Product of composition and differentiation operators and closures of weighted Bergman spaces in Bloch type spaces

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Abstract

Closures of weighted Bergman spaces in Bloch type spaces are investigated in the paper. Moreover, the boundedness and compactness of the product of composition and differentiation operators from Bloch type spaces to closures of weighted Bergman spaces spaces in Bloch type spaces are characterized.

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1 Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the class of all functions analytic in $\mathbb{D}$. For $0 < p < \infty$, H^p denotes the Hardy space, which consists of all functions $f \in H(\mathbb{D})$ for which (see [6])

$$\|f\|_{H^p}^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^p d\theta < \infty.$$

As usual, H^∞ denotes the space of all bounded analytic functions in $\mathbb{D}$.

The Bloch space $\mathcal{B}$ is the set of all functions $f \in H(\mathbb{D})$ that satisfies

$$\|f\|_\beta = \sup_{z\in\mathbb{D}} \left(1 - |z|^2\right) \left|f'(z)\right| < \infty.$$

It is well known that $\mathcal{B}$ is a Banach space if it is equipped with the norm $\|f\|_\mathcal{B} = |f(0)| + \|f\|_\beta$. Note that $H^\infty \subset \mathcal{B}$. The little Bloch space, denoted by $\mathcal{B}_0$, is the subspace of $\mathcal{B}$ consisting of all $f \in H(\mathbb{D})$ such that $\lim_{|z|\to 1}(1 - |z|^2)|f'(z)| = 0$. It is well known that $\mathcal{B}_0$ is the closure of polynomials in $\mathcal{B}$.

Let $0 < \alpha < \infty$. Recall that the Bloch type space, denoted by $\mathcal{B}^\alpha$, is the space of all functions $f \in H(\mathbb{D})$ satisfying

$$\|f\|_{\mathcal{B}^\alpha} = \left|f(0)\right| + \sup_{z\in\mathbb{D}} \left(1 - |z|^2\right)^\alpha \left|f'(z)\right| < \infty.$$

It is easy to see that $\mathcal{B}^\alpha$ is a Banach space under the norm $\|\cdot\|_{\mathcal{B}^\alpha}$. Let n be a positive integer. It is well known that $\|f\|_{\mathcal{B}^\alpha}$ is equivalent to $\|f\|_{\mathcal{B}^{\alpha,n}}$ (see [34, p. 1149]), where

$$\|f\|_{\mathcal{B}^{\alpha,n}} = |f(0)| + |f'(0)| + \cdots + |f^{(n-1)}(0)| + \sup_{z\in\mathbb{D}}\left(1 - |z|^2\right)^{\alpha+n-1}\left|f^{(n)}(z)\right|.$$

The little Bloch type space $\mathcal{B}_0^\alpha$ is the subspace of $\mathcal{B}^\alpha$ consisting of all $f \in H(\mathbb{D})$ such that

$$\lim_{|z|\to 1}\left(1 - |z|^2\right)^\alpha\left|f'(z)\right| = 0.$$

For $0 < p < \infty$ and $s > -1$, the classical weighted Bergman space A_s^p consists of those $f \in H(\mathbb{D})$ such that

$$\|f\|_{A_s^p} = \left((s + 1)\int_{\mathbb{D}}\left|f(z)\right|^p\left(1 - |z|^2\right)^s dA(z)\right)^{1/p} < \infty.$$

Suppose that ω is a positive and integrable function in $\mathbb{D}$. ω is called a radial weight if $\omega(z) = \omega(|z|)$. Let $\hat{\omega}(r) = \int_r^1 \omega(t)\,dt$ for $0 \le r < 1$ and $\hat{\omega}(z) = \hat{\omega}(|z|)$ for $z \in \mathbb{D}$, respectively. ω is called a doubling weight, denoted by $\omega \in \hat{\mathcal{D}}$, if there is a constant $C > 0$ such that

$$\hat{\omega}(r) < C\hat{\omega}\left(\frac{1 + r}{2}\right), \quad \text{when } 0 \le r < 1.$$

ω is called a regular weight, denoted by $\omega \in \mathcal{R}$, if there is a constant $C > 0$ determined by ω such that

$$\frac{1}{C} < \frac{\hat{\omega}(r)}{(1 - r)\omega(r)} < C, \quad \text{when } 0 \le r < 1.$$

ω is called a reverse doubling weight, denoted by $\omega \in \check{\mathcal{D}}$, if there exist constants $K = K(\omega) > 1$ and $C = C(\omega) > 1$ such that

$$\hat{\omega}(r) \ge C\hat{\omega}\left(1 - \frac{1 - r}{K}\right), \quad 0 \le r < 1.$$

We write $\mathcal{D} = \hat{\mathcal{D}} \cap \check{\mathcal{D}}$. Weights in $\mathcal{D}$ are said to be two-sided doubling weights. They originated in the work of Peláez and Rättyä [22, 23].

Let $0 < p < \infty$ and $\omega \in \mathcal{D}$. The weighted Bergman space $A_\omega^p(\mathbb{D}) = A_\omega^p$ is the space of all $f \in H(\mathbb{D})$ for which

$$\|f\|_{A_\omega^p}^p = \int_{\mathbb{D}}\left|f(z)\right|^p\omega(z)\,dA(z) < \infty.$$

It is easy to check that A_ω^p is a Banach space when $p \ge 1$ and a complete metric space with the distance $\rho(f, g) = \|f - g\|_{A_\omega^p}^p$ when $0 < p < 1$. When $\omega(z) = (1 - |z|^2)^s$ ($s > -1$), the space A_ω^p becomes the classical weighted Bergman space A_s^p.

