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Asymptotic properties of wavelet estimators in heteroscedastic semiparametric model based on negatively associated innovations

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Abstract

Consider the heteroscedastic semiparametric regression model $y_i = x_i\beta + g(t_i) + \varepsilon_i$, $i = 1, 2, \ldots, n$, where β is an unknown slope parameter, $\varepsilon_i = \sigma_i e_i$, $\sigma_i^2 = f(u_i)$, (x_i, t_i, u_i) are nonrandom design points, y_i are the response variables, f and g are unknown functions defined on the closed interval [0, 1], random errors $\{e_i\}$ are negatively associated (NA) random variables with zero means. Whereas kernel estimators of β, g, and f have attracted a lot of attention in the literature, in this paper, we investigate their wavelet estimators and derive the strong consistency of these estimators under NA error assumption. At the same time, we also obtain the Berry–Esséen type bounds of the wavelet estimators of β and g.

MSC: 62G05; 60F05

Keywords: Semiparametric regression model; Wavelet estimator; Berry–Esséen bound; Negatively associated random error; Consistency

1 Introduction

Consider the following partially linear or semiparametric regression model:

$$y_i = x_i\beta + g(t_i) + \varepsilon_i, \quad 1 \le i \le n, \tag{1.1}$$

where β is an unknown slope parameter, $\varepsilon_i = \sigma_i e_i$, $\sigma_i^2 = f(u_i)$, (x_i, t_i, u_i) are nonrandom design points, y_i are the response variables, f and g are unknown functions defined on the closed interval $[0, 1]$, and $\{e_i\}$ are random errors.

It is well known that model (1.1) and its particular cases have been widely studied by many authors when the errors e_i are independent identically distributed (i.i.d.). For instance, when $\sigma_i^2 = \sigma^2$, model (1.1) is reduced to the usual partial linear model, which was first introduced by Engle et al. [1], and then various estimation methods have been used to obtain estimators of the unknown quantities in (1.1) and their asymptotic properties, (see [2–5]). Under the mixing assumption, the asymptotic normality of the estimators for β and g were derived in [6–9]. When $\sigma_i^2 = f(u_i)$, model (1.1) becomes the heteroscedastic semiparametric model, Back and Liang [10], Zhang and Liang [11], and Wei and Li [12] established the strong consistency and the asymptotic normality, respectively, for the least-squares estimators (LSEs) and weighted least-squares estimators (WLSEs) of β, based on nonparametric estimators of f and g. If $g(t) \equiv 0$ and $\sigma_i^2 = f(u_i)$, model (1.1) is reduced to the heteroscedastic linear model; when $\beta \equiv 0$ and $\sigma_i^2 = f(u_i)$, model (1.1) boils down to the heteroscedastic nonparametric regression model, whose asymptotic properties of unknown quantities were studied by Robinson [13], Carroll and Härdle [14], and Liang and Qi [15].

In recent years, wavelets techniques, owing to their ability to adapt to local features of curves, have been used extensively in statistics, engineering, and technological fields. Many authors have considered employing wavelet methods to estimate nonparametric and semiparametric models. See Antoniadis et al. [16], Sun and Chai [17], Li et al. [18–20], Xue [21], Zhou et al. [22], among others.

In this paper, we consider NA for the model errors. Let us recall the definition of NA random variables. A finite collection of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be negatively associated (NA) if for all disjoint subsets $A, B \subset \{1, 2, \ldots, n\}$,

$$\operatorname{Cov}\bigl(f(X_i, i \in A), g(X_j, j \in B)\bigr) \leq 0,$$

where f and g are real coordinatewise nondecreasing functions such that their covariance exists. An infinite sequence of random variables $\{X_n, n \geq 1\}$ is said to be NA if for every $n \geq 2$, $X_1, X_2, \ldots, X_n$ are NA. The definition of NA random variables was introduced by Alam and Saxena [23] and carefully studied by Joag-Dev and Proschan [24]. Although i.i.d. random variables are NA, NA variables may be non-i.i.d. according to the definitions. Because of its wide applications in systems reliability and multivariate statistical analysis, recently the notion of NA received a lot of attention. We refer to Matula [25] and Shao [26], amongst others.

In this paper, we aim to derive the least squares estimators, weighted least squares estimators of β, and their strong consistency for the wavelet estimators of f and g. At the same time, Berry–Esséen-type bounds of their wavelet estimators of β and g are investigated for the heteroscedastic semiparametric model under NA random errors.

The structure of the rest is as follows. Some basic assumptions and estimators are listed in Sect. 2. Some notations and main results are given in Sect. 3. Proofs of the main results are provided in Sect. 4. In the Appendix, some preliminary lemmas are stated.

Throughout the paper, $C, C_1, C_2, \ldots$ denote some positive constants not depending on n, which may be different in various places. By $\lfloor x \rfloor$ we denote the largest integer not exceeding x; $(j_1, j_2, \ldots, j_n)$ stands for any permutation of $(1, 2, \ldots, n)$; $a_n = O(b_n)$ means $|a_n| \leq C|b_n|$, and $a_n = o(b_n)$ means that $a_n/b_n \to 0$. By $I(A)$ we denote the indicator function of a set A, $\Phi(x)$ is the standard normal distribution function, and $a^+ = \max(0, a)$, $a^- = \min(0, -a)$. All limits are taken as the sample size n tends to ∞, unless specified otherwise.

2 Estimators and assumptions

In model (1.1), if β is known to be the true parameter, then since $Ee_i = 0$, we have $g(t_i) = E(y_i - x_i\beta)$, $1 \leq i \leq n$. Hence a natural wavelet estimator of g is

$$g_n(t, \beta) = \sum_{i=1}^{n} (y_i - x_i\beta) \int_{A_i} E_m(t, s)\, ds,$$

where the wavelet kernel $E_m(t,s)$ can be defined by $E_m(t,s) = 2^m \sum_{k\in Z} \phi(2^m t - k)\phi(2^m s - k)$, ϕ is a scaling function, $m = m(n) > 0$ is an integer depending only on n, and $A_i = [s_{i-1}, s_i)$ are intervals that partition $[0,1]$ with $t_i \in A_i$, $i = 1, 2, \dots, n$, and $0 \le t_1 \le t_2 \le \cdots \le t_n = 1$. To estimate β, we minimize

$$S(\beta) = \sum_{i=1}^{n} \bigl[y_i - x_i\beta - g_n(t,\beta)\bigr]^2 = \sum_{i=1}^{n} (\tilde{y}_i - \tilde{x}_i\beta)^2. \tag{2.1}$$

The minimizer to (2.1) is found to be

$$\hat{\beta}_L = \sum_{i=1}^{n} \tilde{x}_i\tilde{y}_i / S_n^2, \tag{2.2}$$

where $\tilde{x}_i = x_i - \sum_{j=1}^{n} x_j \int_{A_j} E_m(t_i,s)\,ds$, $\tilde{y}_i = y_i - \sum_{j=1}^{n} y_j \int_{A_j} E_m(t_i,s)\,ds$, and $S_n^2 = \sum_{i=1}^{n} \tilde{x}_i^2$. The estimator $\hat{\beta}_L$ is called the LSE of β.

When the errors are heteroscedastic, we consider two different cases according to f. If $\sigma_i^2 = f(u_i)$ are known, then $\hat{\beta}_L$ is modified to be the WLSE

$$\hat{\beta}_W = \sum_{i=1}^{n} a_i\tilde{x}_i\tilde{y}_i / T_n^2, \tag{2.3}$$

where $a_i = \sigma_i^{-2} = 1/f(u_i)$ and $T_n^2 = \sum_{i=1}^{n} a_i\tilde{x}_i^2$. In fact, f is unknown and must be estimated. When $Ee_i^2 = 1$, we have $E(y_i - x_i\beta - g(t_i))^2 = f(u_i)$. Hence the estimator of f can be defined by

$$\hat{f}_n(u_i) = \sum_{j=1}^{n} (\tilde{y}_j - \tilde{x}_j\hat{\beta}_L)^2 \int_{B_j} E_m(u_i,s)\,ds, \tag{2.4}$$

where $B_i = [s'_{i-1}, s'_i)$ are intervals that partition $[0,1]$ with $u_i \in B_i$, $i = 1, 2, \dots, n$, and $0 \le u_1 \le u_2 \le \cdots \le u_n = 1$.

For convenience, we assume that $\min_{1\le i\le n} |\hat{f}_n(u_i)| > 0$. Consequently, the WLSE of β is

$$\tilde{\beta}_W = \sum_{i=1}^{n} a_{ni}\tilde{x}_i\tilde{y}_i / W_n^2, \tag{2.5}$$

where $a_{ni} = 1/\hat{f}_n(u_i)$, $1 \le i \le n$, and $W_n^2 = \sum_{i=1}^{n} a_{ni}\tilde{x}_i^2$.

We define the plug-in estimators of the nonparametric component g corresponding to $\hat{\beta}_L$, $\hat{\beta}_W$, and $\tilde{\beta}_W$, respectively, by

$$\hat{g}_L(t) = \sum_{i=1}^{n} (y_i - x_i\hat{\beta}_L) \int_{A_i} E_m(t,s)\,ds, \qquad \hat{g}_W(t) = \sum_{i=1}^{n} (y_i - x_i\hat{\beta}_W) \int_{A_i} E_m(t,s)\,ds,$$

and

$$\tilde{g}_W(t) = \sum_{i=1}^{n} (y_i - x_i\tilde{\beta}_W) \int_{A_i} E_m(t,s)\,ds.$$

Now we list some basic assumptions, which will be used in Sect. 3.

(A0) Let $\{e_i, i \geq 1\}$ be a sequence of negatively associated random variables with $Ee_i = 0$ and $\sup_{i\geq 1} E|e_i|^{2+\delta} < \infty$ for some $\delta > 0$.

(A1) There exists a function h on $[0,1]$ such that $x_i = h(t_i) + v_i$, and
  (i) $\lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} v_i^2 = \Sigma_0$ $(0 < \Sigma_0 < \infty)$;
  (ii) $\max_{1\leq i\leq n} |v_i| = O(1)$;
  (iii) $\limsup_{n\to\infty} (\sqrt{n}\log n)^{-1} \max_{1\leq m\leq n} |\sum_{i=1}^{m} v_{j_i}| < \infty$.

