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A note on generalized convex functions

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Abstract

In the article, we provide an example for a η-convex function defined on rectangle is not convex, prove that every η-convex function defined on rectangle is coordinate η-convex and its converse is not true in general, define the coordinate (η_1,η_2)-convex function and establish its Hermite–Hadamard type inequality.

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1 Introduction

Let $I\subseteq\mathbb{R}$ be an interval. Then a real-valued function $\Psi: I\mapsto\mathbb{R}$ is said to be convex on I if the inequality

$$\Psi\left[\lambda a+(1-\lambda)b\right]\le\lambda\Psi(a)+(1-\lambda)\Psi(b) \tag{1.1}$$

holds for all $a,b\in I$ and $\lambda\in(0,1)$. Ψ is said to be concave if inequality (1.1) is reversed.

It is well known that the convexity theory has wide applications in special functions [1–30], differential equations [31–61] and bivariate means [62–67]. Recently, the extensions, generalizations, refinements and variants for the convexity have attracted the attention of many researchers. For example, Schur convexity [68–70], GA-convexity [71], GG-convexity [72], s-convexity [73, 74], preinvexity [75], strong convexity [76–79] and others [80–85].

Dragomir [86] defined the coordinate convex as follows.

Definition 1.1 (See [86]) Let $I_1,I_2\subseteq\mathbb{R}$ be two interval, $\Psi: I_1\times I_2\mapsto\mathbb{R}$ be a real-valued function, and the partial mappings $\Psi_y: I_1\mapsto\mathbb{R}$ and $\Psi_x: I_2\mapsto\mathbb{R}$ be defined by

$$\Psi_y(u)=\Psi(u,y),\qquad \Psi_x(v)=\Psi(x,v),$$

respectively. Then Ψ is said to be coordinate convex on $I_1\times I_2$ if Ψ_y is convex on I_1 for all $y\in I_2$ and Ψ_x is convex on I_2 for all $x\in I_1$.

Remark 1.2 Dragomir [86] proved that every convex function is coordinate convex, but not vice versa.

Next, we recall the concept of η-convexity which can be found in the literature [87].

Definition 1.3 (See [87]) Let $I \subseteq \mathbb{R}$ be an interval, and $\Psi : I \mapsto \mathbb{R}$ and $\eta : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ be two real-valued functions. Then Ψ is said to be η-convex if the inequality

$$\Psi\left[\mu x + (1-\mu)y\right] \leq \Psi(y) + \mu\eta\left[\Psi(x), \Psi(y)\right]$$

holds for all $x, y \in I$ and $\mu \in [0,1]$.

Note that the η-convexity reduces to the usual convexity if $\eta(x,y) = x - y$ in Definition 1.3.

The main purpose of the article is to give a non-trivial example for a η-convex function defined on rectangle is not convex, prove that every η-convex function defined on rectangle is coordinate η-convex but not vice versa, define the coordinate (η_1, η_2)-convex function and establish its Hermite–Hadamard type inequality.

2 Main results

To begin this section, it is interesting to give the definition of η-convex function defined on rectangle, and give a non-trivial example for a η-convex function defined on rectangle is not convex.

Definition 2.1 Let $I_1, I_2 \subseteq \mathbb{R}$ be two intervals, and $\Psi : I_1 \times I_2 \mapsto \mathbb{R}$ and $\eta : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ be two real-valued functions. Then Ψ is said to be η-convex if the inequality

$$\Psi\left[\mu x + (1-\mu)z, \mu y + (1-\mu)w\right] \leq \Psi(z,w) + \mu\eta\left[\Psi(x,y), \Psi(z,w)\right]$$

holds for all $(x,y), (z,w) \in I_1 \times I_2$ and $\mu \in [0,1]$.

Example 2.2 Let $\Psi : [1,5] \times [1,5] \mapsto \mathbb{R}$ and $\eta : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ be defined by

$$\Psi(x,y) = x^2y^2, \qquad \eta(x,y) = 104x + 103y.$$

Then Ψ is η-convex on $[1,5] \times [1,5]$, but it is not convex.