Let φ be an analytic self-map of $\mathbb{D}$. Every φ induces a composition operator C_φ, which is defined on $H(\mathbb{D})$ by

$$C_\varphi(f)(z) = f\left(\varphi(z)\right), \quad z \in \mathbb{D}.$$

Composition operators have been widely studied on various Banach spaces of analytic functions in recent years. One of the main themes when studying composition operators is to characterize the operator properties of C_φ in terms of the function properties of the symbol φ when C_φ acts on several Banach spaces of analytic functions. Readers can refer to [5], which is a standard introductory reference for the theory of composition operators written by Cowen and MacCluer.

It is a well-known consequence of the Schwartz–Pick lemma that the composition operators are bounded on the classical Bloch space. In 1995, Madigan and Matheson [21] characterized continuity and compactness for composition operators on the Bloch space. Another two compactness criteria for composition operators on the Bloch space have been obtained in [28] and [30]. More characterizations of the boundedness and compactness of composition operators between different Bloch type spaces were studied, for instance, in [12, 18, 19, 32] (see also the references therein).

Let D be the differentiation operator $Df = f', f \in H(\mathbb{D})$. It is well known that the differentiation operator is unbounded on various Banach spaces of analytic functions. The products of composition and differentiation operators DC_φ and $C_\varphi D$ are defined by

$$DC_\varphi(f) = (f \circ \varphi)' = f'(\varphi)\varphi' \quad \text{and} \quad C_\varphi D(f) = f'(\varphi).$$

For a nonnegative integer $m \in \mathbb{N}$, we define $D^m f = f^{(m)}, f \in H(\mathbb{D})$. It is natural to define the product of composition operator and mth differentiation operator as follows:

$$C_\varphi D^m f = f^{(m)} \circ \varphi, \quad f \in H(\mathbb{D}).$$

See [13–17, 37] for the study of the product of composition and differentiation operators on various spaces of analytic functions.

Let X and Y be two Banach spaces of analytic functions. For simplicity, the closure of $X \cap Y$ in the norm of Y is denoted by $\mathcal{C}_Y(X \cap Y)$. The question, which was raised by Anderson, Clunie, and Pommerenke in [2], of describing the closure of H^∞ in $\mathcal{B}$ is still open. Anderson in [1] mentioned that Jones gave an unpublished characterization of the closure of BMOA in $\mathcal{B}$. Ghatage and Zheng in [10] provided a complete proof for Jones's description. In 2008, Zhao studied $\mathcal{C}_\mathcal{B}(F(p, p - 2, s))$ when $1 \le p < \infty$ and $0 < s \le 1$ in [31]. Aulaskari and Zhao studied composition operators from the Bloch space to $\mathcal{C}_\mathcal{B}(F(p, p - 2, s))$ in [3]. Monreal Galán and Nicolau in [7] characterized the closure in the Bloch norm of the space $H^p \cap \mathcal{B}$ for $1 < p < \infty$. Recently, Galanopoulos, Monreal Galán, and Pau in [8] extended the above result to the case of the unit ball in $\mathbb{C}^n$ and for $0 < p < \infty$. Bao and Göğüş in [4] characterized the closure of $\mathcal{D}_\alpha^2 \cap \mathcal{B}$ $(-1 < \alpha \le 1)$ in the Bloch space, where $\mathcal{D}_\alpha^2$ is the Dirichlet type space. See [9, 11, 20, 26, 27, 29] for more results of the closure of some function spaces in the Bloch space.

It is well known that in many aspects the Hardy space H^p is the limit of A_α^p as $\alpha \to -1$. However, this is a rough estimate. From [22, 23], we see that A_ω^p induced by rapidly increasing weights lie "closer" to H^p than any A_α^p, i.e.,

$$H^p \subset A_\omega^p \subset A_\alpha^p.$$

In addition, for any $\alpha > -1$ and $0 < p < \infty$, $\mathcal{B} \subset A_\alpha^p$ and $\mathcal{B} \subsetneq A_\omega^p$ for some $\omega \in \hat{\mathcal{D}}$. Motivated by the above observations and [7], we naturally look for a characterization of $\mathcal{C}_\mathcal{B}(A_\omega^p \cap \mathcal{B})$

or, more generally, of $\mathcal{C}_{\mathcal{B}^\beta}(A_\omega^p \cap \mathcal{B}^\beta)$. The purpose of this paper is to study the closure of $A_\omega^p \cap \mathcal{B}^\beta$ in the norm of $\mathcal{B}^\beta$. We give a complete characterization for $\mathcal{C}_{\mathcal{B}^\beta}(A_\omega^p \cap \mathcal{B}^\beta)$. Moreover, we study the boundedness and compactness of the operators $C_\varphi D^m : \mathcal{B}^\alpha(\mathcal{B}_0^\alpha) \to \mathcal{C}_{\mathcal{B}^\beta}(A_\omega^p \cap \mathcal{B}^\beta)$ and $C_\varphi D^m : \mathcal{C}_{\mathcal{B}^\beta}(A_\omega^p \cap \mathcal{B}^\beta) \to \mathcal{C}_{\mathcal{B}^\beta}(A_\omega^p \cap \mathcal{B}^\beta)$.

Throughout this paper, we say that $f \lesssim h$ if there exists a constant $C > 0$ such that $f \leq Ch$. The symbol $f \approx h$ means that $f \lesssim h \lesssim f$.

2 Characterization of $\mathcal{C}_{\mathcal{B}^\beta}(A_\omega^p \cap \mathcal{B}^\beta)$

To state and prove our main results in this paper, we need some lemmas. The following well-known estimate can be found in [35, Lemma 3.10].

Lemma 1 *Suppose $s > 0$ and $t > -1$. Then there exists a positive constant C such that*

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^t}{|1-\bar{z}w|^{2+t+s}}\, dA(w) \leq \frac{C}{(1-|z|^2)^s}$$

for all $z \in \mathbb{D}$.

