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Optimal two-parameter geometric and arithmetic mean bounds for the Sándor–Yang mean

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Abstract

In the article, we provide the sharp bounds for the Sándor–Yang mean in terms of certain families of the two-parameter geometric and arithmetic mean and the one-parameter geometric and harmonic means. As applications, we present new bounds for a certain Yang mean and the inverse tangent function.

MSC: 26E60

Keywords: Arithmetic mean; Geometric mean; Quadratic mean; Yang mean; Sándor–Yang mean

1 Introduction

Let $\nu \in (-\infty, \infty)$ and $\sigma, \tau > 0$ with $\sigma \neq \tau$. Then we denote by

$$\begin{aligned} &\mathbf{G}(\sigma,\tau)=\sigma^{1/2}\tau^{1/2}, \qquad \mathbf{U}(\sigma,\tau)=\frac{\sqrt{2}(\sigma-\tau)}{2\arctan(\frac{\sqrt{2}(\sigma-\tau)}{2\sqrt{\sigma\tau}})}, \\ &\mathbf{Q}(\sigma,\tau)=\left(\frac{\sigma^2+\tau^2}{2}\right)^{1/2}, \end{aligned} \tag{1.1}$$

and

$$\mathbf{H}_\nu(\sigma,\tau)=\left(\frac{\sigma^\nu+\tau^\nu}{2}\right)^{1/\nu} \quad (\nu\neq 0), \qquad \mathbf{H}_0(\sigma,\tau)=\sigma^{1/2}\tau^{1/2}$$

the geometric mean, Yang mean [1], quadratic mean [2], and νth Hölder mean [3] of σ and τ, respectively.

It is not difficult to verify that the νth Hölder mean $H_\nu(\sigma,\tau)$ is strictly increasing with respect to $\nu \in (-\infty, \infty)$ for all distinct positive real numbers σ and τ, and

$$\begin{aligned} &\mathbf{H}_{-1}(\sigma,\tau)=\frac{2\sigma\tau}{\sigma+\tau}=\mathbf{H}(\sigma,\tau), \qquad \mathbf{H}_0(\sigma,\tau)=\sigma^{1/2}\tau^{1/2}=\mathbf{G}(\sigma,\tau), \\ &\mathbf{H}_1(\sigma,\tau)=\frac{\sigma+\tau}{2}=\mathbf{A}(\sigma,\tau), \qquad \mathbf{H}_2(\sigma,\tau)=\left(\frac{\sigma^2+\tau^2}{2}\right)^{1/2}=\mathbf{Q}(\sigma,\tau) \end{aligned}$$

are the classical harmonic, geometric, arithmetic, and quadratic means of σ and τ, respectively.

The bivariate means have in the past decades been the subject of intense research activity [4–13] because many important special functions can be expressed by the bivariate means [14–31] and they have wide applications in mathematics, statistics, physics, economics [32–55], and many other natural and human social sciences [56–76].

Yang, Wu, and Chu [77] proved that $\kappa_1 = 2\log 2/(2\log\pi - \log 2) \simeq 0.8684$ is the largest possible value and $\kappa_2 = 4/3$ is the least possible value such that the two-sided inequality

$$\mathbf{H}_{\kappa_1}(\sigma,\tau) < \mathbf{U}(\sigma,\tau) < \mathbf{H}_{\kappa_2}(\sigma,\tau)$$

takes place for all distinct positive real numbers σ and τ, which leads to the conclusion that

$$\mathbf{G}(\sigma,\tau) < \mathbf{U}(\sigma,\tau) < \mathbf{Q}(\sigma,\tau)$$

for $\sigma,\tau > 0$ with $\sigma \neq \tau$.

In [78], Qian and Chu found that $\lambda = \lambda_0 \simeq 0.5451$ and $\mu = 2$ are the best possible parameters such that the double inequality

$$\mathcal{L}_\lambda(\sigma,\tau) < \mathbf{U}(\sigma,\tau) < \mathcal{L}_\mu(\sigma,\tau)$$

holds for all unequal positive real numbers σ and τ, where

$$\mathcal{L}_\nu(\sigma,\tau) = \left[\frac{\sigma^{\nu+1}-\tau^{\nu+1}}{(\nu+1)(\sigma-\tau)}\right]^{1/\nu} \quad (\nu \neq -1, 0)$$

$$\mathcal{L}_{-1}(\sigma,\tau) = \frac{\sigma-\tau}{\log\sigma - \log\tau}, \qquad \mathcal{L}_0(\sigma,\tau) = \frac{1}{e}\left(\frac{\sigma^\sigma}{\tau^\tau}\right)^{1/(\sigma-\tau)}$$

is the νth generalized logarithmic mean of σ and τ.

