + \frac{(\gamma_3 - \xi_1)}{2}B_3(\lambda_1, \xi_1, \lambda_2, \xi_2) + (\lambda_2 - \gamma_1)(\gamma_3 - \xi_1)B_4(\lambda_1, \xi_1, \lambda_2, \xi_2), \\
 &A_3(\lambda_1, \xi_1, \lambda_2, \xi_2) \\
 &= \frac{1}{4}B_1(\lambda_1, \xi_1, \lambda_2, \xi_2) + \frac{(\gamma_1 - \lambda_1)}{2}B_2(\lambda_1, \xi_1, \lambda_2, \xi_2) \\
 &\quad + \frac{(\xi_2 - \gamma_3)}{2}B_3(\lambda_1, \xi_1, \lambda_2, \xi_2) + (\gamma_1 - \lambda_1)(\xi_2 - \gamma_3)B_4(\lambda_1, \xi_1, \lambda_2, \xi_2)
 \end{aligned}$$

and

$$\begin{aligned}
 &A_4(\lambda_1, \xi_1, \lambda_2, \xi_2) \\
 &= \frac{1}{4}B_1(\lambda_1, \xi_1, \lambda_2, \xi_2) + \frac{(\gamma_1 - \lambda_1)}{2}B_2(\lambda_1, \xi_1, \lambda_2, \xi_2) \\
 &\quad + \frac{(\gamma_3 - \xi_1)}{2}B_3(\lambda_1, \xi_1, \lambda_2, \xi_2) + (\gamma_1 - \lambda_1)(\gamma_3 - \xi_1)B_4(\lambda_1, \xi_1, \lambda_2, \xi_2).
 \end{aligned}$$

*Proof* We clearly see that

$$\begin{aligned}
 &\Phi(\mu, \nu) - \Phi(\gamma_1, \nu) - \Phi(\mu, \gamma_3) + \Phi(\gamma_1, \gamma_3) \\
 &= \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} [\sigma(\mu - \varrho)\sigma(\nu - \varsigma) - \sigma(\gamma_1 - \varrho)\sigma(\nu - \varsigma) \\
 &\quad - \sigma(\mu - \varrho)\sigma(\gamma_3 - \varsigma) + \sigma(\gamma_1 - \varrho)\sigma(\gamma_3 - \varsigma)] \Phi_{\varrho\varsigma}(\varrho, \varsigma) d\varsigma d\varrho. \tag{3.6}
 \end{aligned}$$

It follows from (3.6) that

$$\begin{aligned}
 &\int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi(\mu, \nu)\omega(\mu, \nu) d\nu d\mu - \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi(\gamma_1, \nu)\omega(\mu, \nu) d\nu d\mu \\
 &\quad - \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi(\mu, \gamma_3)\omega(\mu, \nu) d\nu d\mu + \Phi(\gamma_1, \gamma_3) \\
 &= \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \left( \int_{\varrho}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu, \nu) d\nu d\mu - \sigma(\gamma_1 - \varrho) \int_{\lambda_1}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu, \nu) d\nu d\mu \right. \\
 &\quad \left. - \sigma(\gamma_3 - \varsigma) \int_{\varrho}^{\lambda_2} \int_{\xi_1}^{\xi_2} \omega(\mu, \nu) d\nu d\mu + \sigma(\gamma_1 - \varrho)\sigma(\gamma_3 - \varsigma) \right) \\
 &\quad \times \Phi_{\varrho\varsigma}(\varrho, \varsigma) d\varsigma d\varrho. \tag{3.7}
 \end{aligned}$$

From Lemma 2.5 and (3.7) we get

$$\begin{aligned}
 & \left| \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi(\mu, \nu) \omega(\mu, \nu) \, d\nu \, d\mu - \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi(\Upsilon_1, \nu) \omega(\mu, \nu) \, d\nu \, d\mu \right. \\
 & \quad \left. - \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi(\mu, \Upsilon_3) \omega(\mu, \nu) \, d\nu \, d\mu + \Phi(\Upsilon_1, \Upsilon_3) \right| \\
 & \leq \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \left| \int_{\varrho}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu - \sigma(\Upsilon_1 - \varrho) \int_{\lambda_1}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu \right. \\
 & \quad \left. - \sigma(\Upsilon_3 - \varsigma) \int_{\varrho}^{\lambda_2} \int_{\xi_1}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu + \sigma(\Upsilon_1 - \varrho) \sigma(\Upsilon_3 - \varsigma) \right| |\Phi_{\varrho\varsigma}(\varrho, \varsigma)| \, ds \, dt \\
 & \leq \frac{1}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \left[ |\Phi_{\varrho\varsigma}(\lambda_1, \lambda_2)| \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \left| \int_{\varrho}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu \right. \right. \\
 & \quad \left. - \sigma(\Upsilon_1 - \varrho) \int_{\lambda_1}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu - \sigma(\Upsilon_3 - \varsigma) \int_{\varrho}^{\lambda_2} \int_{\xi_1}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu \right. \\
 & \quad \left. + \sigma(\Upsilon_1 - \varrho) \sigma(\Upsilon_3 - \varsigma) \right| (\lambda_2 - \varrho)(\xi_2 - \varsigma) \, d\varsigma \, d\varrho \\
 & \quad + |\Phi_{\varrho\varsigma}(\lambda_1, \xi_2)| \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \left| \int_{\varrho}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu \right. \\
 & \quad \left. - \sigma(\Upsilon_1 - \varrho) \int_{\lambda_1}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu - \sigma(\Upsilon_3 - \varsigma) \int_{\varrho}^{\lambda_2} \int_{\xi_1}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu \right. \\
 & \quad \left. + \sigma(\Upsilon_1 - \varrho) \sigma(\Upsilon_3 - \varsigma) \right| (\lambda_2 - \varrho)(\varsigma - \xi_1) \, d\varsigma \, d\varrho \\
 & \quad + |\Phi_{\varrho\varsigma}(\lambda_2, \xi_1)| \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \left| \int_{\varrho}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu \right. \\
 & \quad \left. - \sigma(\Upsilon_1 - \varrho) \int_{\lambda_1}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu - \sigma(\Upsilon_3 - \varsigma) \int_{\varrho}^{\lambda_2} \int_{\xi_1}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu \right. \\
 & \quad \left. + \sigma(\Upsilon_1 - \varrho) \sigma(\Upsilon_3 - \varsigma) \right| (\varrho - \lambda_1)(\xi_2 - \varsigma) \, d\varsigma \, d\varrho \\
 & \quad + |\Phi_{\varrho\varsigma}(\lambda_2, \xi_2)| \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \left| \int_{\varrho}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu \right. \\
 & \quad \left. - \sigma(\Upsilon_1 - \varrho) \int_{\lambda_1}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu - \sigma(\Upsilon_3 - \varsigma) \int_{\varrho}^{\lambda_2} \int_{\xi_1}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu \right. \\
 & \quad \left. + \sigma(\Upsilon_1 - \varrho) \sigma(\Upsilon_3 - \varsigma) \right| (\varrho - \lambda_1)(\varsigma - \xi_1) \, d\varsigma \, d\varrho \Big] \tag{3.8}
 \end{aligned}$$

and

$$\begin{aligned}
 & \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \left| \int_{\varrho}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu - \sigma(\Upsilon_1 - \varrho) \int_{\lambda_1}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu \right. \\
 & \quad \left. - \sigma(\Upsilon_3 - \varsigma) \int_{\varrho}^{\lambda_2} \int_{\xi_1}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu + \sigma(\Upsilon_1 - \varrho) \sigma(\Upsilon_3 - \varsigma) \right| (\lambda_2 - \varrho)(\xi_2 - \varsigma) \, d\varsigma \, d\varrho
 \end{aligned}$$