Lemma 2 ([35, Exercise 8, p. 63]) *For any $T > 0$, there exists a constant $C > 0$ (depending on T but not on t and z) such that*

$$\int_0^{2\pi} \frac{d\theta}{|1-ze^{i\theta}|^{1+t}} \leq \frac{C\Gamma(t)}{(1-|z|)^t}$$

for all $z \in \mathbb{D}$ and $t \in (0, T)$. Here $\Gamma(t) = \int_0^{+\infty} x^{t-1} e^{-x}\, dx$.

Lemma 3 ([25, Proposition 5]) *Let $0 < p < \infty$ and $\omega \in \hat{\mathcal{D}}$, and write $W(r) = W_\omega(r) = \frac{\hat{\omega}(r)}{1-r}$ for all $0 \leq r < 1$. Then $\|f\|_{A_W^p} \approx \|f\|_{A_\omega^p}$ for all $f \in H(\mathbb{D})$ if and only if $\omega \in \mathcal{D}$. Moreover, if $\omega \in \mathcal{D}$, we have $W \in \mathcal{R}$ and $\hat{W} = \hat{\omega}$.*

Lemma 4 ([24, Theorem 5]) *Let $\omega \in \hat{\mathcal{D}}$, $0 < p < \infty$, and $k \in \mathbb{N}$. Then*

$$\|f\|_{A_\omega^p}^p \approx \int_{\mathbb{D}} |f^{(k)}(z)|^p (1-|z|)^{kp} \omega(z)\, dA(z) + \sum_{j=0}^{k-1} |f^{(j)}(0)|^p, \quad f \in H(\mathbb{D}),$$

if and only if $\omega \in \mathcal{D}$.

Now we are in a position to state and prove our main results in this paper.

Theorem 1 *Let n be a positive integer. Suppose that $0 < p < \infty$, $\omega \in \mathcal{D}$, and $0 < \beta < \infty$. Let $f \in \mathcal{B}^\beta$. Then $f \in \mathcal{C}_{\mathcal{B}^\beta}(A_\omega^p \cap \mathcal{B}^\beta)$ if and only if, for any $\epsilon > 0$,*

$$\int_{\Omega_{n,\beta,\epsilon}(f)} \frac{\hat{\omega}(z)\, dA(z)}{(1-|z|^2)^{p\beta-p+1}} < \infty, \tag{1}$$

where

$$\Omega_{n,\beta,\epsilon}(f) = \left\{ z \in \mathbb{D} : (1-|z|^2)^{\beta+n-1} |f^{(n)}(z)| \geq \epsilon \right\}.$$

Proof Let $W(z) = \frac{\hat{\omega}(z)}{1-|z|^2} = \frac{\hat{\omega}(|z|)}{1-|z|^2}$. By Lemma 3, $\|f\|_{A^p_W} \approx \|f\|_{A^p_\omega}$. Hence,

$$f \in \mathcal{C}_{\mathcal{B}^\beta}\left(A^p_W \cap \mathcal{B}^\beta\right) \quad \Leftrightarrow \quad f \in \mathcal{C}_{\mathcal{B}^\beta}\left(A^p_\omega \cap \mathcal{B}^\beta\right).$$

Take a function f in the closure in the Bloch type norm of $A^p_W \cap \mathcal{B}^\beta$ and $\epsilon > 0$. Then there exists $g \in A^p_W \cap \mathcal{B}^\beta$ such that $\|f - g\|_{\mathcal{B}^{\beta,n}} \le \frac{\epsilon}{2}$. Note that

$$
\begin{aligned}
\left(1 - |z|^2\right)^{\beta+n-1}\left|f^{(n)}(z)\right| &\le \sup_{w \in \mathbb{D}}\left(1 - |w|^2\right)^{\beta+n-1}\left|f^{(n)}(w) - g^{(n)}(w)\right| \\
&\quad + \left(1 - |z|^2\right)^{\beta+n-1}\left|g^{(n)}(z)\right| \\
&\le \frac{\epsilon}{2} + \left(1 - |z|^2\right)^{\beta+n-1}\left|g^{(n)}(z)\right|, \quad z \in \mathbb{D}.
\end{aligned}
$$

This implies that $\Omega_{n,\beta,\epsilon}(f) \subseteq \Omega_{n,\beta,\frac{\epsilon}{2}}(g)$. Then, by Lemmas 3 and 4, we have

$$
\begin{aligned}
\|g\|^p_{A^p_\omega} &\gtrsim \int_{\mathbb{D}}\left|g^{(n)}(z)\right|^p\left(1 - |z|^2\right)^{np}W(z)\,dA(z) \\
&\ge \int_{\Omega_{n,\beta,\frac{\epsilon}{2}}(g)}\left|g^{(n)}(z)\right|^p\left(1 - |z|^2\right)^{np}W(z)\,dA(z) \\
&= \int_{\Omega_{n,\beta,\frac{\epsilon}{2}}(g)}\frac{|g^{(n)}(z)|^p(1 - |z|^2)^{(\beta+n-1)p}W(z)}{(1 - |z|^2)^{(\beta+n-1)p-np}}\,dA(z) \\
&\ge \left(\frac{\epsilon}{2}\right)^p\int_{\Omega_{n,\beta,\frac{\epsilon}{2}}(g)}\frac{W(z)\,dA(z)}{(1 - |z|^2)^{p\beta-p}} \\
&\ge \left(\frac{\epsilon}{2}\right)^p\int_{\Omega_{n,\beta,\epsilon}(f)}\frac{W(z)\,dA(z)}{(1 - |z|^2)^{p\beta-p}},
\end{aligned}
$$

which implies that

$$\int_{\Omega_{n,\beta,\epsilon}(f)}\frac{\hat{\omega}(z)\,dA(z)}{(1 - |z|^2)^{p\beta-p+1}} = \int_{\Omega_{n,\beta,\epsilon}(f)}\frac{W(z)\,dA(z)}{(1 - |z|^2)^{p\beta-p}} < \infty.$$