Proof Let $\mu \in [0,1]$. Then for any $(x,y), (z,w) \in [1,5]$ we have

$$\begin{aligned}
&\Psi\left[\mu x + (1-\mu)z, \mu y + (1-\mu)w\right] \\
&\quad= \left[\mu x + (1-\mu)z\right]^2\left[\mu y + (1-\mu)w\right]^2 \\
&\quad= \left[z^2 + \mu\left(\mu x^2 + \mu z^2 - 2z^2\right) + 2\mu(1-\mu)xz\right] \\
&\qquad\times \left[w^2 + \mu\left(\mu y^2 + \mu w^2 - 2w^2\right) + 2\mu(1-\mu)yw\right] \\
&\quad\leq \left[z^2 + \mu x^2 + 2\mu(1-\mu)xz\right]\left[w^2 + \mu y^2 + 2\mu(1-\mu)yw\right] \\
&\quad\leq \left[z^2 + \mu x^2 + \mu(1-\mu)\left(x^2+z^2\right)\right]\left[w^2 + \mu y^2 + \mu(1-\mu)\left(y^2+w^2\right)\right] \\
&\quad\leq \left[z^2 + \mu\left(x^2 + x^2 + z^2\right)\right]\left[w^2 + \mu\left(y^2 + y^2 + w^2\right)\right] \\
&\quad= z^2w^2 + \mu\left[2y^2z^2 + z^2w^2 + 2x^2w^2 + w^2z^2\right] + \mu^2\left[4x^2y^2 + 2x^2w^2 + 2y^2z^2 + z^2w^2\right] \\
&\quad\leq \Psi(z,w) + \mu\left[2y^2z^2 + z^2w^2 + 2x^2w^2 + w^2z^2\right] + \mu\left[4x^2y^2 + 2x^2w^2 + 2y^2z^2 + z^2w^2\right] \\
&\quad= \Psi(z,w) + \mu\left[4x^2y^2 + 3z^2w^2 + 4\left(z^2y^2 + x^2w^2\right)\right]. \qquad (2.1)
\end{aligned}$$

Note that

$$z^2 \leq 25x^2, \qquad x^2 \leq 25z^2. \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$\begin{aligned}
&\Psi\bigl[\mu x + (1-\mu)z, \mu y + (1-\mu)w\bigr] \\
&\quad \leq \Psi(z,w) + \mu\bigl[104x^2y^2 + 103z^2w^2\bigr] \\
&\quad = \Psi(z,w) + \mu\eta\bigl[\Psi(x,y), \Psi(z,w)\bigr],
\end{aligned}$$

which shows that Ψ is η-convex on $[1,5] \times [1,5]$. It is easily to verify that Ψ is not convex on $[1,5] \times [1,5]$, for details see [79]. □

Next, we introduce the definition of coordinate (η_1, η_2)-convexity.

Definition 2.3 Let $I_1, I_2 \subseteq \mathbb{R}$ be two intervals, $\Psi : I_1 \times I_2 \mapsto \mathbb{R}$, $\eta_1, \eta_2 : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ be three real-valued functions, and the partial mappings $\Psi_y : I_1 \mapsto \mathbb{R}$ and $\Psi_x : I_2 \mapsto \mathbb{R}$ be defined by

$$\Psi_y(u) = \Psi(u,y), \qquad \Psi_x(v) = \Psi(x,v).$$

Then Ψ is said to be coordinate (η_1, η_2)-convex on $I_1 \times I_2$ if Ψ_y is η_1-convex on I_1 and Ψ_x is η_2-convex on I_2. In particular, if $\eta_1 = \eta_2 = \eta$, then Ψ is said to be coordinate η-convex.

Example 2.4 Let $\Psi : [0,\infty) \times [0,\infty) \mapsto \mathbb{R}$ be defined by $\Psi(x,y) = -|x| - y^2$, $\eta_1(x,y) = -x - y$ and $\eta_2(x,y) = -x - 2y$. Then Ψ is coordinate (η_1, η_2)-convex on $[0,\infty) \times [0,\infty)$.

Proof Let $x_1, y_1 \in [0,\infty)$ and $\mu \in [0,1]$. Then for any $(x,y) \in [0,\infty)$ we clearly see that

$$\Psi_y\bigl(\mu x_1 + (1-\mu)x_2\bigr) = -\bigl|\mu x_1 + (1-\mu)x_2\bigr| - y^2, \tag{2.3}$$

$$\begin{aligned}
&\Psi_y(x_2) + \mu\eta_1\bigl(\Psi_y(x_1), \Psi_y(x_2)\bigr) \\
&\quad = -|x_2| - y^2 + \mu\eta_1\bigl(-|x_1| - y^2, -|x_2| - y^2\bigr) \\
&\quad = -|x_2| - y^2 + \mu\bigl(|x_1| + |x_2| + 2y^2\bigr),
\end{aligned} \tag{2.4}$$

$$\Psi_x\bigl(\mu y_1 + (1-\mu)y_2\bigr) = -|x| - \bigl(\mu y_1 + (1-\mu)y_2\bigr)^2, \tag{2.5}$$

$$\begin{aligned}
&\Psi_x(y_2) + \mu\eta_2\bigl(\Psi_x(y_1), \Psi_x(y_2)\bigr) \\
&\quad = -|x| - y_2^2 + \mu\eta_2\bigl(-|x| - y_1^2, -|x| - y_2^2\bigr) \\
&\quad = -|x| - y_2^2 + \mu\bigl(y_1^2 + 2y_2^2 + 3|x|\bigr).
\end{aligned} \tag{2.6}$$