The Sándor–Yang mean $\mathbf{SY}(\sigma,\tau)$ [1] and two-parameter geometric and arithmetic mean $\mathbf{GA}_{\eta,\nu}(\sigma,\tau)$ [79] are defined by

$$\mathbf{SY}(\sigma,\tau) = \mathbf{Q}(\sigma,\tau)e^{\mathbf{G}(\sigma,\tau)/\mathbf{U}(\sigma,\tau)-1} \tag{1.2}$$

and

$$\mathbf{GA}_{\eta,\nu}(\sigma,\tau) = \mathbf{G}^\nu\big[\eta\sigma + (1-\eta)\tau, \eta\tau + (1-\eta)\sigma\big]\mathbf{A}^{1-\nu}(\sigma,\tau), \tag{1.3}$$

respectively.

Identity (1.3) leads to the conclusion that

$$\mathbf{GA}_{p,1}(\sigma,\tau) = \mathbf{G}\big[p\sigma + (1-p)\tau, p\tau + (1-p)\sigma\big], \tag{1.4}$$

$$\mathbf{GA}_{p,2}(\sigma,\tau) = \mathbf{H}\big[p\sigma + (1-p)\tau, p\tau + (1-p)\sigma\big], \tag{1.5}$$

and

$$\mathbf{GA}_{p,0}(\sigma,\tau)=\mathbf{GA}_{1/2,1/2}(\sigma,\tau)=\mathbf{A}(\sigma,\tau). \tag{1.6}$$

Chu et al. [79] proved that the inequalities

$$\mathbf{GA}_{\eta_1,\nu}(\sigma,\tau)>\mathbf{AGM}(\sigma,\tau)$$

and

$$\mathbf{GA}_{\eta_2,\nu}(\sigma,\tau)>\mathbf{L}(\sigma,\tau)$$

are valid for all distinct positive real numbers σ and τ if and only if

$$\eta_1\geq\frac{1}{2}-\frac{\sqrt{2\nu}}{4\nu},\qquad \eta_2\geq\frac{1}{2}-\frac{\sqrt{6\nu}}{6\nu}$$

if $\nu\in[1,\infty)$ and $0<\eta_1,\eta_2<1/2$, where

$$\mathbf{L}(\sigma,\tau)=\mathcal{L}_{-1}(\sigma,\tau)=\frac{\sigma-\tau}{\log\sigma-\log\tau}$$

and

$$\mathbf{AGM}(\sigma,\tau)=\frac{\pi}{2\int_0^{\pi}\frac{dt}{\sqrt{\sigma^2\cos^2 t+\tau^2\sin^2 t}}}$$

are the logarithmic and Gaussian arithmetic-geometric means of σ and τ, respectively.

Zhang, Yang, and Qian [80], and He et al. [81] proved that

$$\lambda_1=\lambda_2=\frac{\sqrt{2}}{e}\simeq 0.5203,\qquad \lambda_3=\frac{2\log 2}{2+\log 2}\simeq 0.5147,\qquad \nu_1=\frac{5}{6},\qquad \nu_2=\nu_3=\frac{2}{3}$$

are the best possible parameters such that the double inequalities

$$\lambda_1\mathbf{A}(\sigma,\tau)+(1-\lambda_1)\mathbf{H}(\sigma,\tau)<\mathbf{SY}(\sigma,\tau)<\nu_1\mathbf{A}(\sigma,\tau)+(1-\nu_1)\mathbf{H}(\sigma,\tau),$$
$$\lambda_2\mathbf{A}(\sigma,\tau)+(1-\lambda_1)\mathbf{G}(\sigma,\tau)<\mathbf{SY}(\sigma,\tau)<\nu_2\mathbf{A}(\sigma,\tau)+(1-\nu_2)\mathbf{G}(\sigma,\tau),$$

and

$$\mathbf{H}_{\lambda_3}(\sigma,\tau)<\mathbf{SY}(\sigma,\tau)<\mathbf{H}_{\nu_3}(\sigma,\tau) \tag{1.7}$$

hold for all $\sigma,\tau>0$ with $\sigma\neq\tau$.