$$\begin{aligned}
 &= \int_{\lambda_1}^{\gamma_1} \int_{\xi_1}^{\gamma_3} \left( \int_{\lambda_1}^{\varrho} \int_{\xi_1}^{\varsigma} \omega(\mu, \nu) \, d\nu \, d\mu \right) (\lambda_2 - \varrho)(\xi_2 - \varsigma) \, d\varsigma \, d\varrho \\
 &\quad + \int_{\gamma_1}^{\lambda_2} \int_{\xi_1}^{\gamma_3} \left( \int_{\varrho}^{\lambda_2} \int_{\xi_1}^{\varsigma} \omega(\mu, \nu) \, d\nu \, d\mu \right) (\lambda_2 - \varrho)(\xi_2 - \varsigma) \, d\varsigma \, d\varrho \\
 &\quad + \int_{\lambda_1}^{\gamma_1} \int_{\gamma_3}^{\xi_2} \left( \int_{\lambda_1}^{\varrho} \int_{\varsigma}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu \right) (\lambda_2 - \varrho)(\xi_2 - \varsigma) \, d\varsigma \, d\varrho \\
 &\quad + \int_{\gamma_1}^{\lambda_2} \int_{\gamma_3}^{\xi_2} \left( \int_{\varrho}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu \right) (\lambda_2 - \varrho)(\xi_2 - \varsigma) \, d\varsigma \, d\varrho \\
 &= \frac{(\xi_2 - \gamma_3)^2 (\lambda_2 - \gamma_1)^2}{4} \int_{\lambda_1}^{\gamma_1} \int_{\xi_1}^{\gamma_3} \omega(\mu, \nu) \, d\nu \, d\mu \\
 &\quad - \frac{(\xi_2 - \gamma_3)^2 (\lambda_2 - \gamma_1)^2}{4} \int_{\gamma_1}^{\lambda_2} \int_{\xi_1}^{\gamma_3} \omega(\mu, \nu) \, d\nu \, d\mu \\
 &\quad - \frac{(\xi_2 - \gamma_3)^2 (\lambda_2 - \gamma_1)^2}{4} \int_{\lambda_1}^{\gamma_1} \int_{\gamma_3}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu \\
 &\quad + \frac{(\xi_2 - \gamma_3)^2 (\lambda_2 - \gamma_1)^2}{4} \int_{\gamma_1}^{\lambda_2} \int_{\gamma_3}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu \\
 &\quad - \frac{(\xi_2 - \gamma_3)^2}{2} \int_{\lambda_1}^{\gamma_1} \int_{\xi_1}^{\gamma_3} \frac{(\lambda_2 - \mu)^2}{2} \omega(\mu, \nu) \, d\nu \, d\mu \\
 &\quad + \frac{(\xi_2 - \gamma_3)^2}{2} \int_{\gamma_1}^{\lambda_2} \int_{\xi_1}^{\gamma_3} \frac{(\lambda_2 - \mu)^2}{2} \omega(\mu, \nu) \, d\nu \, d\mu \\
 &\quad + \frac{(\xi_2 - \gamma_3)^2}{2} \int_{\lambda_1}^{\gamma_1} \int_{\gamma_3}^{\xi_2} \frac{(\lambda_2 - \mu)^2}{2} \omega(\mu, \nu) \, d\nu \, d\mu \\
 &\quad - \frac{(\xi_2 - \gamma_3)^2}{2} \int_{\gamma_1}^{\lambda_2} \int_{\gamma_3}^{\xi_2} \frac{(\lambda_2 - \mu)^2}{2} \omega(\mu, \nu) \, d\nu \, d\mu \\
 &\quad - \frac{(\lambda_2 - \gamma_1)^2}{2} \int_{\lambda_1}^{\gamma_1} \int_{\xi_1}^{\gamma_3} \frac{(\xi_2 - \nu)^2}{2} \omega(\mu, \nu) \, d\nu \, d\mu \\
 &\quad + \frac{(\lambda_2 - \gamma_1)^2}{2} \int_{\gamma_1}^{\lambda_2} \int_{\xi_1}^{\gamma_3} \frac{(\xi_2 - \nu)^2}{2} \omega(\mu, \nu) \, d\nu \, d\mu \\
 &\quad + \frac{(\lambda_2 - \gamma_1)^2}{2} \int_{\lambda_1}^{\gamma_1} \int_{\gamma_3}^{\xi_2} \frac{(\xi_2 - \nu)^2}{2} \omega(\mu, \nu) \, d\nu \, d\mu \\
 &\quad - \frac{(\lambda_2 - \gamma_1)^2}{2} \int_{\gamma_1}^{\lambda_2} \int_{\gamma_3}^{\xi_2} \frac{(\xi_2 - \nu)^2}{2} \omega(\mu, \nu) \, d\nu \, d\mu \\
 &\quad + \int_{\lambda_1}^{\gamma_1} \int_{\xi_1}^{\gamma_3} \frac{(\xi_2 - \nu)^2 (\lambda_2 - \mu)^2}{4} \omega(\mu, \nu) \, d\nu \, d\mu \\
 &\quad - \int_{\gamma_1}^{\lambda_2} \int_{\xi_1}^{\gamma_3} \frac{(\xi_2 - \nu)^2 (\lambda_2 - \nu)^2}{4} \omega(\mu, \nu) \, d\nu \, d\mu \\
 &\quad - \int_{\lambda_1}^{\gamma_1} \int_{\gamma_3}^{\xi_2} \frac{(\xi_2 - \nu)^2 (\lambda_2 - \mu)^2}{4} \omega(\mu, \nu) \, d\nu \, d\mu \\
 &\quad + \int_{\gamma_1}^{\lambda_2} \int_{\gamma_3}^{\xi_2} \frac{(\xi_2 - \nu)^2 (\lambda_2 - \mu)^2}{4} \omega(\mu, \nu) \, d\nu \, d\mu.
 \end{aligned} \tag{3.9}$$



Equation (3.9) can be rewritten as

$$\begin{aligned} & \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \left| \int_{\varrho}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu - \sigma(\Upsilon_1 - \varrho) \int_{\lambda_1}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu \right. \\ & \quad \left. - \sigma(\Upsilon_3 - \varsigma) \int_{\varrho}^{\lambda_2} \int_{\xi_1}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu + \sigma(\Upsilon_1 - \varrho)\sigma(\Upsilon_3 - \varsigma) \right| \\ & \quad \times (\lambda_2 - \varrho)(\xi_2 - \varsigma) \, d\varsigma \, d\varrho \\ & = \frac{1}{4} B_1(\lambda_1, \xi_1, \lambda_2, \xi_2) + \frac{(\lambda_2 - \Upsilon_1)}{2} B_2(\lambda_1, \xi_1, \lambda_2, \xi_2) + \frac{(\xi_2 - \Upsilon_3)}{2} B_3(\lambda_1, \xi_1, \lambda_2, \xi_2) \\ & \quad + (\lambda_2 - \Upsilon_1)(\xi_2 - \Upsilon_3) B_4(\lambda_1, \xi_1, \lambda_2, \xi_2) = A_1(\lambda_1, \xi_1, \lambda_2, \xi_2). \end{aligned} \tag{3.10}$$

Similarly, we have

$$\begin{aligned} & \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \left| \int_{\varrho}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu - \sigma(\Upsilon_1 - \varrho) \int_{\lambda_1}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu \right. \\ & \quad \left. - \sigma(\Upsilon_3 - \varsigma) \int_{\varrho}^{\lambda_2} \int_{\xi_1}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu + \sigma(\Upsilon_1 - \varrho)\sigma(\Upsilon_3 - \varsigma) \right| \\ & \quad \times (\lambda_2 - \varrho)(\varsigma - \xi_1) \, d\varsigma \, d\varrho \\ & = \frac{1}{4} B_1(\lambda_1, \xi_1, \lambda_2, \xi_2) + \frac{(\lambda_2 - \Upsilon_1)}{2} B_2(\lambda_1, \xi_1, \lambda_2, \xi_2) + \frac{(\Upsilon_3 - \xi_1)}{2} B_3(\lambda_1, \xi_1, \lambda_2, \xi_2) \\ & \quad + (\lambda_2 - \Upsilon_1)(\Upsilon_3 - \xi_1) B_4(\lambda_1, \xi_1, \lambda_2, \xi_2) = A_2(\lambda_1, \xi_1, \lambda_2, \xi_2), \end{aligned} \tag{3.11}$$

$$\begin{aligned} & \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \left| \int_{\varrho}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu - \sigma(\Upsilon_1 - \varrho) \int_{\lambda_1}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu \right. \\ & \quad \left. - \sigma(\Upsilon_3 - \varsigma) \int_{\varrho}^{\lambda_2} \int_{\xi_1}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu + \sigma(\Upsilon_1 - \varrho)\sigma(\Upsilon_3 - \varsigma) \right| \\ & \quad \times (\lambda_1 - \varrho)(\xi_2 - \varsigma) \, d\varsigma \, d\varrho \\ & = \frac{1}{4} B_1(\lambda_1, \xi_1, \lambda_2, \xi_2) + \frac{(\Upsilon_1 - \lambda_1)}{2} B_2(\lambda_1, \xi_1, \lambda_2, \xi_2) + \frac{(\xi_2 - \Upsilon_3)}{2} B_3(\lambda_1, \xi_1, \lambda_2, \xi_2) \\ & \quad + (\Upsilon_1 - \lambda_1)(\xi_2 - \Upsilon_3) B_4(\lambda_1, \xi_1, \lambda_2, \xi_2) = A_3(\lambda_1, \xi_1, \lambda_2, \xi_2) \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} & \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \left| \int_{\varrho}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu - \sigma(\Upsilon_1 - \varrho) \int_{\lambda_1}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu \right. \\ & \quad \left. - \sigma(\Upsilon_3 - \varsigma) \int_{\varrho}^{\lambda_2} \int_{\xi_1}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu + \sigma(\Upsilon_1 - \varrho)\sigma(\Upsilon_3 - \varsigma) \right| \\ & \quad \times (\lambda_1 - \varrho)(\varsigma - \xi_1) \, d\varsigma \, d\varrho \\ & = \frac{1}{4} B_1(\lambda_1, \xi_1, \lambda_2, \xi_2) + \frac{(\Upsilon_1 - \lambda_1)}{2} B_2(\lambda_1, \xi_1, \lambda_2, \xi_2) + \frac{(\Upsilon_3 - \xi_1)}{2} B_3(\lambda_1, \xi_1, \lambda_2, \xi_2) \\ & \quad + (\Upsilon_1 - \lambda_1)(\Upsilon_3 - \xi_1) B_4(\lambda_1, \xi_1, \lambda_2, \xi_2) = A_4(\lambda_1, \xi_1, \lambda_2, \xi_2). \end{aligned} \tag{3.13}$$

Therefore, inequality (3.5) follows from (3.8) and (3.10)–(3.13). □

**Corollary 3.2** *If all the conditions of Theorem 3.1 are satisfied and  $\omega(\mu, v)$  is symmetric with respect to  $\frac{\lambda_1+\lambda_2}{2}$  and  $\frac{\xi_1+\xi_2}{2}$  on the co-ordinates. Then one has*

$$\begin{aligned} & \left| \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi(\mu, v) \omega(\mu, v) \, dv \, d\mu - \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi\left(\frac{\lambda_1 + \lambda_2}{2}, v\right) \omega(\mu, v) \, dv \, d\mu \right. \\ & \quad \left. - \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi\left(\mu, \frac{\xi_1 + \xi_2}{2}\right) \omega(\mu, v) \, dv \, d\mu + \Phi\left(\frac{\lambda_1 + \lambda_2}{2}, \frac{\xi_1 + \xi_2}{2}\right) \right| \\ & \leq \left[ |\Phi_{\varrho_S}(\lambda_1, \xi_1)| + |\Phi_{\varrho_S}(\lambda_1, \xi_2)| + |\Phi_{\varrho_S}(\lambda_2, \xi_1)| + |\Phi_{\varrho_S}(\lambda_2, \xi_2)| \right] \\ & \quad \times \int_{\frac{\lambda_1+\lambda_2}{2}}^{\lambda_2} \int_{\frac{\xi_1+\xi_2}{2}}^{\xi_2} \left(\mu - \frac{\lambda_1 + \lambda_2}{2}\right) \left(v - \frac{\xi_1 + \xi_2}{2}\right) \omega(\mu, v) \, dv \, d\mu. \end{aligned} \tag{3.14}$$