It follows from (2.3)–(2.6) that

$$\begin{aligned}
&\Psi_y(x_2) + \mu\eta_1\bigl(\Psi_y(x_1), \Psi_y(x_2)\bigr) - \Psi_y\bigl(\mu x_1 + (1-\mu)x_2\bigr) \\
&\quad = \mu\bigl(|x_1| + |x_2| + 2y^2\bigr) + \bigl|\mu x_1 + (1-\mu)x_2\bigr| - |x_2| \\
&\quad \geq 2\mu y^2 + \mu|x_1| + \mu|x_2| + (1-\mu)|x_2| - \mu|x_1| - |x_2|
\end{aligned}$$

$$= 2\mu y^2 \geq 0, \tag{2.7}$$

$$\begin{aligned} &\Psi_x(y_2) + \mu\eta_2\big(\Psi_x(y_1), \Psi_x(y_2)\big) - \Psi_x\big(\mu y_1 + (1-\mu)y_2\big) \\ &\quad = 3\mu|x| + 2\mu(1-\mu)y_1y_2 + \mu(1+\mu)y_1^2 + \mu^2y_2^2 \geq 0. \end{aligned} \tag{2.8}$$

Therefore, Ψ is coordinate (η_1, η_2)-convex on $[0,\infty) \times [0,\infty)$ follows from (2.7) and (2.8). □

Theorem 2.5 *Let $I_1, I_2 \subseteq \mathbb{R}$ be two interval and $\eta : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ be a real-valued function. Then Ψ is coordinate η-convex on $I_1 \times I_2$ if Ψ is η-convex on $I_1 \times I_2$.*

Proof Let $(x, y) \in I_1 \times I_2$, $u, v \in I_2$ and $z, w \in I_1$. Then it follows from the η-convexity of the function Ψ on $I_1 \times I_2$ that

$$\begin{aligned} \Psi_x\big(\mu v + (1-\mu)u\big) &= \Psi\big(x, \mu v + (1-\mu)u\big) \\ &= \Psi\big(\mu x + (1-\mu)x, \mu v + (1-\mu)u\big) \\ &\leq \Psi(x,u) + \mu\eta\big(\Psi(x,v), \Psi(x,u)\big) \\ &= \Psi_x(u) + \mu\eta\big(\Psi_x(v), \Psi_x(u)\big) \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} \Psi_y\big(\mu z + (1-\mu)w\big) &= \Psi\big(\mu z + (1-\mu)w, y\big) \\ &= \Psi\big(\mu z + (1-\mu)w, \mu y + (1-\mu)y\big) \\ &\leq \Psi(w,y) + \mu\eta\big(\Psi(z,y), \Psi(w,y)\big) \\ &= \Psi_y(w) + \mu\eta\big(\Psi_y(z), \Psi_y(w)\big). \end{aligned} \tag{2.10}$$

Inequalities (2.9) and (2.10) imply that Ψ_x is η-convex on I_2 and Ψ_y is η-convex on I_1. Therefore, Ψ is coordinate η-convex on $I_1 \times I_2$. □

Example 2.6 Let $I_1 = I_2 = [0,\infty)$, $\Psi, \eta : I_1 \times I_2 \mapsto [0,\infty)$ be defined by

$$\Psi(x,y) = xy, \qquad \eta(x,y) = x + y. \tag{2.11}$$

Then Ψ is coordinate η-convex on $I_1 \times I_2$ but it is not η-convex on $I_1 \times I_2$.

Proof Let $x, y, u, v, z, w \in [0,\infty)$ and $\mu \in [0,1]$. Then it follows from (2.11) that

$$\begin{aligned} \Psi_x\big(\mu u + (1-\mu)v\big) &= \Psi\big(x, \mu u + (1-\mu)v\big) \\ &= x\big(\mu u + (1-\mu)v\big) = -\mu xv + x(\mu u + v), \end{aligned} \tag{2.12}$$

$$\begin{aligned} \Psi(x,v) + \mu\eta\big(\Psi(x,u), \Psi(x,v)\big) &= xv + \mu\eta(xu, xv) \\ &= xv + \mu(xu + xv) = \mu xv + x(\mu u + v), \end{aligned} \tag{2.13}$$

$$\Psi_y\big(\mu z + (1-\mu)w\big) = \Psi\big(\mu z + (1-\mu)w, y\big)$$

$$= y\big(\mu z + (1-\mu)w\big) = -\mu y w + y(\mu z + w), \tag{2.14}$$

$$\begin{aligned} \Psi(w,y) + \mu\eta\big(\Psi(z,y), \Psi(w,y)\big) &= wy + \mu\eta(zy, wy) \\ &= wy + \mu(zy + wy) = \mu y w + y(\mu z + w). \end{aligned} \tag{2.15}$$

Inequalities (2.12)–(2.15) imply that

$$\Psi_x\big(\mu u + (1-\mu)v\big) \le \Psi(x,v) + \mu\eta\big(\Psi(x,u), \Psi(x,v)\big) \tag{2.16}$$

and

$$\Psi_y\big(\mu z + (1-\mu)w\big) \le \Psi(w,y) + \mu\eta\big(\Psi(z,y), \Psi(w,y)\big). \tag{2.17}$$