(A2) (i) f, g, and h satisfy the Lipschitz condition of order 1 on $[0,1]$, $h \in H^{\nu}$, $\nu > 3/2$, where H^{ν} is the Sobolev space of order ν;
  (ii) $0 < m_0 \leq \min_{1\leq i\leq n} f(u_i) \leq \max_{1\leq i\leq n} f(u_i) \leq M_0 < \infty$.

(A3) The scaling function ϕ is r-regular (r is a positive integer), satisfies the Lipschitz condition of order 1, and has a compact support. Furthermore, $|\hat{\phi}(\xi) - 1| = O(\xi)$ as $\xi \to \infty$, where $\hat{\phi}$ is the Fourier transform of ϕ.

(A4) (i) $\max_{1\leq i\leq n} |s_i - s_{i-1}| = O(n^{-1})$; $\max_{1\leq i\leq n} |s'_i - s'_{i-1}| = O(n^{-1})$;
  (ii) There exist $d_1 > 0$ and $d_2 > 0$ such that $\min_{1\leq i\leq n} |t_i - t_{i-1}| \geq d_1/n$ and $\min_{1\leq i\leq n} |u_i - u_{i-1}| \geq d_2/n$.

(A5) The spectral density function $\psi(\omega)$ of $\{e_i\}$ satisfies $0 < C_1 \leq \psi(\omega) \leq C_2 < \infty$ for $\omega \in (-\pi, \pi]$.

(A6) There exist positive integers $p := p(n)$, $q := q(n)$, and $k := k_n = \lfloor \frac{n}{p+q} \rfloor$ such that for $p + q \leq n$, $qp^{-1} \leq C < \infty$.

Remark 2.1 Assumptions (A1), (A2), and (A5) are the standard regularity conditions commonly used in the recent literature such as Härdle [5], Liang et al. [6], Liang and Fan [7], Zhang and Liang [11]. Assumption (A3) is a general condition for wavelet estimator. According to Bernstein's blockwise idea, Assumption (A6) is a technical condition and easily satisfied if p, q are chosen reasonably to show Theorem 3.3 (see, e.g., Liang et al. [15, 27] and Li et al. [18, 20]).

Remark 2.2 It can be deduced from (A1)(i), (iii), (A2), (A3), and (A4) that

$$\begin{aligned} &n^{-1}\sum_{i=1}^{n}\tilde{x}_i^2 \to \Sigma_0; \qquad S_n^{-2}\sum_{i=1}^{n}|\tilde{x}_i| \leq C; \\ &C_3 \leq n^{-1}\sum_{i=1}^{n}\sigma_i^{-2}\tilde{x}_i^2 \leq C_4; \qquad T_n^{-2}\sum_{i=1}^{n}\bigl|\sigma_i^{-2}\tilde{x}_i\bigr| \leq C. \end{aligned} \tag{2.6}$$

Remark 2.3 Let $\tilde{\kappa}(t_i) = \kappa(t_i) - \sum_{j=1}^{n}\kappa(t_j)\int_{A_j} E_m(t_i,s)\,ds$, set $\kappa = f, g$, or h. Under assumptions (A2)(i), (A3), and (A4), by Theorem 3.2 in [16] it follows that

$$\sup_t\bigl|\tilde{\kappa}(t)\bigr| = O\bigl(n^{-1} + 2^{-m}\bigr). \tag{2.7}$$

Remark 2.4 By (A1)(ii), (2.7), and Lemma A.6 in the Appendix it is easy to obtain that

$$\max_{1\leq i\leq n}|\tilde{x}_i| \leq \max_{1\leq i\leq n}|\tilde{h}_i| + \max_{1\leq i\leq n}|v_i| + \max_{1\leq j\leq n}|v_j| \max_{1\leq i\leq n}\sum_{j=1}^{n}\left|\int_{A_j} E_m(t_i,s)\,ds\right| = O(1). \tag{2.8}$$

3 Notations and main results

To state our main results, we introduce the following notations. Set

$$\sigma_{1n}^2 = \operatorname{Var}\left(\sum_{i=1}^{n} \tilde{x}_i \sigma_i e_i\right), \qquad \sigma_{2n}^2 = \operatorname{Var}\left(\sum_{i=1}^{n} \tilde{x}_i \sigma_i^{-1} e_i\right),$$

$$\Gamma_n^2(t) = \operatorname{Var}\left(\sum_{i=1}^{n} \sigma_i e_i \int_{A_i} E_m(t,s)\,ds\right); \qquad u(q) = \sup_{j\ge 1} \sum_{j:|j-i|\ge q} \left|\operatorname{Cov}(e_i,e_j)\right|,$$

$$\lambda_{1n} = qp^{-1}, \qquad \lambda_{2n} = pn^{-1}, \qquad \lambda_{3n} = \left(2^{-m}+n^{-1}\right)^2, \qquad \lambda_{4n} = \frac{2^m}{n}\log^2 n;$$

$$\lambda_{5n} = \left(2^{-m}+n^{-1}\right)\log n + \sqrt{n}\left(2^{-m}+n^{-1}\right)^{-2},$$

$$\gamma_{1n} = qp^{-1}2^m, \qquad \gamma_{2n} = pn^{-1}2^m, \qquad \gamma_{3n} = 2^{-m/2} + \sqrt{2^m/n}\log n;$$

$$\mu_{1n} = \sum_{i=1}^{3} \lambda_{in}^{1/3} + 2\lambda_{4n}^{1/3} + \lambda_{5n}; \qquad \mu_{2n} = \sum_{i=1}^{2} \gamma_{in}^{1/3} + \gamma_{3n}^{(2+\delta)/(3+\delta)};$$

$$\upsilon_{1n} = \lambda_{2n}^{\delta/2} + u^{1/3}(q); \qquad \upsilon_{2n} = \gamma_{2n}^{\delta/2} + u^{1/3}(q).$$

Theorem 3.1 *Suppose that* (A0), (A1)(i), *and* (A2)–(A5) *hold. If* $2^m/n = O(n^{-1/2})$, *then*

$$\text{(i)}\quad \hat{\beta}_L \to \beta \quad a.s.; \qquad \text{(ii)}\quad \hat{\beta}_W \to \beta \quad a.s. \tag{3.1}$$

In addition, if $\max_{1\le j\le n} |\sum_{i=1}^n x_i \int_{A_i} E_m(t_j,s)\,ds| = O(1)$, *then*

$$\text{(i)}\quad \max_{1\le i\le n}\left|\hat{g}_L(t_i) - g(t_i)\right| \to 0 \quad a.s.; \qquad \text{(ii)}\quad \max_{1\le i\le n}\left|\hat{g}_W(t_i) - g(t_i)\right| \to 0 \quad a.s. \tag{3.2}$$

Theorem 3.2 *Assume that* (A0), (A1)(i), *and* (A2)–(A5) *are satisfied. If* $Ee_i^2 = 1$, $\sup_i E|e_i|^p < \infty$ *for some* $p > 4$ *and* $2^m/n = O(n^{-1/2})$, *then*

$$\begin{aligned} &\text{(i)}\quad |\hat{\beta}_L - \beta| = o\left(n^{-1/4}\right); \qquad \text{(ii)}\quad \max_{1\le i\le n}\left|\hat{f}_n(u_i) - f(u_i)\right| \to 0 \quad a.s.; \\ &\text{(iii)}\quad \tilde{\beta}_W \to \beta \quad a.s. \end{aligned} \tag{3.3}$$

In addition, if $\max_{1\le j\le n} |\sum_{i=1}^n x_i \int_{A_i} E_m(t_j,s)\,ds| = O(1)$, *then*

$$\max_{1\le i\le n}\left|\tilde{g}_W(t_i) - g(t_i)\right| \to 0 \quad a.s. \tag{3.4}$$

Remark 3.1 When random errors $\{e_i\}$ are i.i.d. random variables with zero means, Chen et al. [3] proved (3.1) and (3.3) under similar conditions. Since independent random samples are a particular case of NA random samples, Theorems 3.1 and 3.2 extend the results of Chen et al. [3]. Back and Liang [10] also obtained (3.1)–(3.4) for the weighted estimators of unknown quantities under NA samples.

Theorem 3.3 *Suppose that* (A0)–(A6) *are satisfied. If* $\mu_{1n} \to 0$ *and* $\upsilon_{1n} \to 0$, *then we have*

$$\begin{aligned} &\text{(i)} \quad \sup_y \left| P\left(\frac{S_n^2(\hat{\beta}_L - \beta)}{\sigma_{1n}} \le y \right) - \Phi(y) \right| = O(\mu_{1n} + \upsilon_{1n}); \\ &\text{(ii)} \quad \sup_y \left| P\left(\frac{T_n^2(\hat{\beta}_W - \beta)}{\sigma_{2n}} \le y \right) - \Phi(y) \right| = O(\mu_{1n} + \upsilon_{1n}). \end{aligned} \tag{3.5}$$

Corollary 3.1 *Suppose that the assumptions of Theorem* 3.3 *are fulfilled. If* $2^m/n = O(n^{-\theta})$, $u(n) = O(n^{-(1-\theta)/(2\theta-1)})$, $\frac{1}{2} < \theta \le \frac{7}{10}$, *then*

$$\begin{aligned} &\text{(i)} \quad \sup_y \left| P\left(\frac{S_n^2(\hat{\beta}_L - \beta)}{\sigma_{1n}} \le y \right) - \Phi(y) \right| = O\bigl(n^{\frac{\theta-1}{3}}\bigr); \\ &\text{(ii)} \quad \sup_y \left| P\left(\frac{T_n^2(\hat{\beta}_W - \beta)}{\sigma_{2n}} \le y \right) - \Phi(y) \right| = O\bigl(n^{\frac{\theta-1}{3}}\bigr). \end{aligned} \tag{3.6}$$

Theorem 3.4 *Under the assumptions of Theorem* 3.3, *if* $\mu_{2n} \to 0$ *and* $\upsilon_{2n} \to 0$, *then for each* $t \in [0,1]$, *we have*

$$\sup_y \left| P\left(\frac{\hat{g}(t) - E\hat{g}(t)}{\Gamma_n(t)} \le y \right) - \Phi(y) \right| = O(\mu_{2n} + \upsilon_{2n}), \tag{3.7}$$

where $\hat{g}(t) = \hat{g}_L(t)$ *or* $\hat{g}_W(t)$.