Conversely, assume that (1) holds. Fix $\epsilon > 0$ and let f satisfy (1). Without loss of generality, we may assume that $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$. For any $z \in \mathbb{D}$, using Proposition 4.27 in [35], we have

$$f(z) = \frac{1}{(\gamma+2)\cdots(\gamma+n)}\int_{\mathbb{D}}\frac{f^{(n)}(w)(1 - |w|^2)^{n+\gamma}}{(1 - z\overline{w})^{2+\gamma}\overline{w}^n}\,dA(w),$$

where $\gamma \ge 0$. Then we will choose γ large enough for our purpose. Following [31], we set $f(z) = f_1(z) + f_2(z)$, where

$$f_1(z) = \frac{1}{(\gamma+2)\cdots(\gamma+n)}\int_{\Omega_{n,\beta,\epsilon}(f)}\frac{f^{(n)}(w)(1 - |w|^2)^{n+\gamma}}{(1 - z\overline{w})^{2+\gamma}\overline{w}^n}\,dA(w)$$

and

$$f_2(z) = \frac{1}{(\gamma+2)\cdots(\gamma+n)}\int_{\mathbb{D}\setminus\Omega_{n,\beta,\epsilon}(f)}\frac{f^{(n)}(w)(1 - |w|^2)^{n+\gamma}}{(1 - z\overline{w})^{2+\gamma}\overline{w}^n}\,dA(w).$$

Obviously,

$$f_1^{(n)}(z) = (\gamma + n + 1) \int_{\Omega_{n,\beta,\epsilon}(f)} \frac{f^{(n)}(w)(1 - |w|^2)^{n+\gamma}}{(1 - z\overline{w})^{n+2+\gamma}} \, dA(w)$$

and

$$f_2^{(n)}(z) = (\gamma + n + 1) \int_{\mathbb{D} \setminus \Omega_{n,\beta,\epsilon}(f)} \frac{f^{(n)}(w)(1 - |w|^2)^{n+\gamma}}{(1 - z\overline{w})^{n+2+\gamma}} \, dA(w).$$

Let $h(z) = f_1(z) - \sum_{k=1}^{n-1} \frac{f_1^{(k)}(0)}{k!} z^k$. Then $h(0) = h'(0) = \cdots = h^{(n-1)}(0) = 0$ and $(f - h)^{(n)}(z) = f_2^{(n)}(z)$. Using Lemma 1, we obtain

$$
\begin{aligned}
\|f - h\|_{\mathcal{B}^{\beta,n}} &= \sup_{z \in \mathbb{D}} \left(1 - |z|^2\right)^{\beta+n-1} \left| f_2^{(n)}(z) \right| \\
&\lesssim \sup_{z \in \mathbb{D}} \left(1 - |z|^2\right)^{\beta+n-1} \int_{\mathbb{D} \setminus \Omega_{n,\beta,\epsilon}(f)} \frac{|f^{(n)}(w)|(1 - |w|^2)^{n+\gamma}}{|1 - z\overline{w}|^{n+2+\gamma}} \, dA(w) \\
&\lesssim \epsilon \sup_{z \in \mathbb{D}} \left(1 - |z|^2\right)^{\beta+n-1} \int_{\mathbb{D}} \frac{(1 - |w|^2)^{\gamma-\beta+1}}{|1 - z\overline{w}|^{n+2+\gamma}} \, dA(w) \\
&\lesssim \epsilon.
\end{aligned}
$$

This means that $h \in \mathcal{B}^\beta$.

Since $W \in \mathcal{R}$, it is well known that, see [22] for example, there exist $-1 < a < b < \infty$ and $\delta \in [0, 1)$ such that

$$\frac{W(r)}{(1 - r)^b} \nearrow \infty, \quad \text{and} \quad \frac{W(r)}{(1 - r)^a} \searrow 0, \quad \text{when } r \geq \delta.$$

Without loss of generality, we can assume $\delta = 0$. Then we can choose $\gamma > 0$ such that $(p - 1)(\beta - 1) + \gamma - |b| > 0$. By Lemmas 1 and 2, we have

$$
\begin{aligned}
&\int_{|z| \leq |w|} \frac{(1 - |z|^2)^{n-(p-1)(\beta-1)} W(z)}{|1 - z\overline{w}|^{n+2+\gamma}} \, dA(z) \\
&\quad \approx \int_0^{|w|} \left(1 - r^2\right)^{n-(p-1)(\beta-1)} W(r) \, dr \int_0^{2\pi} \frac{1}{|1 - r\overline{w}e^{i\theta}|^{n+2+\gamma}} \, d\theta \\
&\quad \approx \int_0^{|w|} \frac{W(r)}{(1 - r)^b} \frac{(1 - r)^{n-(p-1)(\beta-1)+b}}{(1 - r|w|)^{n+\gamma+1}} \, dr \\
&\quad \lesssim \frac{W(w)}{(1 - |w|)^b} \int_0^{|w|} \frac{1}{(1 - r)^{(p-1)(\beta-1)+\gamma-b+1}} \, dr \\
&\quad \lesssim \frac{W(w)}{(1 - |w|)^{(p-1)(\beta-1)+\gamma}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{|w| < |z| < 1} \frac{(1 - |z|^2)^{n-(p-1)(\beta-1)} W(z)}{|1 - z\overline{w}|^{n+2+\gamma}} \, dA(z) \\
&\quad \lesssim \hat{\omega}(w) \int_{|w| < |z| < 1} \frac{(1 - |z|^2)^{n-(p-1)(\beta-1)-1}}{|1 - z\overline{w}|^{n+2+\gamma}} \, dA(z)
\end{aligned}
$$