*Proof* Making use of the hypothesis of Theorem 3.1 and the symmetry of the function  $\omega(\mu, v)$  with respect to  $\frac{\lambda_1+\lambda_2}{2}$  and  $\frac{\xi_1+\xi_2}{2}$  on the co-ordinates that the function  $(\Upsilon_1 - \mu)^2(\Upsilon_3 - v)^2\omega(\mu, v)$  is symmetric with respect to  $\frac{\lambda_1+\lambda_2}{2}$  and  $\frac{\xi_1+\xi_2}{2}$  on the co-ordinates, the function  $(\Upsilon_1 - \mu)(\Upsilon_3 - v)^2\omega(\mu, v)$  is symmetric with respect to  $\frac{\xi_1+\xi_2}{2}$  on the co-ordinates, and the function  $(\Upsilon_1 - \mu)^2(\Upsilon_3 - v)\omega(\mu, v)$  is symmetric with respect to  $\frac{\lambda_1+\lambda_2}{2}$  on the co-ordinates. Therefore,

$$\begin{aligned} & \int_{\lambda_1}^{\Upsilon_1} \int_{\xi_1}^{\Upsilon_3} (\Upsilon_1 - \mu)^2(\Upsilon_3 - v)^2\omega(\mu, v) \, dv \, d\mu \\ & = \int_{\lambda_1}^{\Upsilon_1} \int_{\Upsilon_3}^{\xi_2} (\Upsilon_1 - \mu)^2(\Upsilon_3 - v)^2\omega(\mu, v) \, dv \, d\mu \\ & = \int_{\Upsilon_1}^{\lambda_2} \int_{\xi_1}^{\Upsilon_3} (\Upsilon_1 - \mu)^2(\Upsilon_3 - v)^2\omega(\mu, v) \, dv \, d\mu \\ & = \int_{\Upsilon_1}^{\lambda_2} \int_{\Upsilon_3}^{\xi_2} (\Upsilon_1 - \mu)^2(\Upsilon_3 - v)^2\omega(\mu, v) \, dv \, d\mu, \\ & \int_{\lambda_1}^{\Upsilon_1} \int_{\xi_1}^{\Upsilon_3} (\Upsilon_1 - \mu)(\Upsilon_3 - v)^2\omega(\mu, v) \, dv \, d\mu \\ & = \int_{\lambda_1}^{\Upsilon_1} \int_{\Upsilon_3}^{\xi_2} (\Upsilon_1 - \mu)(\Upsilon_3 - v)^2\omega(\mu, v) \, dv \, d\mu, \\ & \int_{\Upsilon_1}^{\lambda_2} \int_{\xi_1}^{\Upsilon_3} (\Upsilon_1 - \mu)(\Upsilon_3 - v)^2\omega(\mu, v) \, dv \, d\mu \\ & = \int_{\Upsilon_1}^{\lambda_2} \int_{\Upsilon_3}^{\xi_2} (\Upsilon_1 - \mu)(\Upsilon_3 - v)^2\omega(\mu, v) \, dv \, d\mu, \\ & \int_{\lambda_1}^{\Upsilon_1} \int_{\xi_1}^{\Upsilon_3} (\Upsilon_1 - \mu)^2(\Upsilon_3 - v)\omega(\mu, v) \, dv \, d\mu \\ & = \int_{\lambda_1}^{\Upsilon_1} \int_{\Upsilon_3}^{\xi_2} (\Upsilon_1 - \mu)^2(\Upsilon_3 - v)\omega(\mu, v) \, dv \, d\mu \end{aligned}$$

and

$$\int_{\Upsilon_1}^{\lambda_2} \int_{\xi_1}^{\Upsilon_3} (\Upsilon_1 - \mu)^2(\Upsilon_3 - v)\omega(\mu, v) \, dv \, d\mu = \int_{\Upsilon_1}^{\lambda_2} \int_{\Upsilon_3}^{\xi_2} (\Upsilon_1 - \mu)^2(\Upsilon_3 - v)\omega(\mu, v) \, dv \, d\mu.$$

From the above equations, we obtain

$$B_1(\lambda_1, \xi_1, \lambda_2, \xi_2) = B_2(\lambda_1, \xi_1, \lambda_2, \xi_2) = B_3(\lambda_1, \xi_1, \lambda_2, \xi_2) = 0$$

and

$$B_4(\lambda_1, \xi_1, \lambda_2, \xi_2) = 4 \int_{\frac{\lambda_1+\lambda_2}{2}}^b \int_{\frac{\xi_1+\xi_2}{2}}^d \left( \mu - \frac{\lambda_1 + \lambda_2}{2} \right) \left( v - \frac{\xi_1 + \xi_2}{2} \right) \omega(\mu, v) \, dv \, d\mu.$$

Therefore, inequality (3.14) follows from (3.10) and the above quantities. □

*Remark 3.3* Inequality (1.3) can be derived from (3.5) if

$$\omega(\mu, v) = \frac{g(\mu, v)}{\int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} g(\mu, v) \, dv \, d\mu}$$

and  $g(\mu, v)$  is symmetric with respect to  $\frac{\lambda_1+\lambda_2}{2}$  and  $\frac{\xi_1+\xi_2}{2}$  on the co-ordinates.

**Theorem 3.4** *Let  $q > 1$ ,  $\Omega \subseteq \mathbb{R}^2$ ,  $\Omega^\circ$  be the interior of  $\Omega$ ,  $\lambda_1 < \lambda_2$  and  $\xi_1 < \xi_2$  such that  $[\lambda_1, \lambda_2] \times [\xi_1, \xi_2] \subseteq \Omega^\circ$ ,  $\Phi : \Omega \rightarrow \mathbb{R}$  be a twice partially differentiable mapping on  $\Omega^\circ$  such that  $\Phi_{\varrho\varsigma} \in L([\lambda_1, \lambda_2] \times [\xi_1, \xi_2])$  and  $|\Phi_{\varrho\varsigma}|^q$  is co-ordinated convex on  $[\lambda_1, \lambda_2] \times [\xi_1, \xi_2]$ , and  $\omega : [\lambda_1, \lambda_2] \times [\xi_1, \xi_2] \rightarrow [0, \infty)$  be a continuous mapping. Then*

$$\begin{aligned} & \left| \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi(\mu, v) \omega(\mu, v) \, dv \, d\mu - \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi(\Upsilon_1, v) \omega(\mu, v) \, dv \, d\mu \right. \\ & \quad \left. - \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi(\mu, \Upsilon_3) \omega(\mu, v) \, dv \, d\mu + \Phi(\Upsilon_1, \Upsilon_3) \right| \\ & \leq 4^{1-\frac{1}{q}} \left( \int_{\Upsilon_1}^{\lambda_2} \int_{\Upsilon_3}^{\lambda_2} (\mu - \Upsilon_1)(v - \Upsilon_3) \omega(\mu, v) \, dv \, d\mu \right)^{1-\frac{1}{q}} \left( \frac{1}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \right)^{\frac{1}{q}} \\ & \quad \times \left[ |\Phi_{\varrho\varsigma}(\lambda_1, \xi_1)|^q A_1(\lambda_1, \xi_1, \lambda_2, \xi_2) + |\Phi_{\varrho\varsigma}(\lambda_1, \xi_2)|^q A_2(\lambda_1, \xi_1, \lambda_2, \xi_2) \right. \\ & \quad \left. + |\Phi_{\varrho\varsigma}(\lambda_2, \xi_1)|^q A_3(\lambda_1, \xi_1, \lambda_2, \xi_2) + |\Phi_{\varrho\varsigma}(\lambda_2, \xi_2)|^q A_4(\lambda_1, \xi_1, \lambda_2, \xi_2) \right]^{\frac{1}{q}}, \end{aligned} \tag{3.15}$$

where  $A_1(\lambda_1, \xi_1, \lambda_2, \xi_2), A_2(\lambda_1, \xi_1, \lambda_2, \xi_2), A_3(\lambda_1, \xi_1, \lambda_2, \xi_2)$  and  $A_4(\lambda_1, \xi_1, \lambda_2, \xi_2)$  are defined in Theorem 3.1.

*Proof* It follows from (3.8) and the Hölder inequality that

$$\begin{aligned} & \left| \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi(\mu, v) \omega(\mu, v) \, dv \, d\mu - \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi(\Upsilon_1, v) \omega(\mu, v) \, dv \, d\mu \right. \\ & \quad \left. - \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi(\mu, \Upsilon_3) \omega(\mu, v) \, dv \, d\mu + \Phi(\Upsilon_1, \Upsilon_3) \right| \\ & \leq \left( \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \left| \int_{\varrho}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu, v) \, dv \, d\mu - \sigma(\Upsilon_1 - \varrho) \int_{\lambda_1}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu, v) \, dv \, d\mu \right. \right. \\ & \quad \left. \left. - \sigma(\Upsilon_1 - \varsigma) \int_{\varrho}^{\lambda_2} \int_{\xi_1}^{\lambda_2} \omega(\mu, v) \, dv \, d\mu + \sigma(\Upsilon_1 - \varrho) \sigma(\Upsilon_3 - \varsigma) \int_{\varrho}^{\lambda_2} \int_{\xi_1}^{\xi_2} \omega(\mu, v) \, dv \, d\mu \right| d\varsigma \, d\varrho \right)^{1-\frac{1}{q}} \end{aligned}$$

$$\begin{aligned}
 & \times \left( \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \left| \int_{\varrho}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu - \sigma(\Upsilon_1 - \varrho) \int_{\lambda_1}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu \right. \right. \\
 & \left. \left. - \sigma(\Upsilon_3 - \varsigma) \int_{\varrho}^{\lambda_2} \int_{\xi_1}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu + \sigma(\Upsilon_1 - \varrho)\sigma(\Upsilon_3 - \varsigma) \right| \right. \\
 & \left. \times |\Phi_{\varrho\varsigma}(\varrho, \varsigma)|^q \, d\varsigma \, d\varrho \right)^{\frac{1}{q}}. \tag{3.16}
 \end{aligned}$$