Corollary 3.2 *Under the assumption of Theorem* 3.4, *if*

$$2^m/n = O\bigl(n^{-\theta}\bigr), \qquad u(n) = O\bigl(n^{-(\theta-\rho)/(2\rho-1)}\bigr), \qquad \frac{1}{2} < \rho < \theta < 1,$$

then

$$\sup_y \left| P\left(\frac{\hat{g}(t) - E\hat{g}(t)}{\Gamma_n(t)} \le y \right) - \Phi(y) \right| = O\bigl(n^{-\min\{\frac{\theta-\rho}{3}, \frac{3(1-\theta)}{8}\}}\bigr). \tag{3.8}$$

4 Proof of main theorems

Proof of Theorem 3.1 We prove only (3.1)(ii), as the proof of (3.1)(i) is analogous.

Step 1. Set $\tilde{g}_i = g(t_i) - \sum_{j=1}^n g(t_j) \int_{A_j} E_m(t_i,s)\,ds$, $\tilde{\varepsilon}_i = \varepsilon_i - \sum_{j=1}^n \varepsilon_j \int_{A_j} E_m(t_i,s)\,ds$. It easily follows that

$$\begin{aligned} \hat{\beta}_W - \beta &= T_n^{-2} \left[\sum_{i=1}^n a_i \tilde{x}_i \varepsilon_i - \sum_{i=1}^n a_i \tilde{x}_i \left(\sum_{j=1}^n \varepsilon_j \int_{A_j} E_m(t_i,s)\,ds \right) + \sum_{i=1}^n a_i \tilde{x}_i \tilde{g}_i \right] \\ &:= A_{1n} - A_{2n} + A_{3n}. \end{aligned} \tag{4.1}$$

Firstly, we prove $A_{1n} \to 0$ a.s. Note that $A_{1n} = \sum_{i=1}^{n}(T_n^{-2}a_i\tilde{x}_i\sigma_i)e_i =: \sum_{i=1}^{n} c_{ni}e_i$. It follows from (A1)(i), (A2)(ii), (2.6), and $2^m/n = O(n^{-1/2})$ that

$$\max_{1\le i\le n}|c_{ni}| \le \max_{1\le i\le n}\frac{|a_i\tilde{x}_i|}{T_n}\cdot\max_{1\le i\le n}\frac{\sigma_i}{T_n} = O\bigl(n^{-1/2}\bigr),$$
$$\sum_{i=1}^{n} c_{ni}^2 = T_n^{-4}\sum_{i=1}^{n} a_i\tilde{x}_i^2\cdot a_i\sigma_i^2 = O\bigl(n^{-1}\bigr).$$

Hence, according to Lemma A.1, we have $A_{1n} \to 0$ a.s. Obviously,

$$A_{2n} = \sum_{j=1}^{n}\left(\sum_{i=1}^{n} T_n^{-2}a_i\tilde{x}_i\sigma_j\int_{A_j}E_m(t_i,s)\,ds\right)e_j =: \sum_{j=1}^{n} d_{nj}e_j.$$

By (A2)(ii), Lemma A.6, (2.6), and $2^m/n = O(n^{-1/2})$ we can obtain

$$\max_{1\le j\le n}|d_{nj}| \le \Bigl(\max_{1\le j\le n}\sigma_j\Bigr)\left(\max_{1\le i,j\le n}\int_{A_j}\bigl|E_m(t_i,s)\bigr|\,ds\right)\left(T_n^{-2}\sum_{i=1}^{n}|a_i\tilde{x}_i|\right) = O\bigl(n^{-1/2}\bigr),$$
$$\sum_{j=1}^{n} d_{nj}^2 \le CT_n^{-2}\sum_{j=1}^{n}\sum_{i=1}^{n}\left(\int_{A_j}E_m(t_i,s)\,ds\right)^2\left(T_n^{-2}\sum_{i=1}^{n}a_i\tilde{x}_i^2\right)$$
$$\le CT_n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\int_{A_j}E_m(t_i,s)\,ds\right)^2 = O\bigl(2^m/n\bigr) = O\bigl(n^{-1/2}\bigr).$$

Therefore $A_{2n} \to 0$ a.s. by Lemma A.1. Clearly, from (2.6) and (2.7) we obtain

$$|A_{3n}| \le \Bigl(\max_{1\le i\le n}|\tilde{g}_i|\Bigr)\cdot\left(T_n^{-2}\sum_{i=1}^{n}|a_i\tilde{x}_i|\right) = O\bigl(2^{-m}+n^{-1}\bigr) \to 0.$$

Step 2. We prove (3.2)(i), as the proof of (3.2)(ii) is analogous. We can see that

$$\max_{1\le i\le n}\bigl|\hat{g}_L(t_i)-g(t_i)\bigr|$$
$$\le \max_{1\le i\le n}\left\{|\beta-\hat{\beta}_L|\cdot\left|\sum_{j=1}^{n}x_j\int_{A_j}E_m(t_i,s)\,ds\right| + |\tilde{g}_i| + \left|\sum_{j=1}^{n}\sigma_je_j\int_{A_j}E_m(t_i,s)\,ds\right|\right\}$$
$$:= B_{1n}+B_{2n}+B_{3n}.$$

Together with (3.1)(i), under the additional assumption, it follows that $B_{1n} \to 0$ a.s. We obtain from (A2)(ii) and Lemma A.6 that $B_{3n} \to 0$ a.s. by applying Lemma A.2. Hence (3.2)(i) is proved by (2.7). □

Proof of Theorem 3.2 *Step 1.* Firstly, we prove (3.3)(i). We have

$$|\hat{\beta}_L-\beta| = S_n^{-2}\left[\sum_{i=1}^{n}\tilde{x}_i\varepsilon_i - \sum_{i=1}^{n}\tilde{x}_i\left(\sum_{j=1}^{n}\varepsilon_j\int_{A_j}E_m(t_i,s)\,ds\right) + \sum_{i=1}^{n}\tilde{x}_i\tilde{g}_i\right]$$
$$:= C_{1n}-C_{2n}+C_{3n}. \tag{4.2}$$

Note that

$$C_{1n} = \sum_{i=1}^{n} \left(S_n^{-2}\tilde{x}_i\sigma_i\right)e_i := \sum_{i=1}^{n} c'_{ni}e_i$$

and

$$C_{2n} = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} S_n^{-2}\tilde{x}_i\sigma_j \int_{A_j} E_m(t_i,s)\,ds\right)e_j := \sum_{j=1}^{n} d'_{nj}e_j.$$

Similar to the proof of (3.1)(ii), we have

$$\max_{1\le i\le n}\left|c'_{ni}\right| \le \left(\max_{1\le i\le n}\frac{|\tilde{x}_i|}{S_n}\right)\cdot\frac{1}{S_n} = O\left(n^{-1/2}\right), \qquad \sum_{i=1}^{n} c'^2_{ni} \le C\sum_{i=1}^{n}\tilde{x}_i^2/S_n^4 = O\left(n^{-1}\right),$$

$$\max_{1\le j\le n}\left|d'_{nj}\right| \le \left(\max_{1\le j\le n}\sigma_j\right)\left(\max_{1\le i,j\le n}\int_{A_j}\left|E_m(t_i,s)\right|ds\right)\left(S_n^{-2}\sum_{i=1}^{n}|\tilde{x}_i|\right) = O\left(2^m/n\right) = O\left(n^{-1/2}\right),$$

$$\begin{aligned}\sum_{j=1}^{n} d'^2_{nj} &\le CS_n^{-2}\sum_{j=1}^{n}\sum_{i=1}^{n}\left(\int_{A_j}E_m(t_i,s)\,ds\right)^2\left(S_n^{-2}\sum_{i=1}^{n}\tilde{x}_i^2\right)\\ &\le CS_n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\int_{A_j}E_m(t_i,s)\,ds\right)^2 = O\left(2^m/n\right) = O\left(n^{-1/2}\right).\end{aligned}$$

Therefore, applying Lemma A.1 and taking $\alpha = 4$, we obtain that $C_{in} = o(n^{-1/4})$ a.s., $i = 1, 2$. As for C_{3n}, by $2^m/n = O(n^{-1/2})$, (2.6), and (2.7) we easily see that

$$|C_{3n}| \le \left(\max_{1\le i\le n}|\tilde{g}_i|\right)\cdot\left(S_n^{-2}\sum_{i=1}^{n}|\tilde{x}_i|\right) = O\left(2^{-m}+n^{-1}\right) = o\left(n^{-1/4}\right).$$

Step 2. We prove (3.3)(ii). Noting that $\hat{f}_n(u) = \sum_{i=1}^{n}[\tilde{x}_i(\beta-\hat{\beta}_L)+\tilde{g}_i+\tilde{\varepsilon}_i]^2\int_{B_i}E_m(u,s)\,ds$, we can see that

$$\begin{aligned}&\max_{1\le j\le n}\left|\hat{f}_n(u_j)-f(u_j)\right|\\ &\quad\le \max_{1\le j\le n}\left|\sum_{i=1}^{n}\varepsilon_i^2\int_{B_i}E_m(u_j,s)\,ds - f(u_j)\right|\\ &\qquad + 2\max_{1\le j\le n}\left|\sum_{i=1}^{n}\varepsilon_i\int_{B_i}E_m(u_j,s)\,ds\left(\sum_{j=1}^{n}\varepsilon_j\int_{A_j}E_m(t_i,s)\,ds\right)\right|\\ &\qquad + \max_{1\le j\le n}\left|\sum_{i=1}^{n}\int_{B_i}E_m(u_j,s)\,ds\left(\sum_{j=1}^{n}\varepsilon_j\int_{A_j}E_m(t_i,s)\,ds\right)^2\right|\\ &\qquad + 2\max_{1\le j\le n}\left|\sum_{i=1}^{n}\tilde{\varepsilon}_i\tilde{g}_i\int_{B_i}E_m(u_j,s)\,ds\right|\\ &\qquad + 2|\beta-\hat{\beta}_L|\max_{1\le j\le n}\left|\sum_{i=1}^{n}\tilde{x}_i\tilde{\varepsilon}_i\int_{B_i}E_m(u_j,s)\,ds\right| + (\beta-\hat{\beta}_L)^2\max_{1\le j\le n}\left|\sum_{i=1}^{n}\tilde{x}_i^2\int_{B_i}E_m(u_j,s)\,ds\right|\end{aligned}$$

$$
\begin{aligned}
&+2|\beta-\hat{\beta}_L|\max_{1\le j\le n}\left|\sum_{i=1}^n \tilde{x}_i\tilde{g}_i\int_{B_i}E_m(u_j,s)\,ds\right|+\max_{1\le j\le n}\left|\sum_{i=1}^n \tilde{g}_i^2\int_{B_i}E_m(u_j,s)\,ds\right|\\
&:=\sum_{i=1}^8 D_{in}.
\end{aligned}
$$