$$\leq \hat{\omega}(w) \int_{\mathbb{D}} \frac{(1 - |z|^2)^{n - (p-1)(\beta-1) - 1}}{|1 - z\overline{w}|^{n+2+\gamma}} \, dA(z)$$

$$\lesssim \frac{W(w)}{(1 - |w|)^{(p-1)(\beta-1)+\gamma}}.$$

Thus using Fubini's theorem and Lemma 1, we obtain

$$\int_{\mathbb{D}} |h^{(n)}(z)|^p (1 - |z|^2)^{np} W(z) \, dA(z)$$

$$= \int_{\mathbb{D}} |f_1^{(n)}(z)|^p (1 - |z|^2)^{np} W(z) \, dA(z)$$

$$\leq \|f_1\|_{\mathcal{B}^{\beta,n}}^{p-1} \int_{\mathbb{D}} |f_1^{(n)}(z)| (1 - |z|^2)^{n - (p-1)(\beta-1)} W(z) \, dA(z)$$

$$\lesssim \|f_1\|_{\mathcal{B}^{\beta,n}}^{p-1} \int_{\mathbb{D}} (1 - |z|^2)^{n - (p-1)(\beta-1)} W(z)$$

$$\times \left(\int_{\Omega_{n,\beta,\epsilon}(f)} \frac{|f^{(n)}(w)|(1 - |w|^2)^{n+\gamma}}{|1 - z\overline{w}|^{n+2+\gamma}} \, dA(w) \right) dA(z)$$

$$\lesssim \|f_1\|_{\mathcal{B}^{\beta,n}}^{p-1} \int_{\Omega_{n,\beta,\epsilon}(f)} |f^{(n)}(w)| (1 - |w|^2)^{n+\gamma}$$

$$\times \left(\int_{\mathbb{D}} \frac{(1 - |z|^2)^{n - (p-1)(\beta-1)} W(z)}{|1 - z\overline{w}|^{n+2+\gamma}} \, dA(z) \right) dA(w)$$

$$\lesssim \|f_1\|_{\mathcal{B}^{\beta,n}}^{p-1} \int_{\Omega_{n,\beta,\epsilon}(f)} \frac{|f^{(n)}(w)|(1 - |w|^2)^{n+\gamma} W(w)}{(1 - |w|)^{(p-1)(\beta-1)+\gamma}} \, dA(w)$$

$$\lesssim \|f_1\|_{\mathcal{B}^{\beta,n}}^{p-1} \|f\|_{\mathcal{B}^{\beta,n}} \int_{\Omega_{n,\beta,\epsilon}(f)} \frac{W(w) \, dA(w)}{(1 - |w|^2)^{p\beta - p}}$$

$$\lesssim \int_{\Omega_{n,\beta,\epsilon}(f)} \frac{\hat{\omega}(w) \, dA(w)}{(1 - |w|^2)^{p\beta - p + 1}} < \infty.$$

Hence, $h \in A_W^p$. Then, for any $\epsilon > 0$, there exists a function $h \in A_W^p \cap \mathcal{B}^\beta$ such that $\|f - h\|_{\mathcal{B}^{\beta,n}} \lesssim \epsilon$, which means that $f \in \mathcal{C}_{\mathcal{B}^\beta}(A_\omega^p \cap \mathcal{B}^\beta)$. The proof is complete. $\qquad\square$

3 The operator $C_\varphi D^m$ on $\mathcal{C}_{\mathcal{B}^\beta}(A_\omega^p \cap \mathcal{B}^\beta)$

Next, we characterize the boundedness and compactness of the operator $C_\varphi D^m$ from Bloch type spaces $\mathcal{B}^\alpha(\mathcal{B}_0^\alpha)$ to $\mathcal{C}_{\mathcal{B}^\beta}(A_\omega^p \cap \mathcal{B}^\beta)$. We denote $\Omega_{n,\beta,\epsilon}(f)$ by $\Omega_{\beta,\epsilon}(f)$ when $n = 1$.

Theorem 2 *Let φ be an analytic self-map of $\mathbb{D}$ and $m \in \mathbb{N}$. Suppose that $0 < p < \infty$, $\omega \in \mathcal{D}$, and $0 < \alpha, \beta < \infty$. Then $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{C}_{\mathcal{B}^\beta}(A_\omega^p \cap \mathcal{B}^\beta)$ is bounded if and only if, for any $\epsilon > 0$,*

$$\int_{\Gamma_\epsilon(\varphi)} \frac{\hat{\omega}(z) \, dA(z)}{(1 - |z|^2)^{p\beta - p + 1}} < \infty, \tag{2}$$

where

$$\Gamma_\epsilon(\varphi) = \left\{ z \in \mathbb{D} : \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^{\alpha+m}} |\varphi'(z)| \geq \epsilon \right\}.$$

Proof Sufficiency. Assume that (2) holds for any $\epsilon > 0$. Let $f \in \mathcal{B}^\alpha$. Then

$$\left|\left(C_\varphi D^m f\right)'(z)\right|\left(1 - |z|^2\right)^\beta = \left|f^{(m+1)}\left(\varphi(z)\right)\right|\left|\varphi'(z)\right|\left(1 - |z|^2\right)^\beta$$

$$\leq \|f\|_{\mathcal{B}^{\alpha,m+1}} \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^{\alpha+m}}\left|\varphi'(z)\right|.$$