Making use of Lemma 2.5, we have

$$\begin{aligned}
 & \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \left| \int_{\varrho}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu - \sigma(\Upsilon_1 - \varrho) \int_{\lambda_1}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu \right. \\
 & \left. - \sigma(\Upsilon_3 - \varsigma) \int_{\varrho}^{\lambda_2} \int_{\xi_1}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu + \sigma(\Upsilon_1 - \varrho)\sigma(\Upsilon_3 - \varsigma) \right| |\Phi_{\varrho\varsigma}(\varrho, \varsigma)|^q \, d\varsigma \, d\varrho \\
 & \leq \frac{1}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \left[ |\Phi_{\varrho\varsigma}(\lambda_1, \xi_1)|^q A_1(\lambda_1, \xi_1, \lambda_2, \xi_2) + |\Phi_{\varrho\varsigma}(\lambda_1, \xi_2)|^q A_2(\lambda_1, \xi_1, \lambda_2, \xi_2) \right. \\
 & \left. + |\Phi_{\varrho\varsigma}(\lambda_2, \xi_1)|^q A_3(\lambda_1, \xi_1, \lambda_2, \xi_2) + |\Phi_{\varrho\varsigma}(\lambda_2, \xi_2)|^q A_4(\lambda_1, \xi_1, \lambda_2, \xi_2) \right]. \tag{3.17}
 \end{aligned}$$

On the other hand, we also have

$$\begin{aligned}
 & \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \left| \int_{\varrho}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu - \sigma(\Upsilon_1 - \varrho) \int_{\lambda_1}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu \right. \\
 & \left. - \sigma(\Upsilon_3 - \varsigma) \int_{\varrho}^{\lambda_2} \int_{\xi_1}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu + \sigma(\Upsilon_1 - \varrho)\sigma(\Upsilon_3 - \varsigma) \right| \, d\varsigma \, d\varrho \\
 & = \int_{\lambda_1}^{\Upsilon_1} \int_{\xi_1}^{\Upsilon_3} \left( \int_{\lambda_1}^{\varrho} \int_{\xi_1}^{\varsigma} \omega(\mu, \nu) \, d\nu \, d\mu \right) \, d\varsigma \, d\varrho \\
 & \quad + \int_{\Upsilon_1}^{\lambda_2} \int_{\xi_1}^{\Upsilon_3} \left( \int_{\varrho}^{\lambda_2} \int_{\xi_1}^{\varsigma} \omega(\mu, \nu) \, d\nu \, d\mu \right) \, d\varsigma \, d\varrho \\
 & \quad + \int_{\lambda_1}^{\Upsilon_1} \int_{\Upsilon_3}^{\xi_2} \left( \int_{\lambda_1}^{\varrho} \int_{\varsigma}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu \right) \, d\varsigma \, d\varrho \\
 & \quad + \int_{\Upsilon_1}^{\lambda_2} \int_{\Upsilon_3}^{\xi_2} \left( \int_{\varrho}^{\lambda_2} \int_{\varsigma}^{\xi_2} \omega(\mu, \nu) \, d\nu \, d\mu \right) \, d\varsigma \, d\varrho \\
 & = 4 \int_{\Upsilon_1}^{\lambda_2} \int_{\Upsilon_3}^{\xi_2} (\mu - \Upsilon_1)(\nu - \Upsilon_2)\omega(\mu, \nu) \, d\nu \, d\mu. \tag{3.18}
 \end{aligned}$$

Therefore, inequality (3.15) follows from (3.16)–(3.18). □

*Remark 3.5* If  $\omega(\mu, \nu)$  is symmetric with respect to  $\frac{\lambda_1 + \lambda_2}{2}$  and  $\frac{\xi_1 + \xi_2}{2}$  on the co-ordinates, then inequality (3.15) leads to

$$\begin{aligned}
 & \left| \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi(\mu, \nu)\omega(\mu, \nu) \, d\nu \, d\mu - \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi\left(\frac{\lambda_1 + \lambda_2}{2}, \nu\right)\omega(\mu, \nu) \, d\nu \, d\mu \right. \\
 & \left. - \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi\left(\mu, \frac{\xi_1 + \xi_2}{2}\right)\omega(\mu, \nu) \, d\nu \, d\mu + \Phi\left(\frac{\lambda_1 + \lambda_2}{2}, \frac{\xi_1 + \xi_2}{2}\right) \right|
 \end{aligned}$$

$$\begin{aligned} &\leq 4 \left[ \frac{|\Phi_{\rho_S}(\lambda_1, \xi_1)|^q + |\Phi_{\rho_S}(\lambda_1, \xi_2)|^q + |\Phi_{\rho_S}(\lambda_2, \xi_1)|^q + |\Phi_{\rho_S}(\lambda_2, \xi_2)|^q}{4} \right]^{\frac{1}{q}} \\ &\quad \times \int_{\frac{\lambda_1+\lambda_2}{2}}^{\lambda_2} \int_{\frac{\xi_1+\xi_2}{2}}^{\xi_2} \left( \mu - \frac{\lambda_1 + \lambda_2}{2} \right) \left( v - \frac{\xi_1 + \xi_2}{2} \right) \omega(\mu, v) \, dv \, d\mu. \end{aligned} \tag{3.19}$$

*Remark 3.6* If  $g(\mu, v)$  is symmetric with respect to  $\frac{\lambda_1+\lambda_2}{2}$  and  $\frac{\xi_1+\xi_2}{2}$  on the co-ordinates and

$$\omega(\mu, v) = \frac{g(\mu, v)}{\int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} g(\mu, v) \, dv \, d\mu},$$

then inequality (3.15) gives the result proved in [75].

Next, we use the symbols for our upcoming results as follows:

$$\begin{aligned} \Psi(\omega, \Phi) &= \frac{1}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \left[ \Phi(\lambda_1, \xi_1) \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\lambda_2 - \mu)(\xi_2 - v)\omega(\mu, v) \, dv \, d\mu \right. \\ &\quad + \Phi(\lambda_1, \xi_2) \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\lambda_2 - \mu)(v - \xi_1)\omega(\mu, v) \, dv \, d\mu \\ &\quad + \Phi(\lambda_2, \xi_1) \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\mu - \lambda_1)(\xi_2 - v)\omega(\mu, v) \, dv \, d\mu \\ &\quad \left. + \Phi(\lambda_2, \xi_2) \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\mu - \lambda_1)(v - \xi_1)\omega(\mu, v) \, dv \, d\mu \right] \\ &\quad + \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi(\mu, v)\omega(\mu, v) \, dv \, d\mu \\ &\quad - \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\lambda_2 - \mu)\Phi(\lambda_1, v)\omega(\mu, v) \, dv \, d\mu \\ &\quad - \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\mu - \lambda_1)\Phi(\lambda_2, v)\omega(\mu, v) \, dv \, d\mu \\ &\quad - \frac{1}{\xi_2 - \xi_1} \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\xi_2 - v)\Phi(\mu, \xi_1)\omega(\mu, v) \, dv \, d\mu \\ &\quad - \frac{1}{\xi_2 - \xi_1} \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (v - \xi_1)\Phi(\mu, \xi_2)\omega(\mu, v) \, dv \, d\mu. \end{aligned} \tag{3.20}$$

From (3.20) we clearly see that

$$\begin{aligned} &\Psi\left(\frac{1}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)}, \Phi\right) \\ &= \frac{\Phi(\lambda_1, \xi_1) + \Phi(\lambda_1, \xi_2) + \Phi(\lambda_2, \xi_1) + \Phi(\lambda_2, \xi_2)}{4} \\ &\quad - \frac{1}{2(\lambda_2 - \lambda_1)} \int_{\lambda_1}^{\lambda_2} [\Phi(\mu, \xi_2) + \Phi(\mu, \xi_1)] \, d\mu \\ &\quad - \frac{1}{2(\xi_2 - \xi_1)} \int_{\xi_1}^{\xi_2} [\Phi(\lambda_1, v) + \Phi(\lambda_2, v)] \, dv \\ &\quad + \frac{1}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi(\mu, v) \, dv \, d\mu. \end{aligned} \tag{3.21}$$

We also use the following notations:

$$\begin{aligned} \sup\{\Phi_{r\rho}(\lambda_1, \xi_1), \Phi_{r\rho}(\Upsilon_1, \xi_1), \Phi_{r\rho}(\lambda_1, \Upsilon_3), \Phi_{r\rho}(\Upsilon_1, \Upsilon_3)\} &= \eta(\lambda_1, \xi_1, \lambda_2, \xi_2), \\ \sup\{\Phi_{r\rho}(\Upsilon_1, \xi_1), \Phi_{r\rho}(\Upsilon_1, \Upsilon_3), \Phi_{r\rho}(\lambda_2, \xi_1), \Phi_{r\rho}(\lambda_2, \Upsilon_3)\} &= \eta_2(\lambda_1, \xi_1, \lambda_2, \xi_2), \\ \sup\{\Phi_{r\rho}(\lambda_1, \Upsilon_3), \Phi_{r\rho}(\lambda_1, \xi_2), \Phi_{r\rho}(\Upsilon_1, \Upsilon_3), \Phi_{r\rho}(\Upsilon_1, \xi_2)\} &= \eta_3(\lambda_1, \xi_1, \lambda_2, \xi_2), \\ \sup\{\Phi_{r\rho}(\Upsilon_1, \Upsilon_3), \Phi_{r\rho}(\Upsilon_1, \xi_2), \Phi_{r\rho}(\lambda_2, \Upsilon_3), \Phi_{r\rho}(\lambda_2, \xi_2)\} &= \eta_4(\lambda_1, \xi_1, \lambda_2, \xi_2). \end{aligned}$$

The following results present uppermost estimates for  $|\Psi(\omega, \Phi)|$  if the function  $\Phi(\mu, \nu)$  is quasi-convex on the co-ordinates.