Note that

$$\Psi_x\big(\mu u + (1-\mu)v\big) = \Psi\big(\mu x + (1-\mu)x, \mu u + (1-\mu)v\big) \tag{2.18}$$

and

$$\Psi_y\big(\mu z + (1-\mu)w\big) = \Psi\big(\mu z + (1-\mu)w, \mu y + (1-\mu)y\big). \tag{2.19}$$

Therefore, Ψ is coordinate η-convex on $I_1 \times I_2$ follows from (2.16)–(2.19).

Next, we prove that Ψ is not η-convex on $I_1 \times I_2$.

Let $\mu \in (0,1)$, $x = w = 1$ and $y = z = 0$. Then (2.11) leads to

$$\begin{aligned} &\Psi\big(\mu x + (1-\mu)z, \mu y + (1-\mu)w\big) \\ &\quad = \Psi(\mu, 1-\mu) = \mu(1-\mu) > 0, \end{aligned} \tag{2.20}$$

$$\begin{aligned} &\Psi(z,w) + \mu\eta\big(\Psi(x,y), \Psi(z,w)\big) \\ &\quad = \Psi(0,1) + \mu\eta\big(\Psi(1,0), \Psi(0,1)\big) = 0. \end{aligned} \tag{2.21}$$

From (2.20) and (2.21) we clearly see that Ψ is not η-convex on $I_1 \times I_2$. □

Next, we establish a Hermite–Hadamard type inequality for the coordinate (η_1, η_2)-convex function.

Theorem 2.7 *Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, $\Psi : [a,b] \times [c,d] \mapsto \mathbb{R}$, $\eta_1, \eta_2 : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ be three real-valued functions such that Ψ is coordinate (η_1, η_2)-convex on $[a,b] \times [c,d]$ and*

$$\eta_1(x,y) \le M_{\eta_1}, \qquad \eta_2(x,y) \le M_{\eta_2}$$

for all $x, y \in \mathbb{R}$, where M_{η_1} and M_{η_2} are two positive constants. Then

$$\begin{aligned}
&\Psi\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{M_{\eta_1}+M_{\eta_2}}{2}\\
&\quad\le\frac{1}{2}\left[\frac{1}{b-a}\int_a^b\Psi\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d\Psi\left(\frac{a+b}{2},y\right)dy\right]-\frac{M_{\eta_1}+M_{\eta_2}}{4}\\
&\quad\le\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b\Psi(x,y)\,dx\,dy\\
&\quad\le\frac{1}{4}\left[\frac{1}{b-a}\int_a^b\bigl(\Psi(x,c)+\Psi(x,d)\bigr)dx+\frac{1}{d-c}\int_c^d\bigl(\Psi(a,y)+\Psi(b,y)\bigr)dy\right]\\
&\qquad+\frac{M_{\eta_1}+M_{\eta_2}}{4}\\
&\quad\le\frac{1}{4}\bigl[\Psi(a,c)+\Psi(b,c)+\Psi(a,d)+\Psi(b,d)\bigr]+\frac{5}{4}[M_{\eta_1}+M_{\eta_2}]. \qquad (2.22)
\end{aligned}$$

Proof For any fixed $x \in [a,b]$, $\Psi_x(y) = \Psi(x,y)$ is η_2-convex on $[c,d]$ due to Ψ is coordinate (η_1,η_2)-convex on $[a,b]\times[c,d]$. It follows from [77, Theorem 5] that

$$\Psi\left(x,\frac{c+d}{2}\right)-\frac{M_{\eta_2}}{2}\le\frac{1}{d-c}\int_c^d\Psi(x,y)\,dy\le\frac{\Psi(x,c)+\Psi(x,d)}{2}+\frac{M_{\eta_2}}{2}. \qquad (2.23)$$

Integrating each side of inequality (2.23) with respect to the variable x on $[a,b]$ leads to

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b\Psi\left(x,\frac{c+d}{2}\right)dx-\frac{M_{\eta_2}}{2}\\
&\quad\le\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b\Psi(x,y)\,dx\,dy\\
&\quad\le\frac{1}{2(b-a)}\int_a^b\bigl[\Psi(x,c)+\Psi(x,d)\bigr]dx+\frac{M_{\eta_2}}{2}. \qquad (2.24)
\end{aligned}$$

By similar arguments we have

$$\begin{aligned}
&\frac{1}{d-c}\int_c^d\Psi\left(\frac{a+b}{2},y\right)dy-\frac{M_{\eta_1}}{2}\\
&\quad\le\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b\Psi(x,y)\,dx\,dy\\
&\quad\le\frac{1}{2(d-c)}\int_c^d\bigl[\Psi(a,y)+\Psi(b,y)\bigr]dy+\frac{M_{\eta_1}}{2}. \qquad (2.25)
\end{aligned}$$

Adding (2.24) and (2.25) we get the second and third inequalities of (2.22).