As for D_{1n}, we have

$$
\begin{aligned}
D_{1n}&\le \max_{1\le j\le n}\left|\sum_{i=1}^n f(u_i)\bigl(e_i^2-1\bigr)\int_{B_i}E_m(u_j,s)\,ds\right|\\
&\quad+\max_{1\le j\le n}\left|\sum_{i=1}^n f(u_i)\int_{B_i}E_m(u_j,s)\,ds-f(u_j)\right|\\
&=:D_{11n}+D_{12n}.
\end{aligned}
$$

Note that $Ee_i^2=1$, so $e_i^2-1=[(e_i^+)^2-E(e_i^+)^2]+[(e_i^-)^2-E(e_i^-)^2]:=\xi_{i_1}+\xi_{i_2}$, and

$$
D_{11n}\le \max_{1\le j\le n}\left|\sum_{i=1}^n\left(f(u_i)\int_{B_i}E_m(u_j,s)\,ds\right)\xi_{i_1}\right|+\max_{1\le j\le n}\left|\sum_{i=1}^n\left(f(u_i)\int_{B_i}E_m(u_j,s)\,ds\right)\xi_{i_2}\right|.
$$

Since $\{\xi_{i_1},i\ge 1\}$ and $\{\xi_{i_2},i\ge 1\}$ are NA random variables with zero means, $\sup_i E|\xi_{i_j}|^{p/2}\le C\sup_i E|e_i|^p<\infty$, $j=1,2$. By (A2)(ii) and Lemma A.6 we have

$$
\max_{1\le i,j\le n}\left|f(u_i)\int_{B_i}E_m(u_j,s)\,ds\right|=O\bigl(2^m/n\bigr)=O\bigl(n^{-1/2}\bigr)
$$

and

$$
\max_{1\le j\le n}\sum_{i=1}^n\left|f(u_i)\int_{B_i}E_m(u_j,s)\,ds\right|=O(1).
$$

Therefore $D_{11n}\to 0$ a.s. by Lemma A.2. By (2.7) we have $|D_{12n}|=O(2^{-m}+n^{-1})$, so $D_{1n}\to 0$ a.s.

Note that

$$
|D_{2n}|\le 2\left(\max_{1\le i\le n}\left|\sum_{j=1}^n\sigma_j e_j\int_{A_j}E_m(t_i,s)\,ds\right|\right)\left(\max_{1\le j\le n}\left|\sum_{i=1}^n\sigma_i e_i\int_{B_i}E_m(u_j,s)\,ds\right|\right).
$$

Similar to the proof of D_{11n}, we obtain $D_{2n}\to 0$ a.s. by Lemma A.2.

Applying the assumptions, from (2.6), (2.7), (3.3)(i), Lemma A.2, and Lemma A.6 it follows that

$$
|D_{3n}|\le \max_{1\le i\le n}\left(\sum_{j=1}^n\sigma_j e_j\int_{A_j}E_m(t_i,s)\,ds\right)^2\max_{1\le j\le n}\sum_{i=1}^n\left|\int_{B_i}E_m(u_j,s)\,ds\right|\to 0\quad\text{a.s.},
$$

$$
\begin{aligned}
|D_{4n}| \le{}& 2\Bigl(\max_{1\le i\le n}|\tilde{g}_i|\Bigr)\cdot\left(\max_{1\le j\le n}\left|\sum_{i=1}^{n}\varepsilon_i\int_{B_i}E_m(u_j,s)\,ds\right|\right)+2\Bigl(\max_{1\le i\le n}|\tilde{g}_i|\Bigr)\\
&\cdot\left(\max_{1\le i\le n}\left|\sum_{k=1}^{n}\varepsilon_k\int_{A_k}E_m(t_i,s)\,ds\right|\right)\left(\max_{1\le j\le n}\left|\sum_{i=1}^{n}\int_{B_i}E_m(u_j,s)\,ds\right|\right)\\
\to{}& 0\quad\text{a.s.},\\
|D_{6n}| \le{}& (\beta-\hat{\beta}_L)^2\cdot\left(\max_{1\le i,j\le n}\int_{B_i}\bigl|E_m(u_j,s)\bigr|\,ds\right)\left(\sum_{i=1}^{n}\tilde{x}_i^2\right)=o(1)\quad\text{a.s.},\\
|D_{7n}| \le{}& 2|\beta-\hat{\beta}_L|\cdot\Bigl(\max_{1\le i\le n}|\tilde{g}_i|\Bigr)\cdot\left(\max_{1\le i,j\le n}\int_{B_i}\bigl|E_m(u_j,s)\bigr|\,ds\right)\left(\sum_{i=1}^{n}|\tilde{x}_i|\right)\to 0\quad\text{a.s.},\\
|D_{8n}| \le{}& \Bigl(\max_{1\le i\le n}|\tilde{g}_i|\Bigr)^2\cdot\left(\max_{1\le j\le n}\sum_{i=1}^{n}\left|\int_{B_i}E_m(u_j,s)\,ds\right|\right)\to 0,\\
|D_{5n}| \le{}& 2|\beta-\hat{\beta}_L|\cdot\left(\max_{1\le j\le n}\sqrt{\left(\sum_{i=1}^{n}\tilde{x}_i^2\right)\sum_{i=1}^{n}\tilde{\varepsilon}_i^2\left(\int_{B_i}E_m(u_j,s)\,ds\right)^2}\right).
\end{aligned}
$$

To prove that $D_{5n}\to 0$ a.s., it suffices to show that

$$
\max_{1\le j\le n}\sum_{i=1}^{n}\tilde{\varepsilon}_i^2\left(\int_{B_i}E_m(u_j,s)\,ds\right)^2=O\bigl(n^{-1/2}\bigr)\quad\text{a.s.}\tag{4.3}
$$

As for (4.3), we can split

$$
\begin{aligned}
&\max_{1\le j\le n}\sum_{i=1}^{n}\tilde{\varepsilon}_i^2\left(\int_{B_i}E_m(u_j,s)\,ds\right)^2\\
&\quad\le \max_{1\le j\le n}\sum_{i=1}^{n}\varepsilon_i^2\left(\int_{B_i}E_m(u_j,s)\,ds\right)^2\\
&\qquad+2\max_{1\le j\le n}\left|\sum_{i=1}^{n}\varepsilon_i\left(\int_{B_i}E_m(u_j,s)\,ds\right)^2\left(\sum_{k=1}^{n}\varepsilon_k\int_{A_k}E_m(t_i,s)\,ds\right)\right|\\
&\qquad+\left(\max_{1\le j\le n}\sum_{i=1}^{n}\left(\int_{B_i}E_m(u_j,s)\,ds\right)^2\right)\cdot\left(\sum_{k=1}^{n}\varepsilon_k\int_{A_k}E_m(t_i,s)\,ds\right)^2\\
&\quad:=D_{51n}+D_{52n}+D_{53n}.
\end{aligned}
$$

By Lemmas A.2 and A.6, since $2^m/n=O(n^{-1/2})$, we have

$$
\begin{aligned}
|D_{52n}| \le{}& 2\left(\max_{1\le i\le n}\left|\sum_{k=1}^{n}\varepsilon_k\int_{A_k}E_m(t_i,s)\,ds\right|\right)\cdot\left(\max_{1\le i,j\le n}\int_{B_i}\bigl|E_m(u_j,s)\bigr|\,ds\right)\\
&\cdot\left(\max_{1\le j\le n}\left|\sum_{i=1}^{n}\varepsilon_i\int_{B_i}E_m(u_j,s)\,ds\right|\right)\\
={}& o\bigl(n^{-1/2}\bigr)\quad\text{a.s.},
\end{aligned}
$$

$$
\begin{aligned}
|D_{53n}| &\le \left(\max_{1\le i\le n}\left|\sum_{k=1}^{n}\varepsilon_k\int_{A_k}E_m(t_i,s)\,ds\right|^2\right)\cdot\left(\max_{1\le i,j\le n}\int_{B_i}\left|E_m(u_j,s)\right|ds\right)\\
&\quad\cdot\left(\max_{1\le j\le n}\sum_{i=1}^{n}\left|\int_{B_i}E_m(u_j,s)\,ds\right|\right)\\
&= o\left(n^{-1/2}\right)\quad\text{a.s.}
\end{aligned}
$$

Note that

$$
\begin{aligned}
|D_{51n}| &\le \max_{1\le j\le n}\left|\sum_{i=1}^{n}\sigma_i^2\xi_{i_1}\left(\int_{B_i}E_m(u_j,s)\,ds\right)^2\right| + \max_{1\le j\le n}\left|\sum_{i=1}^{n}\sigma_i^2\xi_{i_2}\left(\int_{B_i}E_m(u_j,s)\,ds\right)^2\right|\\
&\quad+\max_{1\le j\le n}\left|\sum_{i=1}^{n}\sigma_i^2Ee_i^2\left(\int_{B_i}E_m(u_j,s)\,ds\right)^2\right|\\
&:= D'_{51n}+D''_{51n}+D'''_{51n},
\end{aligned}
$$

where $\xi_{i_1}=[(e_i^+)^2-E(e_i^+)^2]$ and $\xi_{i_2}=[(e_i^-)^2-E(e_i^-)^2]$ are NA random variables with zero means. Similar to the proof of D_{11n}, we obtain that $D'_{51n}=o(n^{-1/2})$ a.s. and $D''_{51n}=o(n^{-1/2})$ a.s. by Lemma A.2. On the other hand,

$$
\left|D'''_{51n}\right| \le \left(\max_{1\le j\le n}\sigma_i^2\int_{B_i}\left|E_m(u_j,s)\right|ds\right)\left(\max_{1\le j\le n}\sum_{i=1}^{n}\left|\int_{B_i}E_m(u_j,s)\,ds\right|\right) = O\left(n^{-1/2}\right),
$$

and thus we have proved that $D_{51n}=O(n^{-1/2})$ a.s. This completes the proof of (3.3)(ii).