**Theorem 3.7** *Let  $\Omega \subseteq \mathbb{R}^2$ ,  $\Omega^\circ$  be the interior of  $\Omega$ ,  $\lambda_1 < \lambda_2$  and  $\xi_1 < \xi_2$  such that  $[\lambda_1, \lambda_2] \times [\xi_1, \xi_2] \subseteq \Omega^\circ$ ,  $\Phi : \Omega \rightarrow \mathbb{R}$  be a twice partially differentiable mapping on  $\Omega^\circ$  such that  $\Phi_{\rho\varsigma} \in L([\lambda_1, \lambda_2] \times [\xi_1, \xi_2])$  and  $|\Phi_{\rho\varsigma}|$  is quasi-convex on the co-ordinates on  $[\lambda_1, \lambda_2] \times [\xi_1, \xi_2]$ , and  $\omega : [\lambda_1, \lambda_2] \times [\xi_1, \xi_2] \rightarrow [0, \infty)$  be a continuous mapping. Then*

$$\begin{aligned} |\Psi(\omega, \Phi)| &\leq \frac{(\Upsilon_1 - \lambda_1)(\Upsilon_3 - \xi_1)}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \eta(\lambda_1, \xi_1, \lambda_2, \xi_2) \\ &\quad \times \int_0^1 \int_0^1 |H(\omega, \lambda_1, \xi_1, \Upsilon_1, \Upsilon_3; r, \rho)| d\rho dr \\ &\quad + \frac{(\lambda_2 - \Upsilon_1)(\Upsilon_3 - \xi_1)}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \eta_2(\lambda_1, \xi_1, \lambda_2, \xi_2) \\ &\quad \times \int_0^1 \int_0^1 |H(\omega, \Upsilon_1, \xi_1, \lambda_2, \Upsilon_3; r, \rho)| d\rho dr \\ &\quad + \frac{(\Upsilon_1 - \lambda_1)(\xi_2 - \Upsilon_3)}{(\lambda_2 - \lambda_1)(\lambda_2 - \xi_1)} \eta_3(\lambda_1, \xi_1, \lambda_2, \xi_2) \\ &\quad \times \int_0^1 \int_0^1 |H(\omega, \lambda_1, \Upsilon_3, \Upsilon_1, \xi_2; r, \rho)| d\rho dr \\ &\quad + \frac{(\lambda_2 - \Upsilon_1)(\xi_2 - \Upsilon_3)}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \eta_4(\lambda_1, \xi_1, \lambda_2, \xi_2) \\ &\quad \times \int_0^1 \int_0^1 |H(\omega, \Upsilon_1, \Upsilon_3, \lambda_2, \xi_2; r, \rho)| d\rho dr. \end{aligned} \tag{3.22}$$

*Proof* It follows from the quasi-convexity of the function  $|\Phi_{\rho\varsigma}|$  on the co-ordinates that

$$\begin{aligned} &\Phi_{r\rho}((1-r)\lambda_1 + \Upsilon_1 r, (1-\rho)\xi_1 + \Upsilon_3 \rho) \\ &\leq \sup\{\Phi_{r\rho}(\lambda_1, \xi_1), \Phi_{r\rho}(\Upsilon_1, \xi_1), \Phi_{r\rho}(\lambda_1, \Upsilon_3), \Phi_{r\rho}(\Upsilon_1, \Upsilon_3)\} = \eta(\lambda_1, \xi_1, \lambda_2, \xi_2), \\ &\Phi_{r\rho}((1-r)\Upsilon_1 + br, (1-\rho)\xi_1 + \Upsilon_3 \rho) \\ &\leq \sup\{\Phi_{r\rho}(\Upsilon_1, \xi_1), \Phi_{r\rho}(\Upsilon_1, \Upsilon_3), \Phi_{r\rho}(\lambda_2, \xi_1), \Phi_{r\rho}(\lambda_2, \Upsilon_3)\} = \eta_2(\lambda_1, \xi_1, \lambda_2, \xi_2), \\ &\Phi_{r\rho}((1-r)\lambda_1 + \Upsilon_1 r, (1-\rho)\Upsilon_3 + \xi_2 \rho) \\ &\leq \sup\{\Phi_{r\rho}(\lambda_1, \Upsilon_3), \Phi_{r\rho}(\lambda_1, \xi_2), \Phi_{r\rho}(\Upsilon_1, \Upsilon_3), \Phi_{r\rho}(\Upsilon_1, \xi_2)\} = \eta_3(\lambda_1, \xi_1, \lambda_2, \xi_2) \end{aligned}$$

and

$$\begin{aligned} &\Phi_{r\rho}((1-r)\Upsilon_1 + \lambda_2 r, (1-\rho)\Upsilon_3 + \xi_2 \rho) \\ &\leq \sup\{\Phi_{r\rho}(\Upsilon_1, \Upsilon_3), \Phi_{r\rho}(\Upsilon_1, \xi_2), \Phi_{r\rho}(\lambda_2, \Upsilon_3), \Phi_{r\rho}(\lambda_2, \xi_2)\} = \eta_4(\lambda_1, \xi_1, \lambda_2, \xi_2) \end{aligned}$$

for all  $(r, \rho) \in [0, 1] \times [0, 1]$ .

Therefore, inequality (3.22) follows from (2.1) and the above inequalities. □

**Theorem 3.8** *Let  $\Omega \subseteq \mathbb{R}^2$ ,  $\Omega^\circ$  be the interior of  $\Omega$ ,  $\lambda_1 < \lambda_2$  and  $\xi_1 < \xi_2$  such that  $[\lambda_1, \lambda_2] \times [\xi_1, \xi_2] \subseteq \Omega^\circ$ ,  $\Phi : \Omega \rightarrow \mathbb{R}$  be a twice partially differentiable mapping on  $\Omega^\circ$  such that  $\Phi_{\rho\varsigma} \in L([\lambda_1, \lambda_2] \times [\xi_1, \xi_2])$  and  $|\Phi_{\rho\varsigma}|$  is quasi-convex on the co-ordinates on  $[\lambda_1, \lambda_2] \times [\xi_1, \xi_2]$ , and  $\omega : [\lambda_1, \lambda_2] \times [\xi_1, \xi_2] \rightarrow [0, \infty)$  be a continuous function symmetric with respect to  $\frac{\lambda_1 + \lambda_2}{2}$  and  $\frac{\xi_1 + \xi_2}{2}$  on the co-ordinates. Then*

$$\begin{aligned} &\left| \frac{\Phi(\lambda_1, \xi_1) + \Phi(\lambda_1, \xi_2) + \Phi(\lambda_2, \xi_1) + \Phi(\lambda_2, \xi_2)}{4} + \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi(\mu, \nu)\omega(\mu, \nu) \, d\nu \, d\mu \right. \\ &\quad - \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\lambda_2 - \mu)\Phi(\lambda_1, \nu)\omega(\mu, \nu) \, d\nu \, d\mu \\ &\quad - \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\mu - \lambda_1)\Phi(\lambda_2, \nu)\omega(\mu, \nu) \, d\nu \, d\mu \\ &\quad - \frac{1}{\xi_2 - \xi_1} \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\xi_2 - \nu)\Phi(\mu, \xi_1)\omega(\mu, \nu) \, d\nu \, d\mu \\ &\quad \left. - \frac{1}{\xi_2} (\nu - \xi_1)\Phi(\mu, \xi_2)\omega(\mu, \nu) \, d\nu \, d\mu \right| \\ &\leq \left( \int_{\lambda_1}^{\frac{\lambda_1 + \lambda_2}{2}} \int_{\xi_1}^{\frac{\xi_1 + \xi_2}{2}} (\mu - \lambda_1)(\nu - \xi_1)\omega(\mu, \nu) \, d\nu \, d\mu \right) \\ &\quad \times \left[ \sup\left\{ \Phi_{r\rho}(\lambda_1, \xi_1), \Phi_{r\rho}\left(\frac{\lambda_1 + \lambda_2}{2}, \xi_1\right), \right. \right. \\ &\quad \left. \left. \Phi_{r\rho}\left(\lambda_1, \frac{\xi_1 + \xi_2}{2}\right), \Phi_{r\rho}\left(\frac{\lambda_1 + \lambda_2}{2}, \frac{\xi_1 + \xi_2}{2}\right) \right\} \right. \\ &\quad + \sup\left\{ \Phi_{r\rho}\left(\frac{\lambda_1 + \lambda_2}{2}, \xi_1\right), \Phi_{r\rho}\left(\frac{\lambda_1 + \lambda_2}{2}, \frac{\xi_1 + \xi_2}{2}\right), \Phi_{r\rho}(\lambda_2, \xi_1), \Phi_{r\rho}\left(\lambda_2, \frac{\xi_1 + \xi_2}{2}\right) \right\} \\ &\quad + \sup\left\{ \Phi_{r\rho}\left(\lambda_1, \frac{\xi_1 + \xi_2}{2}\right), \Phi_{r\rho}(\lambda_1, \xi_2), \Phi_{r\rho}\left(\frac{\lambda_1 + \lambda_2}{2}, \frac{\xi_1 + \xi_2}{2}\right), \Phi_{r\rho}\left(\frac{\lambda_1 + \lambda_2}{2}, \xi_2\right) \right\} \\ &\quad + \sup\left\{ \Phi_{r\rho}\left(\frac{\lambda_1 + \lambda_2}{2}, \frac{\xi_1 + \xi_2}{2}\right), \Phi_{r\rho}\left(\frac{\lambda_1 + \lambda_2}{2}, \xi_2\right), \right. \\ &\quad \left. \left. \Phi_{r\rho}\left(\lambda_2, \frac{\xi_1 + \xi_2}{2}\right), \Phi_{r\rho}(\lambda_2, \xi_2) \right\} \right]. \tag{3.23} \end{aligned}$$