From (1.4)–(1.7) and the monotonicity of the function $\nu\to\mathbf{H}_\nu(\sigma,\tau)$, we clearly see that

$$\begin{aligned}\mathbf{GA}_{1,2}(\sigma,\tau)&=\mathbf{H}(\sigma,\tau)=\mathbf{H}_{-1}(\sigma,\tau)<\mathbf{G}(\sigma,\tau)=\mathbf{H}_0(\sigma,\tau)\\&<\mathbf{SY}(\sigma,\tau)<\mathbf{H}_1(\sigma,\tau)=\mathbf{A}(\sigma,\tau)=\mathbf{GA}_{p,0}(\sigma,\tau)=\mathbf{GA}_{1/2,1/2}(\sigma,\tau)\end{aligned} \tag{1.8}$$

for all $\sigma,\tau>0$ with $\sigma\neq\tau$.

Motivated by inequality (1.8), we naturally ask the question: For fixed $p \in \mathbb{R}$, what are the best possible parameters λ and μ on the interval $(0, 1/2)$ or $(1/2, 1)$ depending only on the parameter p such that the double inequality

$$\mathbf{GA}_{\lambda,p}(\sigma,\tau) < \mathbf{SY}(\sigma,\tau) < \mathbf{GA}_{\mu,p}(\sigma,\tau)$$

is valid for all unequal positive real numbers σ and τ?

It is the aim of the article to answer the question in the case of $p \in [1, \infty)$ and $\lambda, \mu \in (0, 1/2)$.

2 Lemmas

Lemma 2.1 (see [82, Theorem 1.25]) *Let $\kappa_1, \kappa_2 \in \mathbb{R}$ with $\kappa_1 < \kappa_2$, $\mathcal{F}, \mathcal{G} : [\kappa_1, \kappa_2] \to \mathbb{R}$ be continuous on $[\kappa_1, \kappa_2]$ and differentiable on (κ_1, κ_2) with $\mathcal{G}'(t) \neq 0$ on (κ_1, κ_2). Then both the functions*

$$\frac{\mathcal{F}(t) - \mathcal{F}(\kappa_1)}{\mathcal{G}(t) - \mathcal{G}(\kappa_1)}$$

and

$$\frac{\mathcal{F}(t) - \mathcal{F}(\kappa_2)}{\mathcal{G}(t) - \mathcal{G}(\kappa_2)}$$

are (strictly) increasing (decreasing) on (κ_1, κ_2) if $\mathcal{F}'(t)/\mathcal{G}'(t)$ is (strictly) increasing (decreasing) on (κ_1, κ_2).

Lemma 2.2 *The inequality*

$$\frac{1}{3p} + \left(\frac{2}{e^2}\right)^{1/p} < 1$$

holds for all $p \geq 1$.

Proof Let $p \in [1, \infty)$ and

$$f_1(p) = \frac{1}{3p} + \left(\frac{2}{e^2}\right)^{1/p}. \tag{2.1}$$

Then (2.1) leads to

$$\lim_{p \to \infty} f_1(p) = 1, \tag{2.2}$$

$$\begin{aligned} f_1'(p) &= \frac{2}{p^2} \log\left(\frac{\sqrt{2}e}{2}\right)\left[\left(\frac{\sqrt{2}}{e}\right)^{2/p} - \frac{1}{6\log(\frac{\sqrt{2}e}{2})}\right] \\ &\geq \frac{2}{p^2} \log\left(\frac{\sqrt{2}e}{2}\right)\left[\left(\frac{\sqrt{2}}{e}\right)^{2} - \frac{1}{6\log(\frac{\sqrt{2}e}{2})}\right] \\ &= \frac{12\log(\frac{\sqrt{2}e}{2}) - e^2}{3e^2p^2}. \end{aligned} \tag{2.3}$$

Note that

$$12\log\left(\frac{\sqrt{2}e}{2}\right)-e^2\simeq 0.4521>0. \tag{2.4}$$

Therefore, Lemma 2.2 follows easily from (2.1)–(2.4). □

Lemma 2.3 *The function*

$$f_2(x)=\frac{4(x^2+1)\arctan(x)+x(x^2+2)}{x(3x^2+2)} \tag{2.5}$$

is strictly decreasing from $(0,\infty)$ *on* $(1/3,3)$.