Making use of the (η_1,η_2)-convexity of the function Ψ on $[a,b]\times[c,d]$ and [88, Theorem 5] again we get

$$\Psi\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{M_{\eta_2}}{2}\le\frac{1}{b-a}\int_a^b\Psi\left(x,\frac{c+d}{2}\right)dx, \qquad (2.26)$$

$$\Psi\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{M_{\eta_1}}{2}\le\frac{1}{d-c}\int_c^d\Psi\left(\frac{a+b}{2},y\right)dy, \qquad (2.27)$$

$$\frac{1}{b-a}\int_a^b \Psi(x,c)\,dx \le \frac{\Psi(a,c)+\Psi(b,c)}{2} + \frac{M_{\eta_2}}{2}, \tag{2.28}$$

$$\frac{1}{b-a}\int_a^b \Psi(x,d)\,dx \le \frac{\Psi(a,d)+\Psi(b,d)}{2} + \frac{M_{\eta_2}}{2}, \tag{2.29}$$

$$\frac{1}{d-c}\int_c^d \Psi(a,y)\,dy \le \frac{\Psi(a,c)+\Psi(a,d)}{2} + \frac{M_{\eta_1}}{2}, \tag{2.30}$$

$$\frac{1}{d-c}\int_c^d \Psi(b,y)\,dy \le \frac{\Psi(b,c)+\Psi(b,d)}{2} + \frac{M_{\eta_1}}{2}. \tag{2.31}$$

Therefore, the first inequality of (2.22) follows from (2.26) and (2.27) with adding $-\frac{1}{2}M_{\eta_2}$ and $-\frac{1}{2}M_{\eta_1}$ respectively, and the last inequality in (2.22) can be derived from (2.28)–(2.31) immediately, with adding $\frac{1}{4}[M_{\eta_1}+M_{\eta_2}]$. □

3 Results and discussion

In the article, we establish a non-trivial example for a η-convex function defined on rectangle is not convex, prove that every η-convex function defined on rectangle is coordinate η-convex and its converse is not true in general. Furthermore, we define a new class of function which is coordinate (η_1,η_2)-convex function and prove its well-known Hermite–Hadamard type inequality.

4 Conclusion

We find an example for η-convex function defined on rectangle is not convex. The authors define a coordinate (η_1,η_2)-convex function and prove its results. Our approach may have further applications in the theory of η-convexity.

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References

1. Guessab, A., Schmeisser, G.: Sharp integral inequalities of the Hermite–Hadamard type. J. Approx. Theory **115**(2), 260–288 (2002)
2. Shi, H.-P., Zhang, H.-Q.: Existence of gap solitons in periodic discrete nonlinear Schrödinger equations. J. Math. Anal. Appl. **361**(2), 411–419 (2010)
3. Zhou, W.-J., Zhang, L.: Convergence of a regularized factorized quasi-Newton method for nonlinear least squares problems. Comput. Appl. Math. **29**(2), 195–214 (2010)
4. Yang, X.-S., Zhu, Q.-X., Huang, C.-X.: Generalized lag-synchronization of chaotic mix-delayed systems with uncertain parameters and unknown perturbations. Nonlinear Anal., Real World Appl. **12**(1), 93–105 (2011)