*Proof* From the hypothesis of Theorem 3.8 we have

$$\begin{aligned} \Psi(\omega, \Phi) &= \frac{\Phi(\lambda_1, \xi_1) + \Phi(\lambda_1, \xi_2) + \Phi(\lambda_2, \xi_1) + \Phi(\lambda_2, \xi_2)}{4} \\ &\quad + \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi(\mu, \nu)\omega(\mu, \nu) \, d\nu \, d\mu + \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi(\mu, \nu)\omega(\mu, \nu) \, d\nu \, d\mu \end{aligned}$$

$$\begin{aligned}
 & - \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\lambda_2 - \mu) \Phi(\lambda_1, v) \omega(\mu, v) \, dv \, d\mu \\
 & - \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\mu - \lambda_1) \Phi(\lambda_2, v) \omega(\mu, v) \, dv \, d\mu \\
 & - \frac{1}{\xi_2 - \xi_1} \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (\xi_2 - v) \Phi(\mu, \xi_1) \omega(\mu, v) \, dv \, d\mu \\
 & - \frac{1}{\xi_2 - \xi_1} \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} (v - \xi_1) \Phi(\mu, \xi_2) \omega(\mu, v) \, dv \, d\mu.
 \end{aligned} \tag{3.24}$$

We also observe that

$$\begin{aligned}
 & \frac{(\Upsilon_1 - \lambda_1)(\Upsilon_3 - \xi_1)}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \int_0^1 \int_0^1 |H(\omega, \lambda_1, \xi_1, \Upsilon_1, \Upsilon_3; r, \rho)| \, d\rho \, dr \\
 & = \frac{1}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \int_{\lambda_1}^{\Upsilon_1} \int_{\xi_1}^{\Upsilon_3} \left| \int_{\lambda_1}^{\rho} \int_{\xi_1}^{\sigma} (\mu - \lambda_1)(v - \xi_1) \omega(\mu, v) \, dv \, d\mu \right. \\
 & \quad - \int_{\lambda_1}^{\rho} \int_{\xi_1}^{\xi_2} (\mu - \lambda_1)(v - \xi_1) \omega(\mu, v) \, dv \, d\mu - \int_{\rho}^{\lambda_2} \int_{\xi_1}^{\sigma} (\mu - \lambda_1)(v - \xi_1) \omega(\mu, v) \, dv \, d\mu \\
 & \quad \left. + \int_{\rho}^{\xi_2} \int_{\xi_1}^{\xi_1} (\mu - \lambda_1)(v - \xi_1) \omega(\mu, v) \, dv \, d\mu \right| \, d\zeta \, d\varrho, \\
 & \frac{(\lambda_2 - \Upsilon_1)(\Upsilon_3 - \xi_1)}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \int_0^1 \int_0^1 |H(\omega, \Upsilon_1, \xi_1, \lambda_2, \Upsilon_3; r, \rho)| \, d\rho \, dr \\
 & = \frac{1}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \int_{\Upsilon_1}^{\lambda_2} \int_{\xi_1}^{\Upsilon_3} \left| \int_{\lambda_1}^{\rho} \int_{\xi_1}^{\sigma} (\mu - \lambda_1)(v - \xi_1) \omega(\mu, v) \, dv \, d\mu \right. \\
 & \quad - \int_{\lambda_1}^{\rho} \int_{\xi_1}^{\xi_2} (\mu - \lambda_1)(v - \xi_1) \omega(\mu, v) \, dv \, d\mu - \int_{\rho}^{\lambda_2} \int_{\xi_1}^{\sigma} (\mu - \lambda_1)(v - \xi_1) \omega(\mu, v) \, dv \, d\mu \\
 & \quad \left. + \int_{\rho}^{\xi_2} \int_{\xi_1}^{\xi_1} (\mu - \lambda_1)(v - \xi_1) \omega(\mu, v) \, dv \, d\mu \right| \, d\zeta \, d\varrho, \\
 & \frac{(\Upsilon_1 - \lambda_1)(\xi_2 - \Upsilon_3)}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \int_0^1 \int_0^1 |H(\omega, \Upsilon_1, \xi_1, \lambda_2, \Upsilon_3; r, \rho)| \, d\rho \, dr \\
 & = \frac{1}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \int_{\lambda_1}^{\Upsilon_1} \int_{\Upsilon_3}^{\xi_2} \left| \int_{\lambda_1}^{\rho} \int_{\xi_1}^{\sigma} (\mu - \lambda_1)(v - \xi_1) \omega(\mu, v) \, dv \, d\mu \right. \\
 & \quad - \int_{\lambda_1}^{\rho} \int_{\xi_1}^{\xi_2} (\mu - \lambda_1)(v - \xi_1) \omega(\mu, v) \, dv \, d\mu - \int_{\rho}^{\lambda_2} \int_{\xi_1}^{\sigma} (\mu - \lambda_1)(v - \xi_1) \omega(\mu, v) \, dv \, d\mu \\
 & \quad \left. + \int_{\rho}^{\xi_2} \int_{\xi_1}^{\xi_1} (\mu - \lambda_1)(v - \xi_1) \omega(\mu, v) \, dv \, d\mu \right| \, d\zeta \, d\varrho
 \end{aligned}$$

and

$$\begin{aligned}
 & \frac{(\lambda_2 - \Upsilon_1)(\xi_2 - \Upsilon_3)}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \int_0^1 \int_0^1 |H(\omega, \Upsilon_1, \Upsilon_3, \lambda_2, \xi_2; r, \rho)| \, d\rho \, dr \\
 & = \frac{1}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \int_{\Upsilon_1}^{\lambda_2} \int_{\Upsilon_3}^{\xi_2} \left| \int_{\lambda_1}^{\rho} \int_{\xi_1}^{\sigma} (\mu - \lambda_1)(v - \xi_1) \omega(\mu, v) \, dv \, d\mu \right.
 \end{aligned}$$



$$\begin{aligned}
 & - \int_{\lambda_1}^{\varrho} \int_{\varsigma}^{\xi_2} (\mu - \lambda_1)(v - \xi_1)\omega(\mu, v) \, dv \, d\mu - \int_{\varrho}^{\lambda_2} \int_{\xi_1}^{\varsigma} (\mu - \lambda_1)(v - \xi_1)\omega(\mu, v) \, dv \, d\mu \\
 & + \int_{\varrho}^{\xi_2} \int_{\varsigma}^{\xi_1} (\mu - \lambda_1)(v - \xi_1)\omega(\mu, v) \, dv \, d\mu \Big| d\varsigma \, d\varrho.
 \end{aligned}$$

Let  $p : [\lambda_1, \lambda_2] \times [\xi_1, \xi_2] \rightarrow \mathbb{R}$  defined by

$$\begin{aligned}
 p(\varrho, \varsigma) &= \int_{\lambda_1}^{\varrho} \int_{\xi_1}^{\varsigma} (\mu - \lambda_1)(v - \xi_1)\omega(\mu, v) \, dv \, d\mu \\
 & - \int_{\lambda_1}^{\varrho} \int_{\varsigma}^{\xi_2} (\mu - \lambda_1)(v - \xi_1)\omega(\mu, v) \, dv \, d\mu \\
 & - \int_{\varrho}^{\lambda_2} \int_{\xi_1}^{\varsigma} (\mu - \lambda_1)(v - \xi_1)\omega(\mu, v) \, dv \, d\mu \\
 & + \int_{\varrho}^{\xi_2} \int_{\varsigma}^{\xi_1} (\mu - \lambda_1)(v - \xi_1)\omega(\mu, v) \, dv \, d\mu.
 \end{aligned}$$

Then we clearly see that

$$p_{\varrho\varsigma}(\varrho, \varsigma) = (\lambda_2 - \lambda_1)(\xi_2 - \xi_1)\omega(\varrho, \varsigma) > 0$$

for  $(\varrho, \varsigma) \in [\lambda_1, \lambda_2] \times [\xi_1, \xi_2]$ , which implies that  $p(\varrho, \varsigma)$  is an increasing function on  $[\lambda_1, \lambda_2] \times [\xi_1, \xi_2]$  and

$$p(\Upsilon_1, \Upsilon_3) = 0.$$

Now it is easy to see that

$$\begin{aligned}
 & \frac{(\Upsilon_1 - \lambda_1)(\Upsilon_3 - \xi_1)}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \int_0^1 \int_0^1 |H(\omega, \lambda_1, \xi_1, \Upsilon_1, \Upsilon_3; r, \rho)| \, d\rho \, dr \\
 &= \frac{(\lambda_2 - \Upsilon_1)(\Upsilon_3 - \xi_1)}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \int_0^1 \int_0^1 |H(\omega, \Upsilon_1, \xi_1, \lambda_2, \Upsilon_3; r, \rho)| \, d\rho \, dr \\
 &= \frac{(\Upsilon_1 - \lambda_1)(\xi_2 - \Upsilon_3)}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \int_0^1 \int_0^1 |H(\omega, \Upsilon_1, \xi_1, \lambda_2, \Upsilon_3; r, \rho)| \, d\rho \, dr \\
 &= \frac{(\lambda_2 - \Upsilon_1)(\xi_2 - \Upsilon_3)}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \int_0^1 \int_0^1 |H(\omega, \Upsilon_1, \Upsilon_3, \lambda_2, \xi_2; r, \rho)| \, d\rho \, dr \\
 &= \int_{\lambda_1}^{\frac{\lambda_1 + \lambda_2}{2}} \int_{\xi_1}^{\frac{\xi_1 + \xi_2}{2}} (\mu - \lambda_1)(v - \xi_1)\omega(\mu, v) \, dv \, d\mu.
 \end{aligned}$$