Proof It follows from (2.5) that

$$f_2\left(0^+\right)=3,\qquad \lim_{x\to\infty}f_2(x)=\frac{1}{3}, \tag{2.6}$$

where and in what follows $f(\lambda^+)$ denotes the right limit of the function f at λ.

Let

$$\varphi_1(x)=4\arctan(x)+\frac{x(x^2+2)}{x^2+1},\qquad \varphi_2(x)=\frac{x(3x^2+2)}{x^2+1}.$$

Then we clearly see that

$$\varphi_1\left(0^+\right)=\varphi_2\left(0^+\right)=0,\qquad f_2(x)=\frac{\varphi_1(x)}{\varphi_2(x)},\qquad \frac{\varphi_1'(x)}{\varphi_2'(x)}=\frac{x^2+3}{3x^2+1}. \tag{2.7}$$

It is not difficult to verify that the function $x\to\varphi_1'(x)/\varphi_2'(x)$ is strictly decreasing on $(0,\infty)$.

Therefore, Lemma 2.3 follows from (2.6), (2.7), and Lemma 2.1 together with the monotonicity of the function $\varphi_1'(x)/\varphi_2'(x)$ on the interval $(0,\infty)$. □

Lemma 2.4 *Let* $0<u<1$, $p\geq 1$, *and*

$$g(u,p;x)=\frac{p}{2}\log\left(\frac{(1-u)x^2+2}{x^2+2}\right)+\frac{1}{2}\log\left(\frac{x^2+2}{2(x^2+1)}\right)-\frac{\arctan(x)}{x}+1. \tag{2.8}$$

Then the following statements are true:

(1) $g(u,p;x)>0$ *for all* $x\in(0,\infty)$ *if and only if* $u\leq 1/(3p)$;

(2) $g(u,p;x)<0$ *for all* $x\in(0,\infty)$ *if and only if* $u\geq 1-(2/e^2)^{1/p}$.

Proof From (2.8) we clearly see that

$$g\left(u,p;0^+\right)=0, \tag{2.9}$$

$$\lim_{x\to\infty}g(u,p;x)=\frac{p}{2}\log(1-u)+1-\frac{1}{2}\log 2. \tag{2.10}$$

Let

$$g_0(p,x) = \frac{(x^2+2)[(x^2+2)\arctan(x) - 2x]}{x^2[(x^2+2)\arctan(x) + 2(p-1)x]}. \tag{2.11}$$

Then

$$g_0\big(p,0^+\big) = \frac{1}{3p}, \qquad \lim_{x\to\infty} g_0(p,x) = 1. \tag{2.12}$$

Differentiating $g(u,p;x)$ with respect to x gives

$$\frac{\partial g(u,p;x)}{\partial x} = \frac{(x^2+2)\arctan(x) + 2(p-1)x}{(x^2+2)[(1-u)x^2+2]}\big[g_0(p,x) - u\big]. \tag{2.13}$$

Let

$$g_1(x) = \arctan(x) - \frac{2x}{x^2+2}, \qquad g_2(x) = \frac{x^2}{x^2+2}\left[\arctan(x) + \frac{2(p-1)x}{x^2+2}\right].$$

Then we clearly see that

$$g_0(p,x) = \frac{g_1(x)}{g_2(x)}, \qquad g_1\big(0^+\big) = g_2\big(0^+\big) = 0, \tag{2.14}$$

$$\begin{aligned}\frac{g_1'(x)}{g_2'(x)} &= \frac{x(x^2+2)(3x^2+2)}{4(x^2+1)(x^2+2)\arctan(x) + x[(3-2p)x^4 + 2(5p-3)x^2 + 4(3p-2)]} \\ &= \frac{1}{\frac{4(x^2+1)\arctan(x)+x(x^2+2)}{x(3x^2+2)} + \frac{2(p-1)}{3}\frac{23x^2+22}{(x^2+2)(3x^2+2)} - \frac{2(p-1)}{3}}.\end{aligned} \tag{2.15}$$

It is not difficult to verify that the function $x \to (23x^2+22)/[(x^2+2)(3x^2+2)]$ is strictly decreasing on $(0,\infty)$. Then from Lemma 2.3 and (2.15) we know that $g_1'(x)/g_2'(x)$ is strictly increasing on $(0,\infty)$. Therefore, the fact that the function $x \to g_0(x,p)$ is strictly increasing on $(0,\infty)$ follows from Lemma 2.1 and (2.14) together with the monotonicity of $g_1'(x)/g_2'(x)$ on the interval $(0,\infty)$.