5. Zhu, Q.-X., Huang, C.-X., Yang, X.-S.: Exponential stability for stochastic jumping BAM neural networks with time-varying and distributed delays. Nonlinear Anal. Hybrid Syst. **5**(1), 52–77 (2011)
6. Dai, Z.-F., Wen, F.-H.: A modified CG-DESCENT method for unconstrained optimization. J. Comput. Appl. Math. **235**(11), 3332–3341 (2011)
7. Gou, K., Sun, B.: Numerical solution of the Goursat problem on a triangular domain with mixed boundary conditions. Appl. Math. Comput. **217**(21), 8765–8777 (2011)
8. Lin, L., Liu, Z.-Y.: An alternating projected gradient algorithm for nonnegative matrix factorization. Appl. Math. Comput. **217**(24), 9997–10002 (2011)
9. Zhang, L., Li, J.-L.: A new globalization technique for nonlinear conjugate gradient methods for nonconvex minimization. Appl. Math. Comput. **217**(24), 10295–10304 (2011)
10. Xiao, C.-E., Liu, J.-B., Liu, Y.-L.: An inverse pollution problem in porous media. Appl. Math. Comput. **218**(7), 3649–3653 (2011)
11. Huang, C.-X., Liu, L.-Z.: Sharp function inequalities and boundness for Toeplitz type operator related to general fractional singular integral operator. Publ. Inst. Math. **92**(106), 165–176 (2012)
12. Zhou, W.-J.: On the convergence of the modified Levenberg–Marquardt method with a nonmonotone second order Armijo type line search. J. Comput. Appl. Math. **239**, 152–161 (2013)
13. Zhang, L., Jian, S.-Y.: Further studies on the Wei–Yao–Liu nonlinear conjugate gradient method. Appl. Math. Comput. **219**(14), 7616–7621 (2013)
14. Li, X.-F., Tang, G.-J., Tang, B.-Q.: Stress field around a strike-slip fault in orthotropic elastic layers via a hypersingular integral equation. Comput. Math. Appl. **66**(11), 2317–2326 (2013)
15. Qin, G.-X., Huang, C.-X., Xie, Y.-Q., Wen, F.-H.: Asymptotic behavior for third-order quasi-linear differential equations. Adv. Differ. Equ. **2013**, Article ID 305 (2013)
16. Zhou, W.-J., Chen, X.-L.: On the convergence of a modified regularized Newton method for convex optimization with singular solutions. J. Comput. Appl. Math. **239**, 179–188 (2013)
17. Wang, M.-K., Chu, Y.-M.: Asymptotical bounds for complete elliptic integrals of the second kind. J. Math. Anal. Appl. **402**(1), 119–126 (2013)
18. Wang, G.-D., Zhang, X.-H., Chu, Y.-M.: A power mean inequality involving the complete elliptic integrals. Rocky Mt. J. Math. **44**(5), 1661–1667 (2014)
19. Wang, M.-K., Chu, Y.-M., Jiang, Y.-P.: Ramanujan's cubic transformation inequalities for zero-balanced hypergeometric functions. Rocky Mt. J. Math. **46**(2), 679–691 (2016)
20. Yang, Z.-H., Qian, W.-M., Chu, Y.-M., Zhang, W.: Monotonicity rule for the quotient of two functions and its application. J. Inequal. Appl. **2017**, Article ID 106 (2017)
21. Yang, Z.-H., Qian, W.-M., Chu, Y.-M., Zhang, W.: On rational bounds for the gamma function. J. Inequal. Appl. **2017**, Article ID 210 (2017)
22. Huang, T.-R., Han, B.-W., Ma, X.-Y., Chu, Y.-M.: Optimal bounds for the generalized Euler–Mascheroni constant. J. Inequal. Appl. **2018**, Article ID 118 (2018)
23. Huang, T.-R., Tan, S.-Y., Ma, X.-Y., Chu, Y.-M.: Monotonicity properties and bounds for the complete p-elliptic integrals. J. Inequal. Appl. **2018**, Article ID 239 (2018)
24. Yang, Z.-H., Qian, W.-M., Chu, Y.-M.: Monotonicity properties and bounds involving the complete elliptic integrals of the first kind. Math. Inequal. Appl. **21**(4), 1185–1199 (2018)
25. Yang, Z.-H., Chu, Y.-M., Zhang, W.: High accuracy asymptotic bounds for the complete elliptic integral of the second kind. Appl. Math. Comput. **348**, 552–564 (2019)
26. Qiu, S.-L., Ma, X.-Y., Chu, Y.-M.: Sharp Landen transformation inequalities for hypergeometric functions, with applications. J. Math. Anal. Appl. **474**(2), 1306–1337 (2019)
27. Wang, M.-K., Chu, Y.-M., Zhang, W.: Monotonicity and inequalities involving zero-balanced hypergeometric function. Math. Inequal. Appl. **22**(2), 601–617 (2019)
28. Wang, M.-K., Chu, Y.-M., Zhang, W.: Precise estimates for the solution of Ramanujan's generalized modular equation. Ramanujan J. **49**(3), 653–668 (2019)
29. Wang, M.-K., Zhang, W., Chu, Y.-M.: Monotonicity, convexity and inequalities involving the generalized elliptic integrals. Acta Math. Sci. **39B**(5), 1440–1450 (2019)
30. Wang, M.-K., Chu, H.-H., Chu, Y.-M.: Precise bounds for the weighted Hölder mean of the complete p-elliptic integrals. J. Math. Anal. Appl. https://doi.org/10.1016/j.jmaa.2019.123388
31. Huang, C.-X., Yang, Z.-C., Yi, T.-S., Zou, X.-F.: On the basins of attraction for a class of delay differential equations with non-monotone bistable nonlinearities. J. Differ. Equ. **256**(7), 2101–2114 (2014)
32. Tang, W.-S., Sun, Y.-J.: Construction of Runge–Kutta type methods for solving ordinary differential equations. Appl. Math. Comput. **234**, 179–191 (2014)
33. Huang, C.-X., Guo, S., Liu, L.-Z.: Boundedness on Morrey space for Toeplitz type operator associated to singular integral operator with variable Calderón–Zygmund kernel. J. Math. Inequal. **8**(3), 453–464 (2014)
34. Xie, D.-X., Li, J.: A new analysis of electrostatic free energy minimization and Poisson–Boltzmann equation for protein in ionic solvent. Nonlinear Anal., Real World Appl. **21**, 185–196 (2015)
35. Zhou, W.-J., Wang, F.: A PRP-based residual method for large-scale monotone nonlinear equations. Appl. Math. Comput. **261**, 1–7 (2015)
36. Dai, Z.-F., Chen, X.-H., Wen, F.-H.: A modified Perry's conjugate gradient method-based derivative-free method for solving large-scale nonlinear monotone equations. Appl. Math. Comput. **270**, 378–386 (2015)
37. Liu, Y.-C., Wu, J.: Multiple solutions of ordinary differential systems with min-max terms and applications to the fuzzy differential equations. Adv. Differ. Equ. **2015**, Article ID 379 (2015)
38. Fang, X.-P., Deng, Y.-J., Li, J.: Plasmon resonance and heat generation in nanostructures. Math. Methods Appl. Sci. **38**(18), 4663–4672 (2015)
39. Dai, Z.-F.: Comments on a new class of nonlinear conjugate gradient coefficients with global convergence properties. Appl. Math. Comput. **276**, 297–300 (2016)
40. Li, J.-L., Sun, G.-Y., Zhang, R.-M.: The numerical solution of scattering by infinite rough interfaces based on the integral equation method. Comput. Math. Appl. **71**(7), 1491–1502 (2016)