Thus, for any $\delta > 0$, if $|(C_\varphi D^m f)'(z)|(1 - |z|^2)^\beta > \delta$, we have that

$$\frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^{\alpha+m}}\left|\varphi'(z)\right| \geq \frac{\delta}{\|f\|_{\mathcal{B}^{\alpha,m+1}}} = \epsilon.$$

Therefore,

$$\infty > \int_{\Gamma_\epsilon(\varphi)} \frac{\hat{\omega}(z)\,dA(z)}{(1 - |z|^2)^{p\beta-p+1}} \gtrsim \int_{\Omega_{\beta,\delta}(C_\varphi D^m f)} \frac{\hat{\omega}(z)\,dA(z)}{(1 - |z|^2)^{p\beta-p+1}}.$$

According to Theorem 1, we get that

$$C_\varphi D^m f \in \mathcal{C}_{\mathcal{B}^\beta}\left(A_\omega^p \cap \mathcal{B}^\beta\right).$$

This means that $C_\varphi D^m f : \mathcal{B}^\alpha \to \mathcal{C}_{\mathcal{B}^\beta}(A_\omega^p \cap \mathcal{B}^\beta)$ is bounded.

Necessity. Suppose that $C_\varphi D^m f : \mathcal{B}^\alpha \to \mathcal{C}_{\mathcal{B}^\beta}(A_\omega^p \cap \mathcal{B}^\beta)$ is bounded. It is well known that, for any positive integer m, there exist two functions $f_1, f_2 \in \mathcal{B}^\alpha$ such that (see [36])

$$\left|f_1^{(m)}(z)\right| + \left|f_2^{(m)}(z)\right| \geq \frac{1}{(1 - |z|^2)^{\alpha+m-1}}.$$

Due to our assumption, we get that $f_1^{(m)} \circ \varphi$, $f_2^{(m)} \circ \varphi \in \mathcal{C}_{\mathcal{B}^\beta}(A_\omega^p \cap \mathcal{B}^\beta)$. Thus, for any $\epsilon > 0$, we have

$$\int_{\Omega_{\beta,\frac{\epsilon}{2}}(f_1^{(m)} \circ \varphi)} \frac{\hat{\omega}(z)\,dA(z)}{(1 - |z|^2)^{p\beta-p+1}} < \infty$$

and

$$\int_{\Omega_{\beta,\frac{\epsilon}{2}}(f_2^{(m)} \circ \varphi)} \frac{\hat{\omega}(z)\,dA(z)}{(1 - |z|^2)^{p\beta-p+1}} < \infty.$$

If $z \in \Gamma_\epsilon(\varphi)$, then we have

$$\left(\left|\left(C_\varphi D^m f_1\right)'(z)\right| + \left|\left(C_\varphi D^m f_2\right)'(z)\right|\right)\left(1 - |z|^2\right)^\beta$$

$$= \left(\left|f_1^{(m+1)}\left(\varphi(z)\right)\right| + \left|f_2^{(m+1)}\left(\varphi(z)\right)\right|\right)\left|\varphi'(z)\right|\left(1 - |z|^2\right)^\beta$$

$$\geq \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^{\alpha+m}}\left|\varphi'(z)\right| \geq \epsilon.$$

This means that, either

$$\left|\left(C_\varphi D^m f_1\right)'(z)\right|\left(1 - |z|^2\right)^\beta \geq \frac{\epsilon}{2}$$

or

$$\left| \left(C_\varphi D^m f_2 \right)'(z) \right| \left(1 - |z|^2 \right)^\beta \geq \frac{\epsilon}{2}.$$

Therefore,

$$\int_{\Gamma_\epsilon(\varphi)} \frac{\hat{\omega}(z)\,dA(z)}{(1-|z|^2)^{p\beta-p+1}} \leq \int_{\Omega_{\beta,\frac{\epsilon}{2}}(f_1^{(m)}\circ\varphi)\cup\Omega_{\beta,\frac{\epsilon}{2}}(f_2^{(m)}\circ\varphi)} \frac{\hat{\omega}(z)\,dA(z)}{(1-|z|^2)^{p\beta-p+1}}$$

$$\leq \int_{\Omega_{\beta,\frac{\epsilon}{2}}(f_1^{(m)}\circ\varphi)} \frac{\hat{\omega}(z)\,dA(z)}{(1-|z|^2)^{p\beta-p+1}} + \int_{\Omega_{\beta,\frac{\epsilon}{2}}(f_2^{(m)}\circ\varphi)} \frac{\hat{\omega}(z)\,dA(z)}{(1-|z|^2)^{p\beta-p+1}}$$

$$< \infty.$$

The proof is complete.   □

Theorem 3 *Let φ be an analytic self-map of $\mathbb{D}$ and $m \in \mathbb{N}$. Suppose that $0 < p < \infty$, $\omega \in \mathcal{D}$, and $0 < \alpha, \beta < \infty$. Then $C_\varphi D^m : \mathcal{B}_0^\alpha \to \mathcal{C}_{\mathcal{B}^\beta}(A_\omega^p \cap \mathcal{B}^\beta)$ is bounded if and only if $\varphi \in \mathcal{C}_{\mathcal{B}^\beta}(A_\omega^p \cap \mathcal{B}^\beta)$ and*

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+m}} |\varphi'(z)| < \infty. \tag{3}$$

Proof The necessity of the conditions can be proved immediately. In fact, we suppose that $C_\varphi D^m : \mathcal{B}_0^\alpha \to \mathcal{C}_{\mathcal{B}^\beta}(A_\omega^p \cap \mathcal{B}^\beta)$ is bounded. Notice that $f_m(z) = \frac{z^{m+1}}{(m+1)!} \in \mathcal{B}_0^\alpha$, then we have

$$\varphi = C_\varphi D^m f \in \mathcal{C}_{\mathcal{B}^\beta}\left(A_\omega^p \cap \mathcal{B}^\beta\right).$$

Since $C_\varphi D^m : \mathcal{B}_0^\alpha \to \mathcal{C}_{\mathcal{B}^\beta}(A_\omega^p \cap \mathcal{B}^\beta)$ is bounded and $\mathcal{C}_{\mathcal{B}^\beta}(A_\omega^p \cap \mathcal{B}^\beta) \subseteq \mathcal{B}^\beta$, then $C_\varphi D^m : \mathcal{B}_0^\alpha \to \mathcal{B}^\beta$ is bounded. It is easy to see that (3) holds according to [33, Theorem 2.1].