Thus, inequality (3.23) can be derived from (3.24). □

#### 4 Applications to random variables

Let  $\alpha, \beta \in \mathbb{R}$ ,  $0 < \lambda_1 < \lambda_2$ ,  $0 < \xi_1 < \xi_2$ ,  $\mathcal{X}$  and  $\mathcal{Y}$  be two independent continuous random variables with the common continuous probability density function  $\omega : [\lambda_1, \lambda_2] \times [\xi_1, \xi_2] \rightarrow [0, \infty)$ . Then the  $\alpha$ -moment of  $\mathcal{X}$  and  $\beta$ -moment of  $\mathcal{Y}$  about the origin are, respectively,

defined by

$$\mathcal{E}_\alpha(\mathcal{X}) = \int_{\lambda_1}^{\lambda_2} \varrho^\alpha \omega_1(\varrho) d\varrho, \mathcal{E}_\beta(\mathcal{Y}) = \int_{\xi_1}^{\xi_2} \varsigma^\beta \omega_2(\varsigma) d\varsigma$$

if both  $\mathcal{E}_\alpha(\mathcal{X})$  and  $\mathcal{E}_\beta(\mathcal{Y})$  are finite, where  $\omega_1 : [\lambda_1, \lambda_2] \rightarrow [0, \infty)$  and  $\omega_2 : [\xi_1, \xi_2] \rightarrow [0, \infty)$  are, respectively, the marginal probability density functions of  $\mathcal{X}$  and  $\mathcal{Y}$ . Since  $\mathcal{X}$  and  $\mathcal{Y}$  are two independent random variables, one has

$$\omega(\varrho, \varsigma) = \omega_1(\varrho)\omega_2(\varsigma)$$

for all  $(\varrho, \varsigma) \in [\lambda_1, \lambda_2] \times [\xi_1, \xi_2]$ , and

$$\begin{aligned} \mathcal{E}_{\alpha,\beta}(\mathcal{X}\mathcal{Y}) &= \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \varrho^\alpha \varsigma^\beta \omega(\varrho, \varsigma) d\varsigma d\varrho \\ &= \int_{\lambda_1}^{\lambda_2} \varrho^\alpha \omega_1(\varrho) d\varrho \int_{\xi_1}^{\xi_2} \varsigma^\beta \omega_2(\varsigma) d\varsigma = \mathcal{E}_\alpha(\mathcal{X})\mathcal{E}_\beta(\mathcal{Y}). \end{aligned}$$

Making use of the above notations, we clearly see that

$$\begin{aligned} B_1(\lambda_1, \xi_1, \lambda_2, \xi_2) &= \int_{\lambda_1}^{\mathcal{E}(\mathcal{X})} \int_{\xi_1}^{\mathcal{E}(\mathcal{Y})} (\mathcal{E}(\mathcal{X}) - \mu)^2 (\mathcal{E}(\mathcal{Y}) - v)^2 \omega(\mu, v) dv d\mu \\ &\quad - \int_{\lambda_1}^{\mathcal{E}(\mathcal{X})} \int_{\mathcal{E}(\mathcal{Y})}^{\xi_2} (\mathcal{E}(\mathcal{X}) - \mu)^2 (\mathcal{E}(\mathcal{Y}) - v)^2 \omega(\mu, v) dv d\mu \\ &\quad - \int_{\mathcal{E}(\mathcal{X})}^{\lambda_2} \int_{\xi_1}^{\mathcal{E}(\mathcal{Y})} (\mathcal{E}(\mathcal{X}) - \mu)^2 (\mathcal{E}(\mathcal{Y}) - v)^2 \omega(\mu, v) dv d\mu \\ &\quad + \int_{\mathcal{E}(\mathcal{X})}^{\lambda_2} \int_{\mathcal{E}(\mathcal{Y})}^{\xi_2} (\mathcal{E}(\mathcal{X}) - \mu)^2 (\mathcal{E}(\mathcal{Y}) - v)^2 \omega(\mu, v) dv d\mu, \end{aligned}$$

$$\begin{aligned} B_2(\lambda_1, \xi_1, \lambda_2, \xi_2) &= \int_{\lambda_1}^{\mathcal{E}(\mathcal{X})} \int_{\xi_1}^{\mathcal{E}(\mathcal{Y})} (\mathcal{E}(\mathcal{X}) - \mu)(\mathcal{E}(\mathcal{Y}) - v)^2 \omega(\mu, v) dv d\mu \\ &\quad - \int_{\lambda_1}^{\mathcal{E}(\mathcal{X})} \int_{\mathcal{E}(\mathcal{Y})}^{\xi_2} (\mathcal{E}(\mathcal{X}) - \mu)(\mathcal{E}(\mathcal{Y}) - v)^2 \omega(\mu, v) dv d\mu \\ &\quad - \int_{\mathcal{E}(\mathcal{X})}^{\lambda_2} \int_{\xi_1}^{\mathcal{E}(\mathcal{Y})} (\mathcal{E}(\mathcal{X}) - \mu)(\mathcal{E}(\mathcal{Y}) - v)^2 \omega(\mu, v) dv d\mu \\ &\quad + \int_{\mathcal{E}(\mathcal{X})}^{\lambda_2} \int_{\mathcal{E}(\mathcal{Y})}^{\xi_2} (\mathcal{E}(\mathcal{X}) - \mu)(\mathcal{E}(\mathcal{Y}) - v)^2 \omega(\mu, v) dv d\mu, \end{aligned}$$

$$\begin{aligned} B_3(\lambda_1, \xi_1, \lambda_2, \xi_2) &= \int_{\lambda_1}^{\mathcal{E}(\mathcal{X})} \int_{\xi_1}^{\mathcal{E}(\mathcal{Y})} (\mathcal{E}(\mathcal{X}) - \mu)^2 (\mathcal{E}(\mathcal{Y}) - v) \omega(\mu, v) dv d\mu \\ &\quad - \int_{\lambda_1}^{\mathcal{E}(\mathcal{X})} \int_{\mathcal{E}(\mathcal{Y})}^{\xi_2} (\mathcal{E}(\mathcal{X}) - \mu)^2 (\mathcal{E}(\mathcal{Y}) - v) \omega(\mu, v) dv d\mu \end{aligned}$$

$$\begin{aligned}
 & - \int_{\mathcal{E}(\mathcal{X})}^{\lambda_2} \int_{\xi_1}^{\mathcal{E}(\mathcal{Y})} (\mathcal{E}(\mathcal{X}) - \mu)^2 (\mathcal{E}(\mathcal{Y}) - \nu) \omega(\mu, \nu) \, d\nu \, d\mu \\
 & + \int_{\mathcal{E}(\mathcal{X})}^{\lambda_2} \int_{\mathcal{E}(\mathcal{Y})}^{\xi_2} (\mathcal{E}(\mathcal{X}) - \mu)^2 (\mathcal{E}(\mathcal{Y}) - \nu) \omega(\mu, \nu) \, d\nu \, d\mu
 \end{aligned}$$

and

$$\begin{aligned}
 & B_4(\lambda_1, \xi_1, \lambda_2, \xi_2) \\
 & = \int_{\lambda_1}^{\mathcal{E}(\mathcal{X})} \int_{\xi_1}^{\mathcal{E}(\mathcal{Y})} (\mathcal{E}(\mathcal{X}) - \mu) (\mathcal{E}(\mathcal{Y}) - \nu) \omega(\mu, \nu) \, d\nu \, d\mu \\
 & \quad - \int_{\lambda_1}^{\mathcal{E}(\mathcal{X})} \int_{\mathcal{E}(\mathcal{Y})}^{\xi_2} (\mathcal{E}(\mathcal{X}) - \mu) (\mathcal{E}(\mathcal{Y}) - \nu) \omega(\mu, \nu) \, d\nu \, d\mu \\
 & \quad - \int_{\mathcal{E}(\mathcal{X})}^{\lambda_2} \int_{\xi_1}^{\mathcal{E}(\mathcal{Y})} (\mathcal{E}(\mathcal{X}) - \mu) (\mathcal{E}(\mathcal{Y}) - \nu) \omega(\mu, \nu) \, d\nu \, d\mu \\
 & \quad + \int_{\mathcal{E}(\mathcal{X})}^{\lambda_2} \int_{\mathcal{E}(\mathcal{Y})}^{\xi_2} (\mathcal{E}(\mathcal{X}) - \mu) (\mathcal{E}(\mathcal{Y}) - \nu) \omega(\mu, \nu) \, d\nu \, d\mu.
 \end{aligned}$$

Moreover, we have

$$\begin{aligned}
 & A_1(\lambda_1, \xi_1, \lambda_2, \xi_2) \\
 & = \frac{1}{4} B_1(\lambda_1, \xi_1, \lambda_2, \xi_2) + \frac{(\lambda_2 - \mathcal{E}(\mathcal{X}))}{2} B_2(\lambda_1, \xi_1, \lambda_2, \xi_2) \\
 & \quad + \frac{(\xi_2 - \mathcal{E}(\mathcal{Y}))}{2} B_3(\lambda_1, \xi_1, \lambda_2, \xi_2) + (\lambda_2 - \mathcal{E}(\mathcal{X})) (\xi_2 - \mathcal{E}(\mathcal{Y})) B_4(\lambda_1, \xi_1, \lambda_2, \xi_2),
 \end{aligned}$$

$$\begin{aligned}
 & A_2(\lambda_1, \xi_1, \lambda_2, \xi_2) \\
 & = \frac{1}{4} B_1(\lambda_1, \xi_1, \lambda_2, \xi_2) + \frac{(\lambda_2 - \mathcal{E}(\mathcal{X}))}{2} B_2(\lambda_1, \xi_1, \lambda_2, \xi_2) \\
 & \quad + \frac{(\mathcal{E}(\mathcal{Y}) - \xi_1)}{2} B_3(\lambda_1, \xi_1, \lambda_2, \xi_2) + (\lambda_2 - \mathcal{E}(\mathcal{X})) (\mathcal{E}(\mathcal{Y}) - \xi_1) B_4(\lambda_1, \xi_1, \lambda_2, \xi_2),
 \end{aligned}$$