41. Tan, Y.-X., Jing, K.: Existence and global exponential stability of almost periodic solution for delayed competitive neural networks with discontinuous activations. Math. Methods Appl. Sci. **39**(11), 2821–2839 (2016)
42. Duan, L., Huang, C.-X.: Existence and global attractivity of almost periodic solutions for a delayed differential neoclassical growth model. Math. Methods Appl. Sci. **40**(3), 814–822 (2017)
43. Duan, L., Huang, L.-H., Guo, Z.-Y., Fang, X.-W.: Periodic attractor for reaction-diffusion high-order Hopfield neural networks with time-varying delays. Comput. Math. Appl. **73**(2), 233–245 (2017)
44. Wang, W.-S., Chen, Y.-Z.: Fast numerical valuation of options with jump under Merton's model. J. Comput. Appl. Math. **318**, 79–92 (2017)
45. Huang, C.-X., Liu, L.-Z.: Boundedness of multilinear singular integral operator with a non-smooth kernel and mean oscillation. Quaest. Math. **40**(3), 295–312 (2017)
46. Hu, H.-J., Liu, L.-Z.: Weighted inequalities for a general commutator associated to a singular integral operator satisfying a variant of Hörmander's condition. Math. Notes **101**(5–6), 830–840 (2017)
47. Cai, Z.-W., Huang, J.-H., Huang, L.-H.: Generalized Lyapunov–Razumikhin method for retarded differential inclusions: applications to discontinuous neural networks. Discrete Contin. Dyn. Syst. **22B**(9), 3591–3614 (2017)
48. Wang, W.-S.: On A-stable one-leg methods for solving nonlinear Volterra functional differential equations. Appl. Math. Comput. **314**, 380–390 (2017)
49. Hu, H.-J., Zou, X.-F.: Existence of an extinction wave in the Fisher equation with a shifting habitat. Proc. Am. Math. Soc. **145**(11), 4763–4771 (2017)
50. Tan, Y.-X., Huang, C.-X., Sun, B., Wang, T.: Dynamics of a class of delayed reaction-diffusion systems with Neumann boundary condition. J. Math. Anal. Appl. **458**(2), 1115–1130 (2018)
51. Tang, W.-S., Zhang, J.-J.: Symplecticity-preserving continuous-stage Runge–Kutta–Nyström methods. Appl. Math. Comput. **323**, 204–219 (2018)
52. Duan, L., Fang, X.-W., Huang, C.-X.: Global exponential convergence in a delayed almost periodic Nicholson's blowflies model with discontinuous harvesting. Math. Methods Appl. Sci. **41**(5), 1954–1965 (2018)
53. Liu, Z.-Y., Wu, N.-C., Qin, X.-R., Zhang, Y.-L.: Trigonometric transform splitting methods for real symmetric Toeplitz systems. Comput. Math. Appl. **75**(8), 2782–2794 (2018)
54. Huang, C.-X., Qiao, Y.-C., Huang, L.-H., Agarwal, R.P.: Dynamical behaviors of a food-chain model with stage structure and time delays. Adv. Differ. Equ. **2018**, Article ID 186 (2018)
55. Cai, Z.-W., Huang, J.-H., Huang, L.-H.: Periodic orbit analysis for the delayed Filippov system. Proc. Am. Math. Soc. **146**(11), 4667–4682 (2018)
56. Wang, J.-F., Chen, X.-Y., Huang, L.-H.: The number and stability of limit cycles for planar piecewise linear systems of node-saddle type. J. Math. Anal. Appl. **469**(1), 405–427 (2019)
57. Wang, J.-F., Huang, C.-X., Huang, L.-H.: Discontinuity-induced limit cycles in a general planar piecewise linear system of saddle-focus type. Nonlinear Anal. Hybrid Syst. **33**, 162–178 (2019)
58. Jiang, Y.-J., Xu, X.-J.: A monotone finite volume method for time fractional Fokker–Planck equations. Sci. China Math. **62**(4), 783–794 (2019)
59. Peng, J., Zhang, Y.: Heron triangles with figurate number sides. Acta Math. Hung. **157**(2), 478–488 (2019)