From Lemma 2.2 we know that the interval $(0,1)$ can be expressed by

$$(0,1) = \left(0,\frac{1}{3p}\right] \cup \left(\frac{1}{3p}, 1-\left(\frac{2}{e^2}\right)^{1/p}\right) \cup \left[1-\left(\frac{2}{e^2}\right)^{1/p}, 1\right).$$

We divide the proof into three cases.

Case 1: $0 < u \le 1/(3p)$. Then (2.12) and (2.13) together with the monotonicity of the function $x \to g_0(x,p)$ lead to the conclusion that the function $x \to g(u,p;x)$ is strictly increasing on $(0,\infty)$. Therefore $g(u,p;x) > 0$ for all $x \in (0,\infty)$ follows from (2.9) and the monotonicity of the function $x \to g(u,p;x)$ on the interval $(0,\infty)$.

Case 2: $1-(2/e^2)^{1/p} \le u < 1$. Then from (2.10), (2.12), (2.13), Lemma 2.2, and the monotonicity of the function $x \to g_0(x,p)$, we clearly see that

$$\lim_{x\to\infty} g(u,p;x) \le 0, \tag{2.16}$$

and there exists $x_0 \in (0,\infty)$ such that the function $x \to g(u,p;x)$ is strictly decreasing on $(0,x_0)$ and strictly increasing on (x_0,∞). Therefore $g(u,p;x) < 0$ for all $x \in (0,\infty)$ follows from (2.9) and (2.16) together with the piecewise monotonicity of the function $x \to g(u,p;x)$ on the interval $(0,\infty)$.

Case 3: $1/(3p) < u < 1-(2/e^2)^{1/p}$. Then it follows from (2.10), (2.12), (2.13), and the monotonicity of the function $x \to g_0(x,p)$ that

$$\lim_{x\to\infty} g(u,p;x) > 0, \tag{2.17}$$

and there exists $x^* \in (0,\infty)$ such that the function $x \to g(u,p;x)$ is strictly decreasing on $(0,x^*)$ and strictly increasing on (x^*,∞). Therefore, there exists $\tau \in (0,\infty)$ such that $g(u,p;x) < 0$ for $x \in (0,\tau)$ and $g(u,p;x) > 0$ for $x \in (\tau,\infty)$ follows from (2.9) and (2.17) together with the piecewise monotonicity of the function $x \to g(u,p;x)$ on the interval $(0,\infty)$. □

3 Main result

Theorem 3.1 *Let $p \geq 1$, $0 < \lambda,\mu < 1/2$, and σ and τ be any two different positive real numbers. Then the double inequality*

$$\mathbf{GA}_{\lambda,p}(\sigma,\tau) < \mathbf{SY}(\sigma,\tau) < \mathbf{GA}_{\mu,p}(\sigma,\tau)$$

holds if and only if

$$\lambda \leq \frac{1}{2} - \frac{1}{2}\sqrt{1-\left(\frac{2}{e^2}\right)^{1/p}}, \qquad \mu \geq \frac{1}{2} - \frac{\sqrt{3p}}{6p}.$$

Proof From (1.1)–(1.3) we clearly see that both $\mathbf{GA}_{\theta,p}(\sigma,\tau)$ and $\mathbf{SY}(\sigma,\tau)$ are symmetric and homogenous of degree one with respect to their variables σ and τ. Without loss of generality, we assume that $\sigma > \tau > 0$. Let $0 < \theta < 1/2$ and $x = (\sigma-\tau)/\sqrt{2\sigma\tau} > 0$. Then (1.1)–(1.3) lead to

$$\begin{aligned}
&\mathbf{SY}(\sigma,\tau) = \mathbf{G}(\sigma,\tau)\sqrt{1+x^2}e^{\frac{\arctan(x)}{x}-1},\\
&\mathbf{GA}_{\theta,p}(\sigma,\tau) = \mathbf{G}(\sigma,\tau)\sqrt{1+\frac{x^2}{2}}\left[\frac{(1-(1-2\theta)^2)x^2+2}{x^2+2}\right]^{p/2},\\
&\log\bigl[\mathbf{GA}_{\theta,p}(\sigma,\tau)\bigr] - \log\bigl[\mathbf{SY}(\sigma,\tau)\bigr]\\
&\quad = \frac{p}{2}\log\left[\frac{(1-(1-2\theta)^2)x^2+2}{x^2+2}\right] + \frac{1}{2}\log\left(\frac{x^2+2}{2(x^2+1)}\right) - \frac{\arctan(x)}{x} + 1\\
&\quad = g\bigl((1-2\theta)^2,p;x\bigr),
\end{aligned} \tag{3.1}$$

where $g(\cdot,p;x)$ is defined by (2.8).