Step 3. Next, we prove (3.3)(iii) by means of (A2)(ii) and (3.3)(ii). When n is large enough, it easily follows that

$$
0\le m'_0\le\min_{1\le i\le n}\hat{f}_n(u_i)\le\max_{1\le i\le n}\hat{f}_n(u_i)\le M'_0\le\infty, \tag{4.4}
$$

$C_5\le W_n^2/n\le C_6$, and $W_n^{-2}\sum_{i=1}^{n}|a_{ni}\tilde{x}_i|\le C$. Hence we have

$$
|\tilde{\beta}_n-\beta|\le\frac{n}{W_n^2}\left|\frac{1}{n}\sum_{i=1}^{n}a_{ni}\tilde{x}_i\tilde{\varepsilon}_i\right|+\frac{1}{W_n^2}\left|\sum_{i=1}^{n}a_{ni}\tilde{x}_i\tilde{g}_i\right| =: \frac{n}{W_n^2}E_{1n}+E_{2n}. \tag{4.5}
$$

Together with (2.7) and (4.5), we get

$$
\begin{aligned}
|E_{2n}| &\le \left(\max_{1\le i\le n}|\tilde{g}_i|\right)\cdot\left(W_n^{-2}\sum_{i=1}^{n}|a_{ni}\tilde{x}_i|\right)\to 0,\\
|E_{1n}| &\le \left|\frac{1}{n}\sum_{i=1}^{n}(a_{ni}-a_i)\tilde{x}_i\tilde{\varepsilon}_i\right|+\left|\frac{1}{n}\sum_{i=1}^{n}a_i\tilde{x}_i\tilde{\varepsilon}_i\right|\\
&\le \left|\frac{1}{n}\sum_{i=1}^{n}(a_{ni}-a_i)\tilde{x}_i\varepsilon_i\right|+\left|\frac{1}{n}\sum_{i=1}^{n}(a_{ni}-a_i)\tilde{x}_i\left(\sum_{j=1}^{n}\varepsilon_j\int_{A_j}E_m(t_i,s)\,ds\right)\right|\\
&\quad+\left|\frac{1}{n}\sum_{i=1}^{n}a_i\tilde{x}_i\tilde{\varepsilon}_i\right|\\
&=: E_{11n}+E_{12n}+E_{13n}.
\end{aligned}
$$

We know from Lemma A.2 that

$$\begin{aligned} n^{-1}\sum_{i=1}^{n} e_i^2 &= n^{-1}\sum_{i=1}^{n}\bigl[\bigl(e_i^+\bigr)^2 - E\bigl(e_i^+\bigr)^2\bigr] + n^{-1}\sum_{i=1}^{n}\bigl[\bigl(e_i^-\bigr)^2 - E\bigl(e_i^-\bigr)^2\bigr] + n^{-1}\sum_{i=1}^{n} Ee_i^2 \\ &= O(1) \quad \text{a.s.} \end{aligned} \tag{4.6}$$

Applying Lemma A.2 and combining (4.4)–(4.6) with (3.3)(ii), we obtain

$$\begin{aligned} |E_{11n}| &\le \left(\max_{1\le i\le n}\frac{|\hat{f}_n(u_i)-f(u_i)|}{\hat{f}_n(u_i)f(u_i)}\right)\sqrt{\left(\frac{1}{n}\sum_{i=1}^{n}\tilde{x}_i^2\right)\left(\frac{1}{n}\sum_{i=1}^{n}e_i^2\right)} \to 0 \quad \text{a.s.,} \\ |E_{12n}| &\le \left(\max_{1\le i\le n}\frac{|\hat{f}_n(u_i)-f(u_i)|}{\hat{f}_n(u_i)f(u_i)}\right)\cdot\left(\frac{1}{n}\sum_{i=1}^{n}|\tilde{x}_i|\right)\cdot\left(\max_{1\le i\le n}\left|\sum_{j=1}^{n}\varepsilon_j\int_{A_j}E_m(t_i,s)\,ds\right|\right) \\ &\to 0 \quad \text{a.s.} \end{aligned}$$

As for E_{13n}, we have $E_{13n} = \frac{T_n^2}{n}(|A_{1n}+A_{2n}|) \to 0$ a.s., and therefore $E_{1n} \to 0$ a.s. □

The proof of (3.4) is similar to that of (3.2)(i), and hence we omit it.

Proof of Theorem 3.3 We prove only (3.5)(i), as the proof of (3.5)(ii) is analogous. From the definition of $\hat{\beta}_L$ we have

$$\begin{aligned} S_n^2(\hat{\beta}_L-\beta) &= \sum_{i=1}^{n}\tilde{x}_i\sigma_i e_i - \sum_{i=1}^{n}\tilde{x}_i\left(\sum_{j=1}^{n}\sigma_j e_j\int_{A_j}E_m(t_i,s)\,ds\right) + \sum_{i=1}^{n}\tilde{x}_i\tilde{g}_i \\ &:= L_{1n} - L_{2n} + L_{3n}. \end{aligned} \tag{4.7}$$

Setting $Z_{ni} = \frac{\tilde{x}_i\sigma_i e_i}{\sigma_{1n}}$, we employ Bernstein's big-block and small-block procedure. Let $y_{nm} = \sum_{i=k_m}^{k_m+p-1} Z_{ni}$, $y'_{nm} = \sum_{i=l_m}^{l_m+q-1} Z_{ni}$, $y'_{nk+1} = \sum_{i=k(p+q)+1}^{n} Z_{ni}$, $k_m = (m-1)(p+q)+1$, $l_m = (m-1)(p+q)+p+1$, $m = 1,2,\ldots,k$. Then

$$\sigma_{1n}^{-1}L_{1n} := \tilde{L}_{1n} = \sum_{i=1}^{n} Z_{ni} = \sum_{m=1}^{k} y_{nm} + \sum_{m=1}^{k} y'_{nm} + y'_{nk+1} =: L_{11n} + L_{12n} + L_{13n}.$$

We observe that

$$\begin{aligned} L_{2n} &= \sum_{i=1}^{n}\left[\tilde{h}_i + v_i - \left(\sum_{k=1}^{n} v_k\int_{A_k}E_m(t_i,s)\,ds\right)\right]\cdot\left(\sum_{j=1}^{n}\sigma_j e_j\int_{A_j}E_m(t_i,s)\,ds\right) \\ &= \sum_{i=1}^{n}\tilde{h}_i\left(\sum_{j=1}^{n}\sigma_j e_j\int_{A_j}E_m(t_i,s)\,ds\right) + \sum_{i=1}^{n} v_i\left(\sum_{j=1}^{n}\sigma_j e_j\int_{A_j}E_m(t_i,s)\,ds\right) \\ &\quad - \sum_{i=1}^{n}\left(\sum_{k=1}^{n} v_k\int_{A_k}E_m(t_i,s)\,ds\right)\left(\sum_{j=1}^{n}\sigma_j e_j\int_{A_j}E_m(t_i,s)\,ds\right) \\ &:= L_{21n} + L_{22n} - L_{23n}. \end{aligned}$$

So we can write

$$\frac{S_n^2(\hat{\beta}_L-\beta)}{\sigma_{1n}}=L_{11n}+L_{12n}+L_{13n}+\sigma_{1n}^{-1}(L_{21n}+L_{22n}-L_{23n}+L_{3n}).$$

By applying Lemma A.4 we have

$$\begin{aligned}
&\sup_y\left|P\left(\frac{S_n^2(\hat{\beta}_L-\beta)}{\sigma_{1n}}\le y\right)-\Phi(y)\right| \\
&\quad\le \sup_y\bigl|P(L_{11n}\le y)-\Phi(y)\bigr|+P\bigl(|L_{12n}|>\lambda_{1n}^{1/3}\bigr)+P\bigl(|L_{13n}|>\lambda_{2n}^{1/3}\bigr) \\
&\qquad+P\bigl(\sigma_{1n}^{-1}|L_{21n}|>\lambda_{3n}^{1/3}\bigr)+P\bigl(\sigma_{1n}^{-1}|L_{22n}|>\lambda_{4n}^{1/3}\bigr)+P\bigl(\sigma_{1n}^{-1}|L_{23n}|>\lambda_{4n}^{1/3}\bigr) \\
&\qquad+\frac{1}{\sqrt{2\pi}}\left(\sum_{k=1}^{3}\lambda_{kn}^{1/3}+2\lambda_{4n}^{1/3}\right)+\frac{1}{\sqrt{2\pi}}\sigma_{1n}^{-1}|L_{3n}| \\
&\quad=\sum_{k=1}^{8}I_{kn}.
\end{aligned}\tag{4.8}$$

Therefore, to prove (3.5)(i), it suffices to show that $\sum_{k=2}^{8}I_{kn}=O(\mu_{1n})$ and $I_{1n}=O(\upsilon_{1n}+\lambda_{1n}^{1/2}+\lambda_{2n}^{1/2})$. Here we need to the following Abel inequality (see Härdle et al. [5]). Let $A_1,\ldots,A_n$ and $B_1,\ldots,B_n$ ($B_1\ge B_2\ge\cdots\ge B_n\ge 0$) be two sequences of real numbers, and let $S_k=\sum_{i=1}^{k}A_i$, $M_1=\min_{1\le k\le n}S_k$, and $M_2=\max_{1\le k\le n}S_k$. Then

$$B_1M_1\le\sum_{k=1}^{n}A_kB_k\le B_1M_2.\tag{4.9}$$

Note that

$$\sigma_{1n}^2=E\left(\sum_{i=1}^{n}\tilde{x}_i\sigma_ie_i\right)^2=\int_{-\pi}^{\pi}\psi(\omega)\left|\sum_{k=1}^{n}\tilde{x}_k\sigma_ke^{-ik\omega}\right|^2d\omega$$

and

$$\Gamma_n^2(t)=E\left(\sum_{i=1}^{n}\sigma_ie_i\int_{A_i}E_m(t,s)\,ds\right)^2=\int_{-\pi}^{\pi}\psi(\omega)\left|\sum_{k=1}^{n}\sigma_k\int_{A_k}E_m(t,s)\,dse^{-ik\omega}\right|^2d\omega.$$