$$\begin{aligned}
 & A_3(\lambda_1, \xi_1, \lambda_2, \xi_2) \\
 & = \frac{1}{4} B_1(\lambda_1, \xi_1, \lambda_2, \xi_2) + \frac{(\mathcal{E}(\mathcal{X}) - \lambda_1)}{2} B_2(\lambda_1, \xi_1, \lambda_2, \xi_2) \\
 & \quad + \frac{(\xi_2 - \mathcal{E}(\mathcal{Y}))}{2} B_3(\lambda_1, \xi_1, \lambda_2, \xi_2) + (\mathcal{E}(\mathcal{X}) - \lambda_1) (\xi_2 - \mathcal{E}(\mathcal{Y})) B_4(\lambda_1, \xi_1, \lambda_2, \xi_2)
 \end{aligned}$$

and

$$\begin{aligned}
 & A_4(\lambda_1, \xi_1, \lambda_2, \xi_2) \\
 & = \frac{1}{4} B_1(\lambda_1, \xi_1, \lambda_2, \xi_2) + \frac{(\mathcal{E}(\mathcal{X}) - \lambda_1)}{2} B_2(\lambda_1, \xi_1, \lambda_2, \xi_2) \\
 & \quad + \frac{(\mathcal{E}(\mathcal{Y}) - \xi_1)}{2} B_3(\lambda_1, \xi_1, \lambda_2, \xi_2) + (\mathcal{E}(\mathcal{X}) - \lambda_1) (\mathcal{E}(\mathcal{Y}) - \xi_1) B_4(\lambda_1, \xi_1, \lambda_2, \xi_2).
 \end{aligned}$$

Now, we give an application for our obtained results to the random variables.

**Theorem 4.1** *The inequality*

$$\begin{aligned}
 & \left| \mathcal{E}_\alpha(\mathcal{X})\mathcal{E}_\beta(\mathcal{Y}) - [\mathcal{E}(\mathcal{X})]^\alpha \mathcal{E}_\beta(\mathcal{Y}) - \mathcal{E}_\alpha(\mathcal{X})[\mathcal{E}(\mathcal{Y})]^\beta + [\mathcal{E}(\mathcal{X})]^\alpha [\mathcal{E}(\mathcal{Y})]^\beta \right| \\
 & \leq \frac{\alpha\beta}{(\lambda_2 - \lambda_1)(\xi_2 - \xi_1)} \left[ \lambda_1^{\alpha-1} \xi_1^{\beta-1} \Upsilon_1(\lambda_1, \xi_1, \lambda_2, \xi_2) + \lambda_1^{\alpha-1} \xi_2^{\beta-1} A_2(\lambda_1, \xi_1, \lambda_2, \xi_2) \right. \\
 & \quad \left. + \lambda_2^{\alpha-1} \xi_1^{\beta-1} A_3(\lambda_1, \xi_1, \lambda_2, \xi_2) + \lambda_2^{\alpha-1} \xi_2^{\beta-1} A_4(\lambda_1, \xi_1, \lambda_2, \xi_2) \right] \tag{4.1}
 \end{aligned}$$

holds for  $0 < \lambda_1 < \lambda_2$ ,  $0 < \xi_1 < \xi_2$  and  $\alpha, \beta \geq 2$ , where  $\Upsilon_1(\lambda_1, \xi_1, \lambda_2, \xi_2)$ ,  $A_2(\lambda_1, \xi_1, \lambda_2, \xi_2)$ ,  $A_3(\lambda_1, \xi_1, \lambda_2, \xi_2)$  and  $A_4(\lambda_1, \xi_1, \lambda_2, \xi_2)$  are defined by the previously shown equations.

*Proof* Let  $\alpha, \beta \geq 2$ ,  $\Phi(\varrho, \varsigma) = \varrho^\alpha \varsigma^\beta$  be defined on  $[\lambda_1, \lambda_2] \times [\xi_1, \xi_2]$ . Then we clearly see that  $|\Phi_{\varrho\varsigma}(\varrho, \varsigma)| = \alpha\beta\varrho^{\alpha-1}\varsigma^{\beta-1}$  is co-ordinated convex on  $[\lambda_1, \lambda_2] \times [\xi_1, \xi_2]$  and

$$\begin{aligned}
 & \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi(\varrho, \varsigma) \omega(\varrho, \varsigma) d\varsigma d\varrho \\
 & = \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \varrho^\alpha \varsigma^\beta \omega_1(\varrho) \omega_2(\varsigma) d\varsigma d\varrho \\
 & = \int_{\lambda_1}^{\lambda_2} \varrho^\alpha \omega_1(\varrho) d\varrho \int_{\xi_1}^{\xi_2} \varsigma^\beta \omega_2(\varsigma) d\varsigma = \mathcal{E}_\alpha(\mathcal{X})\mathcal{E}_\beta(\mathcal{Y}), \\
 & \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi(\Upsilon_1, \varrho) \omega(\varrho, \varsigma) d\varsigma d\varrho = [\mathcal{E}(\mathcal{X})]^\alpha \mathcal{E}_\beta(\mathcal{Y}), \\
 & \int_{\lambda_1}^{\lambda_2} \int_{\xi_1}^{\xi_2} \Phi(\varrho, \Upsilon_3) \omega(\varrho, \varsigma) d\varsigma d\varrho = \mathcal{E}_\alpha(\mathcal{X}) [\mathcal{E}(\mathcal{Y})]^\beta
 \end{aligned}$$

and

$$\Phi(\Upsilon_1, \Upsilon_3) = [\mathcal{E}(\mathcal{X})]^\alpha [\mathcal{E}(\mathcal{Y})]^\beta.$$

Therefore, inequality (4.1) follows from (3.5) and the above identities. □

Next, we provide an example to support our obtained results.

*Example 4.2* Let

$$\Phi(\mu, \nu) = \begin{cases} \frac{4}{9}\mu\nu, & 1 \leq \mu, \nu \leq 2, \\ 0, & \text{otherwise,} \end{cases}$$

be the joint probability density function of the random variables  $\mathcal{X}$  and  $\mathcal{Y}$ . Then the marginal probability density functions of the random variables  $\mathcal{X}$  and  $\mathcal{Y}$  are given by

$$\mathcal{G}(\mu) = \begin{cases} \frac{2}{3}\mu, & 1 \leq \mu \leq 2, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\mathcal{H}(\nu) = \begin{cases} \frac{2}{3}\nu, & 1 \leq \nu \leq 2, \\ 0, & \text{otherwise,} \end{cases}$$

respectively, and the random variables  $\mathcal{X}$  and  $\mathcal{Y}$  are independent due to  $\Phi(\mu, \nu) = \mathcal{G}(\mu) \times \mathcal{H}(\nu)$ . Elaborate computations give

$$\begin{aligned} \mathcal{E}(\mathcal{X}) = \mathcal{E}(\mathcal{Y}) &= \frac{14}{9}, \\ B_1(\lambda_1, \xi_1, \lambda_2, \xi_2) &= \frac{597,529}{13,947,137,604}, \\ B_2(\lambda_1, \xi_1, \lambda_2, \xi_2) &= \frac{618,400}{387,420,489}, \\ B_3(\lambda_1, \xi_1, \lambda_2, \xi_2) &= \frac{618,400}{387,420,489}, \end{aligned}$$

and

$$B_4(\lambda_1, \xi_1, \lambda_2, \xi_2) = \frac{2,560,000}{43,046,721}.$$

Hence,

$$\begin{aligned} A_1(\lambda_1, \xi_1, \lambda_2, \xi_2) &= \frac{77,281,681}{6,198,727,824}, \\ A_2(\lambda_1, \xi_1, \lambda_2, \xi_2) &= \frac{288,107,443}{18,596,183,472}, \\ A_3(\lambda_1, \xi_1, \lambda_2, \xi_2) &= \frac{288,107,443}{18,596,183,472}, \end{aligned}$$

and

$$A_4(\lambda_1, \xi_1, \lambda_2, \xi_2) = \frac{1,074,069,529}{55,788,550,416}.$$

Therefore, it follows from (4.1) that

$$\begin{aligned} &\left| \frac{4}{9} \left( \int_1^2 \mu^{\alpha+1} d\mu \right) \left( \int_1^2 \mu^{\beta+1} d\mu \right) - \frac{2}{3} \left( \frac{14}{9} \right)^\alpha \int_1^2 \mu^{\beta+1} d\mu \right. \\ &\quad \left. - \frac{2}{3} \left( \frac{14}{9} \right)^\beta \int_1^2 \mu^{\alpha+1} d\mu + \left( \frac{14}{9} \right)^{\alpha+\beta} \right| \\ &\leq \alpha\beta \left[ \frac{77,281,681}{6,198,727,824} + (2^{\beta-1} + 2^{\alpha-1}) \times \frac{288,107,443}{18,596,183,472} \right. \\ &\quad \left. + 2^{\alpha+\beta-2} \times \frac{1,074,069,529}{55,788,550,416} \right] \end{aligned}$$

for  $\alpha, \beta \geq 2$ , that is,

$$\left| \frac{[2(9^\alpha - 2^{2+\alpha} \times 9^\alpha + 3 \times 14^\alpha) + 3 \times 14^\alpha \times \alpha][2(9^\beta - 2^{2+\beta} \times 9^\beta + 3 \times 14^\beta) + 3 \times 14^\beta \times \beta]}{9^{1+\alpha+\beta}(2 + \alpha)(2 + \beta)} \right| \leq \frac{\alpha\beta(32,773 \times 2^\alpha + 52,746)(32,773 \times 2^\beta + 52,746)}{223,154,201,664}$$

for  $\alpha, \beta \geq 2$ .

### 5 Conclusion

In the article, we have established several weighted Hermite–Hadamard type inequalities for the co-ordinated convex and quasi-convex functions, provided an application to the moment of continuous random variables of bivariate distribution functions in the probability theory and presented an example on the probability distribution to support our results. Our results are generalizations of some previous results, and our approach may have further applications in the theory of convexity and Hermite–Hadamard inequality.

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#### Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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