Therefore, Theorem 3.1 follows easily from Lemma 2.4 and (3.1). □

4 Applications

Let $p = 1,2$. Then Theorem 3.1 leads to Theorem 4.1 immediately, which provides the sharp bounds for the Sándor–Yang mean in terms of the one-parameter geometric and harmonic means.

Theorem 4.1 *Let $0 < \lambda_1, \lambda_2, \mu_1, \mu_2 < 1/2$, and σ and τ be any two distinct positive real numbers. Then the double inequalities*

$$\mathbf{G}\big[\lambda_1\sigma + (1-\lambda_1)\tau, \lambda_1\tau + (1-\lambda_1)\sigma\big] < \mathbf{SY}(\sigma,\tau) < \mathbf{G}\big[\mu_1\sigma + (1-\mu_1)\tau, \mu_1\tau + (1-\mu_1)\sigma\big]$$

and

$$\mathbf{H}\big[\lambda_2\sigma + (1-\lambda_2)\tau, \lambda_2\tau + (1-\lambda_2)\sigma\big] < \mathbf{SY}(\sigma,\tau) < \mathbf{H}\big[\mu_2\sigma + (1-\mu_2)\tau, \mu_2\tau + (1-\mu_2)\sigma\big]$$

hold if and only if

$$\lambda_1 \le \frac{1}{2} - \frac{1}{2}\sqrt{1-\frac{2}{e^2}} \simeq 0.0730, \qquad \mu_1 \ge \frac{1}{2} - \frac{\sqrt{3}}{6} \simeq 0.2113,$$
$$\lambda_2 \le \frac{1}{2} - \frac{1}{2}\sqrt{1-\frac{\sqrt{2}}{e}} \simeq 0.1537, \qquad \mu_2 \ge \frac{1}{2} - \frac{\sqrt{6}}{12} \simeq 0.2959.$$

Theorem 3.1 and (1.2) also lead to Theorem 4.2, which gives the sharp bounds for the Yang mean in terms of the two-parameter geometric and arithmetic mean and the quadratic and geometric means.

Theorem 4.2 *Let $p \ge 1$, $0 < \alpha, \beta < 1/2$, and σ and τ be any two different positive real numbers. Then the two-sided inequality*

$$\frac{\mathbf{G}(\sigma,\tau)}{\log[\mathbf{GA}_{\alpha,p}(\sigma,\tau)] - \log[\mathbf{Q}(\sigma,\tau)] + 1} < \mathbf{U}(\sigma,\tau) < \frac{\mathbf{G}(\sigma,\tau)}{\log[\mathbf{GA}_{\beta,p}(\sigma,\tau)] - \log[\mathbf{Q}(\sigma,\tau)] + 1}$$

takes place if and only if

$$\alpha \ge \frac{1}{2} - \frac{\sqrt{3p}}{6p}, \qquad \beta \le \frac{1}{2} - \frac{1}{2}\sqrt{1-\left(\frac{2}{e^2}\right)^{1/p}}.$$

Let $\sigma > \tau = 1/2$, $\alpha = 1/2 - \sqrt{3p}/(6p)$, and $\beta = 1/2 - \sqrt{1-(2/e^2)^{1/p}}/2$. Then it follows from (1.1), (1.3) that

$$\mathbf{U}\left(\sigma, \frac{1}{2}\right) = \frac{2\sigma - 1}{2\sqrt{2}\arctan(\frac{2\sigma-1}{2\sqrt{\sigma}})}, \tag{4.1}$$

$$\mathbf{GA}_{1/2-\sqrt{3p}/(6p),p}\left(\sigma, \frac{1}{2}\right) = \left[\frac{4(3p-1)\sigma^2 + 4(3p+1)\sigma + 3p - 1}{48p}\right]^p \left(\frac{2\sigma+1}{4}\right)^{1-p}, \tag{4.2}$$

$$\mathbf{GA}_{1/2-\sqrt{1-(2/e^2)^{1/p}}/2,p}\left(\sigma, \frac{1}{2}\right) = \left[\frac{4\times 2^{1/p}\sigma^2 + 4(2e^{2/p} - 2^{1/p})\sigma + 2^{1/p}}{16e^{2/p}}\right]^p \left(\frac{2\sigma+1}{4}\right)^{1-p}. \tag{4.3}$$