To prove the sufficiency, we assume that $\varphi \in \mathcal{C}_{\mathcal{B}^\beta}(A_\omega^p \cap \mathcal{B}^\beta)$ and

$$Q := \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+m}} |\varphi'(z)| < \infty.$$

Let $f \in \mathcal{B}_0^\alpha$. Then, for any $\epsilon > 0$, there is a constant r $(0 < r < 1)$ such that

$$\left| f^{(m)}(z) \right| \left(1 - |z|^2 \right)^{\alpha+m-1} < \frac{\epsilon}{Q}, \quad \text{whenever } |z| > r.$$

Let $z \in \Omega_{\beta,\epsilon}(C_\varphi D^m f)$. Then we have

$$Q \left| f^{(m+1)}\left(\varphi(z) \right) \right| \left(1 - \left| \varphi(z) \right|^2 \right)^{\alpha+m}$$

$$\geq \left| f^{(m+1)}\left(\varphi(z) \right) \right| \left(1 - \left| \varphi(z) \right|^2 \right)^{\alpha+m} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+m}} |\varphi'(z)|$$

$$= \left| \left(C_\varphi D^m f \right)'(z) \right| \left(1 - |z|^2 \right)^\beta \geq \epsilon.$$

This means that $|\varphi(z)| \le r$. Thus,

$$\frac{\|f\|_{\mathcal{B}^{\alpha,m+1}}}{(1-r^2)^{\alpha+m}}\left(1-|z|^2\right)^\beta\left|\varphi'(z)\right|$$

$$\ge \left|f^{(m+1)}\big(\varphi(z)\big)\right|\left(1-\left|\varphi(z)\right|^2\right)^{\alpha+m}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+m}}\left|\varphi'(z)\right|$$

$$= \left|\big(C_\varphi D^m f\big)'(z)\right|\left(1-|z|^2\right)^\beta \ge \epsilon.$$

Let $\delta = \frac{(1-r^2)^{\alpha+m}\epsilon}{\|f\|_{\mathcal{B}^{\alpha,m+1}}}$. Then $|\varphi'(z)|(1-|z|^2)^\beta \ge \delta$. Hence, $\Omega_{\beta,\epsilon}(C_\varphi D^m f) \subseteq \Omega_{\beta,\delta}(\varphi)$. Due to $\varphi \in \mathcal{C}_{\mathcal{B}^\beta}(A_\omega^p \cap \mathcal{B}^\beta)$, we obtain

$$\infty > \int_{\Omega_{\beta,\delta}(\varphi)} \frac{\hat{\omega}(z)\,dA(z)}{(1-|z|^2)^{p\beta-p+1}} \ge \int_{\Omega_{\beta,\epsilon}(C_\varphi D^m f)} \frac{\hat{\omega}(z)\,dA(z)}{(1-|z|^2)^{p\beta-p+1}}.$$

According to Theorem 1, we get that $C_\varphi D^m f \in \mathcal{C}_{\mathcal{B}^\beta}(A_\omega^p \cap \mathcal{B}^\beta)$. Therefore, $C_\varphi D^m : \mathcal{B}_0^\alpha \to \mathcal{C}_{\mathcal{B}^\beta}(A_\omega^p \cap \mathcal{B}^\beta)$ is bounded. The proof is complete. □

Theorem 4 *Let φ be an analytic self-map of $\mathbb{D}$ and $m \in \mathbb{N}$. Suppose that $0 < p < \infty, \omega \in \mathcal{D}$, and $0 < \alpha, \beta < \infty$. Then the following statements are equivalent*:

(i) $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{C}_{\mathcal{B}^\beta}(A_\omega^p \cap \mathcal{B}^\beta)$ *is compact*;

(ii) $C_\varphi D^m : \mathcal{B}_0^\alpha \to \mathcal{C}_{\mathcal{B}^\beta}(A_\omega^p \cap \mathcal{B}^\beta)$ *is compact*;

(iii) $\varphi \in \mathcal{C}_{\mathcal{B}^\beta}(A_\omega^p \cap \mathcal{B}^\beta)$ *and*

$$\lim_{|\varphi(z)| \to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+m}}\left|\varphi'(z)\right| = 0. \tag{4}$$

Proof (i) $\Rightarrow$ (ii). The implication is obvious because $\mathcal{B}_0^\alpha \subseteq \mathcal{B}^\alpha$.

(ii) $\Rightarrow$ (iii). Assume that $C_\varphi D^m : \mathcal{B}_0^\alpha \to \mathcal{C}_{\mathcal{B}^\beta}(A_\omega^p \cap \mathcal{B}^\beta)$ is compact. Obviously, $C_\varphi D^m : \mathcal{B}_0^\alpha \to \mathcal{C}_{\mathcal{B}^\beta}(A_\omega^p \cap \mathcal{B}^\beta)$ is bounded. According to Theorem 3, we obtain $\varphi \in \mathcal{C}_{\mathcal{B}^\beta}(A_\omega^p \cap \mathcal{B}^\beta)$. On the other hand, it is obvious that $\mathcal{C}_{\mathcal{B}^\beta}(A_\omega^p \cap \mathcal{B}^\beta) \subseteq \mathcal{B}^\beta$. Then $C_\varphi D^m : \mathcal{B}_0^\alpha \to \mathcal{B}^\beta$ is compact. This clearly implies that (4) holds by [33, Theorem 2.2].