By (A2)(ii), (A5), (2.6), and Lemma A.6 it follows that

$$C_7n\le C_1\sum_{i=1}^{n}\tilde{x}_i^2\le\sigma_{1n}^2\le C_2\sum_{i=1}^{n}\tilde{x}_i^2\le C_8n,\tag{4.10}$$

$$C_9\sum_{i=1}^{n}\left(\int_{A_i}E_m(t,s)\,ds\right)^2\le\Gamma_n^2(t)\le C_{10}\sum_{i=1}^{n}\left(\int_{A_i}E_m(t,s)\,ds\right)^2=O\bigl(2^m/n\bigr).\tag{4.11}$$

Step 1. We first prove $\sum_{k=2}^{8} I_{kn} = O(\mu_{1n})$. Using Lemma A.3, (2.7), and (2.8), from (A0)(i), (A1)–(A6), (4.10), and (4.11) it follows that

$$I_{2n} \le \frac{E|L_{12n}|^2}{\lambda_{1n}^{2/3}} \le \frac{C}{n\lambda_{1n}^{2/3}} \sum_{m=1}^{k} \sum_{i=l_m}^{l_m+q-1} \tilde{x}_i^2 \sigma_i^2 \le \frac{Ckq}{n\lambda_{1n}^{2/3}} \le C\lambda_{1n}^{1/3},$$

$$I_{3n} \le \frac{E|L_{13n}|^2}{\lambda_{2n}^{2/3}} \le \frac{C}{n\lambda_{2n}^{2/3}} \sum_{i=k(p+q)+1}^{n} \tilde{x}_i^2 \sigma_i^2 \le \frac{Cp}{n\lambda_{2n}^{2/3}} \le C\lambda_{2n}^{1/3},$$

$$\begin{aligned}
I_{4n} &\le \frac{\sigma_{1n}^{-2} E|L_{21n}|^2}{\lambda_{3n}^{2/3}} \le \frac{C}{n\lambda_{3n}^{2/3}} \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \tilde{h}_i \int_{A_j} E_m(t_i,s)\,ds \right)^2 \sigma_j^2 \\
&\le \frac{C}{n\lambda_{3n}^{2/3}} \left(\max_{1\le i\le n} |\tilde{h}_i|^2 \right) \cdot \sum_{j=1}^{n} \left(\max_{1\le j\le n} \sum_{i=1}^{n} \int_{A_j} \left|E_m(t_i,s)\right| ds \right)^2 \\
&\le \frac{C(2^{-m}+n^{-1})^2}{\lambda_{3n}^{2/3}} \le C\lambda_{3n}^{1/3},
\end{aligned}$$

$$\begin{aligned}
I_{5n} &\le \frac{\sigma_{1n}^{-2} E|L_{22n}|^2}{\lambda_{4n}^{2/3}} \le \frac{C}{n\lambda_{4n}^{2/3}} \sum_{j=1}^{n} \left(\sum_{i=1}^{n} v_i \int_{A_j} E_m(t_i,s)\,ds \right)^2 \sigma_j^2 \\
&\le \frac{C}{n\lambda_{4n}^{2/3}} \max_{1\le i,j\le n} \left(\int_{A_j} \left|E_m(t_i,s)\right| ds \right) \cdot \max_{1\le i\le n} \sum_{j=1}^{n} \left| \int_{A_j} \left|E_m(t_i,s)\right| ds \right| \cdot \left(\max_{1\le m\le n} \left| \sum_{i=1}^{m} v_{j_i} \right| \right)^2 \\
&\le \frac{C(2^m n^{-1} \log^2 n)}{\lambda_{4n}^{2/3}} \le C\lambda_{4n}^{1/3},
\end{aligned}$$

$$\begin{aligned}
I_{6n} &\le \frac{\sigma_{1n}^{-2} E|L_{23n}|^2}{\lambda_{4n}^{2/3}} \le \frac{C}{n\lambda_{4n}^{2/3}} \sum_{j=1}^{n} \left(\sum_{l=1}^{n} v_l \int_{A_l} E_m(t_i,s)\,ds \cdot \sum_{i=1}^{n} \int_{A_j} E_m(t_i,s)\,ds \right)^2 \sigma_j^2 \\
&\le \frac{C}{n\lambda_{4n}^{2/3}} \max_{1\le i,j\le n} \left(\int_{A_j} \left|E_m(t_i,s)\right| ds \right) \cdot \max_{1\le i\le n} \sum_{j=1}^{n} \left| \int_{A_j} E_m(t_i,s)\,ds \right| \\
&\quad \cdot \left(\max_{1\le l\le n} \sum_{i=1}^{n} \left| \int_{A_l} E_m(t_i,s)\,ds \right| \max_{1\le m\le n} \left| \sum_{i=1}^{m} v_{j_i} \right| \right)^2 \\
&\le \frac{C(2^m n^{-1} \log^2 n)}{\lambda_{4n}^{2/3}} \le C\lambda_{4n}^{1/3}.
\end{aligned}$$

As for I_{8n}, we have

$$\begin{aligned}
I_{8n} &= \sigma_{1n}^{-1} \left| \sum_{i=1}^{n} \tilde{x}_i \tilde{g}_i \right| \le \frac{C}{\sqrt{n}} \left(\left| \sum_{i=1}^{n} v_i \tilde{g}_i \right| + \left| \sum_{i=1}^{n} \tilde{h}_i \tilde{g}_i \right| + \left| \sum_{i=1}^{n} \tilde{g}_i \sum_{j=1}^{n} v_j \int_{A_j} E_m(t_i,s)\,ds \right| \right) \\
&\le \frac{C}{\sqrt{n}} \max_{1\le i\le n} |\tilde{g}_i| \max_{1\le m\le n} \left| \sum_{i=1}^{m} v_{j_i} \right| + \frac{Cn}{\sqrt{n}} \max_{1\le i\le n} |\tilde{h}_i| \max_{1\le i\le n} |\tilde{g}_i| \\
&\quad + \frac{C}{\sqrt{n}} \max_{1\le i\le n} |\tilde{g}_i| \cdot \max_{1\le j\le n} \sum_{i=1}^{n} \int_{A_j} \left|E_m(t_i,s)\right| ds \cdot \max_{1\le m\le n} \left| \sum_{i=1}^{m} v_{j_i} \right| \\
&\le C\bigl(2^{-m}+n^{-1}\bigr)\bigl(\log n + \sqrt{n}\bigl(2^{-m}+n^{-1}\bigr)\bigr) = C\lambda_{5n}.
\end{aligned}$$

Hence from the previous estimates we obtain that $\sum_{k=2}^{8} I_{kn} = O(\mu_{1n})$.

Step 2. We verify $I_{1n} = O(\lambda_{1n}^{1/2} + \lambda_{2n}^{1/2} + \upsilon_{1n})$. Let $\{\eta_{nm} : m = 1, 2, \ldots, k\}$ be independent random variables with the same distributions as y_{nm}, $m = 1, 2, \ldots, k$. Set $H_n = \sum_{m=1}^{k} \eta_{nm}$ and $s_n^2 = \sum_{m=1}^{k} \operatorname{Var}(y_{nm})$. Following the method of the proof of Theorem 2.1 in Liang and Li [27] and Li et al. [20], we easily see that

$$\begin{aligned} I_{1n} &= \sup_y \bigl|P(L_{11n} \le y) - \Phi(y)\bigr| \\ &\le \sup_y \bigl|P(L_{11n} \le y) - P(H_n \le y)\bigr| \\ &\quad + \sup_y \bigl|P(H_n \le y) - \Phi(y/s_n)\bigr| + \sup_y \bigl|\Phi(y/s_n) - \Phi(y)\bigr| \\ &:= I_{11n} + I_{12n} + I_{13n}. \end{aligned} \tag{4.12}$$

(i) We evaluate s_n^2. Noticing that $s_n^2 = EL_{11n}^2 - 2\sum_{1\le i<j\le k} \operatorname{Cov}(y_{ni}, y_{nj})$ and $E\tilde{L}_{1n}^2 = 1$, we can get

$$\bigl|E(L_{11n})^2 - 1\bigr| \le C\bigl(\lambda_{1n}^{1/2} + \lambda_{2n}^{1/2}\bigr). \tag{4.13}$$

On the other hand, from (A1), (A2), (4.10), and (2.8) it follows that

$$\begin{aligned} \biggl|\sum_{1\le i<j\le k} \operatorname{Cov}(y_{ni}, y_{nj})\biggr| &\le Cn^{-1} \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \sum_{s=k_i}^{k_i+p-1} \sum_{t=k_j}^{k_j+p-1} |\tilde{x}_s \tilde{x}_t \sigma_s \sigma_t| \cdot \bigl|\operatorname{Cov}(e_s, e_t)\bigr| \\ &\le Ckpn^{-1} u(q) \le Cu(q). \end{aligned} \tag{4.14}$$

Thus, from (4.13) and (4.14) it follows that $|s_n^2 - 1| \le C(\lambda_{1n}^{1/2} + \lambda_{2n}^{1/2} + u(q))$.