From Theorem 4.2 and (4.1)–(4.3) we obtain Theorem 4.3, which presents new one-parameter bounds for the inverse tangent function.

Theorem 4.3 *The double inequality*

$$\begin{aligned}
&\frac{2\sigma-1}{2\sqrt{\sigma}}\Big[p\log\big(4\times 2^{1/p}\sigma^2+4\big(2e^{2/p}-2^{1/p}\big)\sigma+2^{1/p}\big)\\
&\quad+(1-p)\log(2\sigma+1)-\frac{1}{2}\log\big(4\sigma^2+1\big)-p\log 4-1-\frac{1}{2}\log 2\Big]\\
&<\arctan\left(\frac{2\sigma-1}{2\sqrt{\sigma}}\right)<\frac{2\sigma-1}{2\sqrt{\sigma}}\Big[p\log\big(4(3p-1)\sigma^2+4(3p+1)\sigma+3p-1\big)\\
&\quad+(1-p)\log(2\sigma+1)-\frac{1}{2}\log\big(4\sigma^2+1\big)-p(\log p+\log 3+2\log 2)+1-\frac{1}{2}\log 2\Big]
\end{aligned}$$

holds for all $\sigma>1/2$ *and* $p\geq 1$.

5 Consequences and discussion

In the article, we have given the sharp bounds for the Sándor–Yang mean

$$\mathbf{SY}(\sigma,\tau)=\mathbf{Q}(\sigma,\tau)e^{\mathbf{G}(\sigma,\tau)/\mathbf{U}(\sigma,\tau)-1}$$

in terms of the two-parameter geometric and arithmetic mean

$$\mathbf{GA}_{\eta,\nu}(\sigma,\tau)=\mathbf{G}^{\nu}\big[\eta\sigma+(1-\eta)\tau,\eta\tau+(1-\eta)\sigma\big]\mathbf{A}^{1-\nu}(\sigma,\tau)$$

and the one-parameter geometric and harmonic means

$$\mathbf{G}\big[\lambda\sigma+(1-\lambda)\tau,\lambda\tau+(1-\lambda)\sigma\big]$$

and

$$\mathbf{H}\big[\mu\sigma+(1-\mu)\tau,\mu\tau+(1-\mu)\sigma\big],$$

and have found the new bounds for the Yang mean

$$\mathbf{U}(\sigma,\tau)=\frac{\sqrt{2}(\sigma-\tau)}{2\arctan(\frac{\sqrt{2}(\sigma-\tau)}{2\sqrt{\sigma\tau}})}$$

and the inverse tangent function $\arctan[(2\sigma-1)/(2\sqrt{\sigma})]$.

6 Conclusion

In the article, we have proved that the double inequalities

$$\mathbf{GA}_{\lambda,p}(\sigma,\tau)<\mathbf{SY}(\sigma,\tau)<\mathbf{GA}_{\mu,p}(\sigma,\tau)$$

and

$$\frac{\mathbf{G}(\sigma,\tau)}{\log[\mathbf{GA}_{\mu,p}(\sigma,\tau)]-\log[\mathbf{Q}(\sigma,\tau)]+1}<\mathbf{U}(\sigma,\tau)<\frac{\mathbf{G}(\sigma,\tau)}{\log[\mathbf{GA}_{\lambda,p}(\sigma,\tau)]-\log[\mathbf{Q}(\sigma,\tau)]+1}$$

are valid for all distinct positive real numbers σ and τ if and only if

$$\lambda \le \frac{1}{2} - \frac{1}{2}\sqrt{1 - \left(\frac{2}{e^2}\right)^{1/p}}, \qquad \mu \ge \frac{1}{2} - \frac{\sqrt{3p}}{6p}.$$

if $p \ge 1$ and $\lambda, \mu \in (0, 1/2)$.

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All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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