(iii) $\Rightarrow$ (i). According to the assumed condition, we see that there exists r $(0 < r < 1)$ such that

$$\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+m}}\left|\varphi'(z)\right| < \frac{\epsilon}{2}, \quad \text{whenever } \left|\varphi(z)\right| > r.$$

Let $z \in \Gamma_\epsilon(\varphi)$, then $|\varphi(z)| \le r$. Therefore,

$$\frac{(1-|z|^2)^\beta}{(1-r^2)^{\alpha+m}}\left|\varphi'(z)\right| \ge \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+m}}\left|\varphi'(z)\right| \ge \epsilon.$$

Thus

$$\left|\varphi'(z)\right|\left(1-|z|^2\right)^\beta \ge \left(1-r^2\right)^{\alpha+m}\epsilon.$$

Set $\delta = (1-r^2)^{\alpha+m}\epsilon$. Then $z \in \Omega_{\beta,\delta}(\varphi)$. Since $\varphi \in \mathcal{C}_{\mathcal{B}^\beta}(A_\omega^p \cap \mathcal{B}^\beta)$, we have

$$\infty > \int_{\Omega_{\beta,\delta}(\varphi)} \frac{\hat{\omega}(z)\,dA(z)}{(1-|z|^2)^{p\beta-p+1}} \gtrsim \int_{\Gamma_\epsilon(\varphi)} \frac{\hat{\omega}(z)\,dA(z)}{(1-|z|^2)^{p\beta-p+1}}.$$

According to Theorem 2, $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{C}_{\mathcal{B}^\beta}(A^p_\omega \cap \mathcal{B}^\beta)$ is bounded. We know that $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is compact by [33, Theorem 2.2]. Therefore, $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{C}_{\mathcal{B}^\beta}(A^p_\omega \cap \mathcal{B}^\beta)$ is compact. The proof is complete.   □

Theorem 5 *Let φ be an analytic self-map of $\mathbb{D}$ and $m \in \mathbb{N}$. Suppose that $0 < p < \infty$, $\omega \in \mathcal{D}$, and $0 < \beta < \infty$. Then $C_\varphi D^m : \mathcal{C}_{\mathcal{B}^\beta}(A^p_\omega \cap \mathcal{B}^\beta) \to \mathcal{C}_{\mathcal{B}^\beta}(A^p_\omega \cap \mathcal{B}^\beta)$ is compact if and only if $\varphi \in \mathcal{C}_{\mathcal{B}^\beta}(A^p_\omega \cap \mathcal{B}^\beta)$ and*

$$\lim_{|\varphi(z)| \to 1} \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^{\beta+m}} |\varphi'(z)| = 0. \tag{5}$$

Proof Assume that $C_\varphi D^m : \mathcal{C}_{\mathcal{B}^\beta}(A^p_\omega \cap \mathcal{B}^\beta) \to \mathcal{C}_{\mathcal{B}^\beta}(A^p_\omega \cap \mathcal{B}^\beta)$ is compact. Thus $C_\varphi D^m : \mathcal{C}_{\mathcal{B}^\beta}(A^p_\omega \cap \mathcal{B}^\beta) \to \mathcal{C}_{\mathcal{B}^\beta}(A^p_\omega \cap \mathcal{B}^\beta)$ is bounded. So we obtain $\varphi \in \mathcal{C}_{\mathcal{B}^\beta}(A^p_\omega \cap \mathcal{B}^\beta)$ since $\frac{z^{m+1}}{(m+1)!} \in \mathcal{C}_{\mathcal{B}^\beta}(A^p_\omega \cap \mathcal{B}^\beta)$. It is well known that $\mathcal{B}^\beta_0$ is the closure of all polynomials in $\mathcal{B}^\beta$ and the space A^p_ω contains all polynomials. Therefore, $C_\varphi D^m : \mathcal{B}^\beta_0 \to \mathcal{C}_{\mathcal{B}^\beta}(A^p_\omega \cap \mathcal{B}^\beta)$ is compact. According to Theorem 4, we see that (5) holds.

Conversely, suppose that $\varphi \in \mathcal{C}_{\mathcal{B}^\beta}(A^p_\omega \cap \mathcal{B}^\beta)$ and (5) holds. By [33, Theorem 2.2], we see that $C_\varphi D^m : \mathcal{B}^\beta \to \mathcal{B}^\beta$ is compact. By Theorem 4, we know that $C_\varphi D^m : \mathcal{B}^\beta \to \mathcal{C}_{\mathcal{B}^\beta}(A^p_\omega \cap \mathcal{B}^\beta)$ is compact. Since $\mathcal{C}_{\mathcal{B}^\beta}(A^p_\omega \cap \mathcal{B}^\beta) \subseteq \mathcal{B}^\beta$, we obtain $C_\varphi D^m : \mathcal{C}_{\mathcal{B}^\beta}(A^p_\omega \cap \mathcal{B}^\beta) \to \mathcal{C}_{\mathcal{B}^\beta}(A^p_\omega \cap \mathcal{B}^\beta)$ is compact. The proof is complete.   □

Availability of data and materials
Not applicable.

Competing interests
The author declares that he has no competing interests.

Authors' contributions
The author has contributed to the writing of this paper. He read and approved the manuscript.

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