(ii) Applying the Berry–Esséen inequality (see Petrov [28], Theorem 5.7), for $\delta > 0$, we get

$$\sup_y \bigl|P(H_n/s_n \le y) - \Phi(y)\bigr| \le C \sum_{m=1}^{k} \bigl(E|y_{nm}|^{2+\delta}/s_n^{2+\delta}\bigr). \tag{4.15}$$

By Lemma A.3 from (A0), (A1), (A2), (4.10), and (2.8) we can deduce that

$$\begin{aligned} \sum_{m=1}^{k} E|y_{nm}|^{2+\delta} &= \sum_{m=1}^{k} E\left|\sum_{j=k_m}^{k_m+p-1} Z_{ni}\right|^{2+\delta} \\ &\le C\sigma_{1n}^{-(2+\delta)} \sum_{m=1}^{k} \left\{ \sum_{i=k_m}^{k_m+p-1} E|\tilde{x}_i \sigma_i e_i|^{2+\delta} + \left[\sum_{i=k_m}^{k_m+p-1} E(\tilde{x}_i \sigma_i e_i)^2\right]^{1+\delta/2} \right\} \\ &\le C\bigl(kpn^{-1}\bigr)\bigl(n^{-\delta/2} + (p/n)^{\delta/2}\bigr) \le C\lambda_{2n}^{\delta/2}. \end{aligned} \tag{4.16}$$

Since $s_n \to 1$ by (4.13) and (4.14), from (4.15) and (4.16) we easily see that $I_{12n} \le C\lambda_{2n}^{\delta/2}$. Note that $I_{13n} = O(|s_n^2 - 1|) = O(\lambda_{1n}^{1/2} + \lambda_{2n}^{1/2} + u(q))$.

(iii) Next, we evaluate I_{11n}. Let $\varphi_1(t)$ and $\varphi_2(t)$ are the characteristic functions of L_{11n} and H_n, respectively. Thus applying the Esséen inequality (see Petrov [28], Theorem 5.3), for any $T > 0$, we have

$$\begin{aligned}
&\sup_t \bigl|P(L_{11n} \le t) - P(H_n \le t)\bigr| \\
&\quad \le \int_{-T}^{T} \left|\frac{\varphi_1(t) - \varphi_2(t)}{t}\right| dt + T \sup_t \int_{|u|\le C/T} \bigl|P(H_n \le u + t) - P(H_n \le t)\bigr|\, du \\
&\quad := I'_{11n} + I''_{11n}.
\end{aligned} \tag{4.17}$$

From Lemma A.5 and (4.14) it follows that

$$\begin{aligned}
\bigl|\varphi_1(t) - \varphi_2(t)\bigr| &= \left|E\exp\left(it\sum_{m=1}^{k} y_{nm}\right) - \prod_{m=1}^{k} E\exp(ity_{nm})\right| \\
&\le 4t^2 \sum_{1\le i<j\le k} \sum_{s_1=k_i}^{k_i+p-1} \sum_{t_1=k_j}^{k_j+p-1} \bigl|\operatorname{Cov}(Z_{ns_1}, Z_{nt_1})\bigr| \\
&\le 4Ct^2 u(q),
\end{aligned}$$

which implies that

$$I'_{11n} = \int_{-T}^{T} \left|\frac{\varphi_1(t) - \varphi_2(t)}{t}\right| dt \le Cu(q)T^2. \tag{4.18}$$

Therefore by (4.15) and (4.16) we have

$$\begin{aligned}
&\sup_t \bigl|P(H_n \le t + u) - P(H_n \le t)\bigr| \\
&\quad \le \sup_t \left|P\left(\frac{H_n}{s_n} \le \frac{t+u}{s_n}\right) - \Phi\left(\frac{t+u}{s_n}\right)\right| \\
&\qquad + \sup_t \left|P\left(\frac{H_n}{s_n} \le \frac{t}{s_n}\right) - \Phi\left(\frac{t}{s_n}\right)\right| + \sup_t \left|\Phi\left(\frac{t+u}{s_n}\right) - \Phi\left(\frac{t}{s_n}\right)\right| \\
&\quad \le 2\sup_t \left|P\left(\frac{H_n}{s_n} \le t\right) - \Phi(t)\right| + \sup_t \left|\Phi\left(\frac{t+u}{s_n}\right) - \Phi\left(\frac{t}{s_n}\right)\right| \\
&\quad \le C\left(\lambda_{2n}^{\delta/2} + \left|\frac{u}{s_n}\right|\right) \le C\bigl(\lambda_{2n}^{\delta/2} + |u|\bigr).
\end{aligned} \tag{4.19}$$

From (4.19) it follows that

$$I''_{11n} = T\sup_t \int_{|u|\le C/T} \bigl|P(H_n \le t + u) - P(H_n \le t)\bigr|\, du \le C\bigl(\lambda_{2n}^{\delta/2} + 1/T\bigr). \tag{4.20}$$

Combining (4.17), (4.18) with (4.20) and choosing $T = u^{-1/3}(q)$, we easily see that $I_{11n} \le C(u^{1/3}(q) + \lambda_{2n}^{\delta/2})$. So $I_{1n} \le C(\lambda_{1n}^{1/2} + \lambda_{2n}^{1/2} + \upsilon_{1n})$. This completes the proof of Theorem 3.3 from Step 1 and Step 2. □

Proof of Corollary 3.1 In Theorem 3.3, choosing $p = \lfloor n^{\theta}\rfloor$, $q = \lfloor n^{2\theta-1}\rfloor$, $\delta = 1$, when $1/2 < \theta \le 7/10$, we have $\mu_{1n} = O(n^{-(\theta-1)/3})$ and $\upsilon_{1n} = O(n^{-(\theta-1)/3})$. Therefore (3.6) directly follows from Theorem 3.3. □

Proof of Theorem 3.4 We prove only the case of $\hat{g}(t) = \hat{g}_L(t)$, as the proof of $\hat{g}(t) = \hat{g}_W(t)$ is analogous.

By the definition of $\hat{g}_L(t)$ we easily see that

$$\begin{aligned}\Gamma_n^{-1}(t)\big(\hat{g}_L(t) - E\hat{g}_L(t)\big) &= \Gamma_n^{-1}(t)\left(\sum_{i=1}^n \varepsilon_i \int_{A_i} E_m(t,s)\,ds\right) \\ &\quad + \Gamma_n^{-1}(t)\left(\sum_{i=1}^n x_i(E\hat{\beta}_L - \beta)\int_{A_i} E_m(t,s)\,ds\right) \\ &\quad + \Gamma_n^{-1}(t)\left(\sum_{i=1}^n x_i(\beta - \hat{\beta}_L)\int_{A_i} E_m(t,s)\,ds\right) \\ &:= J_{1n} + J_{2n} + J_{3n}.\end{aligned}$$

Set $Z'_{ni} = \frac{\sigma_i e_i \int_{A_i} E_m(t,s)\,ds}{\Gamma_n(t)}$. Similar to $\tilde{L}_{1n}$, we can split J_{1n} as $J_{1n} = \sum_{i=1}^n Z'_{ni} := J_{11n} + J_{12n} + J_{13n}$, where $J_{11n} = \sum_{m=1}^k \chi_{nm}$, $J_{12n} = \sum_{m=1}^k \chi'_{nm}$, $J_{13n} = \chi'_{nk+1}$, $\chi_{nm} = \sum_{i=k_m}^{k_m+p-1} Z'_{ni}$, $\chi'_{nm} = \sum_{i=l_m}^{l_m+q-1} Z'_{ni}$, $\chi'_{nk+1} = \sum_{i=k(p+q)+1}^{n} Z'_{ni}$, $k_m = (m-1)(p+q)+1$, $l_m = (m-1)(p+q)+p+1$, $m = 1,2,\ldots,k$.

Applying Lemma A.4, we have

$$\begin{aligned}&\sup_y\left|P\left(\frac{\hat{g}_L(t) - E\hat{g}_L(t)}{\Gamma_n(t)} \le y\right) - \Phi(y)\right| \\ &\quad\le \sup_y\big|P(J_{11n} \le y) - \Phi(y)\big| + P\big(|J_{12n}| > \gamma_{1n}^{1/3}\big) + P\big(|J_{13n}| > \gamma_{2n}^{1/3}\big) \\ &\quad\quad + \frac{|J_{2n}|}{\sqrt{2\pi}} + P\big(|J_{3n}| > \gamma_{3n}^{(2+\delta)/(3+\delta)}\big) + \frac{1}{\sqrt{2\pi}}\left(\sum_{k=1}^2 \gamma_{kn}^{1/3} + \gamma_{3n}^{\frac{2+\delta}{3+\delta}}\right) \\ &\quad= \sum_{k=1}^6 G_{kn}.\end{aligned} \tag{4.21}$$

Hence it suffices to show that $\sum_{k=2}^6 G_{kn} = O(\mu_{2n})$ and $G_{1n} = O(\gamma_{1n}^{1/2} + \gamma_{2n}^{1/2} + \upsilon_{2n})$.

Step 1. We first prove $\sum_{k=2}^6 G_{kn} = O(\mu_{2n})$. Similar to the proof for $I_{2n} - I_{8n}$ in Theorem 3.3, we have

$$\begin{aligned}G_{2n} &\le \frac{E|J_{12n}|^2}{\gamma_{1n}^{2/3}} \le \frac{C}{\Gamma_n^2(t)\gamma_{1n}^{2/3}}\sum_{m=1}^k\sum_{i=l_m}^{l_m+q-1}\left(\int_{A_i} E_m(t,s)\,ds\right)^2\sigma_i^2 \le \frac{Ckq2^m}{n\gamma_{1n}^{2/3}} \le C\gamma_{1n}^{1/3}, \\ G_{3n} &\le \frac{E|J_{13n}|^2}{\gamma_{2n}^{2/3}} \le \frac{C}{\Gamma_n^2(t)\gamma_{2n}^{2/3}}\sum_{i=k(p+q)+1}^{n}\left(\int_{A_i} E_m(t,s)\,ds\right)^2\sigma_i^2 \le \frac{C2^m p}{n\gamma_{2n}^{2/3}} \le C\gamma_{2n}^{1/3}.\end{aligned}$$

Note that if $\xi_n \to \xi \sim N(0,1)$, then $E|\xi_n| \to E|\xi| = \sqrt{2/\pi}$ and $E|\xi_n|^{2+\delta} \to E|\xi|^{2+\delta}$. By Theorem 3.3(i) and (2.6) it follows that

$$|\beta - E\hat{\beta}_L| \le E|\beta - \hat{\beta}_L| = O\big(\sigma_{1n}/S_n^2\big) = O\big(n^{-1/2}\big), \tag{4.22}$$

$$E|\beta - \hat{\beta}_L|^{2+\delta} \le O\big(\big(\sigma_{1n}/S_n^2\big)^{2+\delta}\big) = O\big(n^{-(1+\delta/2)}\big). \tag{4.23}$$

Therefore, applying the Abel inequality (4.9) and combining (A1)(iii) and (A2)(i) with Lemma A.6, from (4.22) and (4.23) we get

$$\begin{aligned}|G_{4n}| &= \frac{1}{\sqrt{2\pi}\,\Gamma_n(t)}\cdot|\beta - E\hat{\beta}_L|\cdot\left|\sum_{i=1}^{n} x_i\int_{A_i}E_m(t,s)\,ds\right| \\ &\le C\Gamma_n^{-1}(t)n^{-1/2}\left(\sup_{0\le t\le 1}\bigl|h(t)\bigr| + \max_{1\le i\le n}\int_{A_i}\bigl|E_m(t,s)\bigr|\,ds\cdot\max_{1\le l\le n}\left|\sum_{i=1}^{l}\nu_{j_i}\right|\right) \\ &\le C\bigl(2^{-m/2}+\sqrt{2^m/n}\log n\bigr) = C\gamma_{3n} \end{aligned} \tag{4.24}$$

and

$$\begin{aligned}|J_{3n}|^{2+\delta} &= \Gamma_n^{-(2+\delta)}(t)E|\beta - \hat{\beta}_L|^{2+\delta}\left|\sum_{i=1}^{n} x_i\int_{A_i}E_m(t,s)\,ds\right|^{2+\delta} \\ &\le C\Gamma_n^{-(2+\delta)}(t)n^{-(2+\delta)/2}\left(\sup_{0\le t\le 1}\bigl|h(t)\bigr| + \max_{1\le i\le n}\int_{A_i}\bigl|E_m(t,s)\bigr|\,ds\cdot\max_{1\le l\le n}\left|\sum_{i=1}^{l}\nu_{j_i}\right|\right)^{2+\delta} \\ &\le C\gamma_{3n}^{2+\delta}, \end{aligned} \tag{4.25}$$

which implies that $G_{5n}\le c\gamma_{3n}^{(2+\delta)/(3+\delta)}$. So we get $\sum_{k=2}^{6}G_{kn} = O(\mu_{2n})$.

Step 2. We verify $G_{1n} = O(\gamma_{1n}^{1/2}+\gamma_{2n}^{1/2}+\upsilon_{2n})$. Let $\{\zeta_{nm}: m = 1,2,\ldots,k\}$ be independent random variables and ζ_{nm} have the same distribution as χ_{nm}, $m = 1,2,\ldots,k$. Set $T_n = \sum_{m=1}^{k}\zeta_{nm}$ and $t_n^2 = \sum_{m=1}^{k}\operatorname{Var}(\chi_{nm})$. Similar to the proof of (4.17), we easily see that

$$\begin{aligned}G_{1n} &= \sup_y\bigl|P(J_{11n}\le y) - \Phi(y)\bigr| \\ &\le \sup_y\bigl|P(J_{11n}\le y) - P(T_n\le y)\bigr| \\ &\quad + \sup_y\bigl|P(T_n\le y) - \Phi(y/t_n)\bigr| + \sup_y\bigl|\Phi(y/t_n) - \Phi(y)\bigr| \\ &= G_{11n}+G_{12n}+G_{13n}. \end{aligned} \tag{4.26}$$

Similar to the proof of (4.13)–(4.20), we can obtain $|t_n^2-1|\le C(\gamma_{1n}^{1/2}+\gamma_{2n}^{1/2}+u(q))$, $|G_{12n}|\le C\gamma_{2n}^{\delta/2}$, $|G_{13n}|\le C(\gamma_{1n}^{1/2}+\gamma_{2n}^{1/2}+u(q))$, and $|G_{11n}|\le C\upsilon_{2n}$. Thus it follows that $G_{1n} = O(\gamma_{1n}^{1/2}+\gamma_{2n}^{1/2}+\upsilon_{2n})$. The proof of Theorem 3.4 is completed. □

Proof of Corollary 3.2 Letting $p = \lfloor n^\rho\rfloor$, $q = \lfloor n^{2\rho-1}\rfloor$, $\delta = 1$, when $1/2<\rho<\theta<1$, we have $\gamma_{1n}^{1/3} = O(n^{-(\theta-\rho)/3})$, $\gamma_{2n}^{1/3} = O(n^{-(\theta-\rho)/3})$, $\gamma_{3n}^{3/4} = O(n^{-3(1-\theta)/8})$, and $u^{1/3}(q) = O(n^{-(\theta-\rho)/3})$. Therefore (3.8) directly follows from Theorem 3.4. □

Appendix

Lemma A.1 (Back and Liang [10]) *Let $\{X_n, n\ge 1\}$ be a sequence of NA random variables with zero means and $\alpha>2$. Assume that $\{a_{ni}, 1\le i\le n, n\ge 1\}$ is a triangular array of real numbers with $\max_{1\le i\le n}|a_{ni}| = O(n^{-1/2})$ and $\sum_{i=1}^{n}a_{ni}^2 = o(n^{-2/\alpha}(\log n)^{-1})$. If $\sup_i \mathrm{E}|X_i|^p<\infty$ for some $p>2\alpha/(\alpha-2)$, then $\sum_{i=1}^{n}a_{ni}X_i = o(n^{-1/\alpha})$ a.s.*

Lemma A.2 (Back and Liang [10]) *Let $\{X_n, n \geq 1\}$ be a sequence of NA random variables with zero means. Assume that $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ is a function array defined on the closed interval I of R satisfying $\max_{1\leq i,j\leq n} |a_{ni}(u_j)| = O(n^{-1/2})$ and $\max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ni}(u_j)| = O(1)$. If $\sup_i \mathrm{E}|X_i|^p < \infty$ for some $p > 2$, then*

$$\text{(i)}\quad \max_{1\leq j\leq n}\left|\sum_{i=1}^{n} a_{ni}(u_j)X_i\right| = o\bigl(L(n)\bigr)\quad a.s.,$$

$$\text{(ii)}\quad \max_{1\leq j\leq n}\left(\sum_{i=1}^{n}\bigl|a_{ni}(u_j)X_i\bigr|\right) = O(1)\quad a.s.,$$

where $L(x) > 0$ is a slowly varying function as $x \to \infty$, and $\sqrt{x}L(x)$ is nondecreasing for $x \geq x_0 > 0$.

Lemma A.3 (Liang and Li [27]) *Let $\{X_n; n \geq 1\}$ be a sequence of NA random variables with zero means and $\mathrm{E}|X_n|^p < \infty$ for some $p > 1$, and let $\{b_i, i \geq 1\}$ be a sequence of real numbers. Then there exists a positive constant C_p such that*

$$\mathrm{E}\max_{1\leq m\leq n}\left|\sum_{i=1}^{m} b_iX_i\right|^p \leq C_p\left\{\sum_{i=1}^{n}\mathrm{E}|b_iX_i|^p + I(p>2)\left(\sum_{i=1}^{n}\mathrm{E}(b_iX_i)^2\right)^{p/2}\right\}.$$

Lemma A.4 (Yang [29]) *Suppose that $\{\varsigma_n, n \geq 1\}$, $\{\eta_n, n \geq 1\}$, and $\{\xi_n, n \geq 1\}$ are three random variable sequences, $\{\gamma_n, n \geq 1\}$ is a positive nonrandom sequence, and $\gamma_n \to 0$. If $\sup_x |F_{\varsigma_n}(x) - \Phi(x)| \leq C\gamma_n$, then for any $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$,*

$$\sup_x\bigl|F_{\varsigma_n+\eta_n+\xi_n}(x) - \Phi(x)\bigr| \leq C\bigl\{\gamma_n + \varepsilon_1 + \varepsilon_2 + P\bigl(|\eta_n| \geq \varepsilon_1\bigr) + P\bigl(|\xi_n| \geq \varepsilon_2\bigr)\bigr\}.$$

Lemma A.5 (Liang and Fan [7]) *Suppose that X_n, $n \geq 1$ is a sequence of NA random variables with finite second moments. Let $\{a_j, j \geq 1\}$ be a real sequence, and let $1 = m_0 < m_1 < \cdots < m_k = n$. Define $Y_l = \sum_{j=m_{l-1}+1}^{m_l} a_jX_j$ for $1 \leq l \leq k$. Then*

$$\left|E\exp\left\{\mathrm{i}t\sum_{l=1}^{k} Y_l\right\} - \prod_{i=1}^{k}\exp\{\mathrm{i}tY_l\}\right| \leq 4t^2\sum_{1\leq s<j\leq k}\ \sum_{l_1=m_{s-1}+1}^{m_s}\ \sum_{l_2=m_{j-1}+1}^{m_j}|a_{l_1}a_{l_2}|\bigl|\mathrm{Cov}(X_{l_1}, X_{l_2})\bigr|.$$

Lemma A.6 (Wei and Li [12]) *Assume that Assumptions* (A3) *and* (A4) *hold. Then*

(i) $\sup_m \int_0^1 |\mathrm{E}_m(t,s)|\,ds \leq C$;
(ii) $\sum_{i=1}^{n} |\int_{A_i} \mathrm{E}_m(t,s)\,ds| \leq C$;
(iii) $\sup_{0\leq s,t\leq 1} |\mathrm{E}_m(t,s)| = O(2^m)$;
(iv) $|\int_{A_i} \mathrm{E}_m(t,s)\,ds| = O(\frac{2^m}{n})$, $i = 1, 2, \ldots, n$;
(v) $\sum_{i=1}^{n}(\int_{A_i} \mathrm{E}_m(t,s)\,ds)^2 = O(\frac{2^m}{n})$;
(vi) $\max_{1\leq i\leq n} \sum_{j=1}^{n} \int_{A_j} |\mathrm{E}_m(t_i,s)|\,ds \leq C$;
(vii) $\max_{1\leq i\leq n} \sum_{j=1}^{n} \int_{A_i} |\mathrm{E}_m(t_j,s)|\,ds \leq C$.

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