60. Tian, Z.-L., Liu, Y., Zhang, Y., Liu, Z.-Y., Tian, M.-Y.: The general inner-outer iteration method based on regular splittings for the PageRank problem. Appl. Math. Comput. **356**, 479–501 (2019)
61. Wang, W.-S., Chen, Y.-Z., Fang, H.: On the variable two-step IMEX BDF method for parabolic integro-differential equations with nonsmooth initial data arising in finance. SIAM J. Numer. Anal. **57**(3), 1289–1317 (2019)
62. Chu, Y.-M., Wang, M.-K., Qiu, S.-L.: Optimal combinations bounds of root-square and arithmetic means for Toader mean. Proc. Indian Acad. Sci. Math. Sci. **122**(1), 41–51 (2012)
63. Zhao, T.-H., Zhou, B.-C., Wang, M.-K., Chu, Y.-M.: On approximating the quasi-arithmetic mean. J. Inequal. Appl. **2019**, Article ID 42 (2019)
64. Wang, J.-L., Qian, W.-M., He, Z.-Y., Chu, Y.-M.: On approximating the Toader mean by other bivariate means. J. Funct. Spaces **2019**, Article ID 6082413 (2019)
65. Qian, W.-M., Xu, H.-Z., Chu, Y.-M.: Improvements of bounds for the Sándor–Yang means. J. Inequal. Appl. **2019**, Article ID 73 (2019)
66. He, X.-H., Qian, W.-M., Xu, H.-Z., Chu, Y.-M.: Sharp power mean bounds for two Sándor–Yang means. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **113**(3), 2627–2638 (2019)
67. Qian, W.-M., He, Z.-Y., Zhang, H.-W., Chu, Y.-M.: Sharp bounds for Neuman means in terms of two-parameter contraharmonic and arithmetic mean. J. Inequal. Appl. **2019**, Article ID 168 (2019)
68. Chu, Y.-M., Wang, G.-D., Zhang, X.-H.: The Schur multiplicative and harmonic convexities of the complete symmetric function. Math. Nachr. **284**(5–6), 653–663 (2011)
69. Chu, Y.-M., Xia, W.-F., Zhang, X.-H.: The Schur concavity, Schur multiplicative and harmonic convexities of the second dual form of the Hamy symmetric function with applications. J. Multivar. Anal. **105**, 412–421 (2012)
70. Wu, S.-H., Chu, Y.-M.: Schur m-power convexity of generalized geometric Bonferroni mean involving three parameters. J. Inequal. Appl. **2019**, Article ID 57 (2019)
71. Zhang, X.-M., Chu, Y.-M., Zhang, X.-H.: The Hermite–Hadamard type inequality of GA-convex functions and its applications. J. Inequal. Appl. **2010**, Article ID 507560 (2010)
72. Khurshid, Y., Adil Khan, M., Chu, Y.-M.: Conformable integral inequalities of the Hermite–Hadamard type in terms of GG- and GA-convexities. J. Funct. Spaces **2019**, Article ID 6926107 (2019)
73. Adil Khan, M., Chu, Y.-M., Khan, T.U., Khan, J.: Some new inequalities of Hermite–Hadamard type for s-convex functions with applications. Open Math. **15**(1), 1414–1430 (2017)
74. Adil Khan, M., Hanif, M., Khan, Z.A., Ahmad, K., Chu, Y.-M.: Association of Jensen's inequality for s-convex function with Csiszár divergence. J. Inequal. Appl. **2019**, Article ID 162 (2019)
75. Khurshid, Y., Adil Khan, M., Chu, Y.-M., Khan, Z.A.: Hermite–Hadamard–Fejér inequalities for conformable fractional integrals via preinvex functions. J. Funct. Spaces **2019**, Article ID 3146210 (2019)
76. Song, Y.-Q., Adil Khan, M., Zaheer Ullah, S., Chu, Y.-M.: Integral inequalities involving strongly convex functions. J. Funct. Spaces **2018**, Article ID 6595921 (2018)
77. Zaheer Ullah, S., Adil Khan, M., Chu, Y.-M.: Majorization theorems for strongly convex functions. J. Inequal. Appl. **2019**, Article ID 58 (2019)

78. Zaheer Ullah, S., Adil Khan, M., Khan, Z.A., Chu, Y.-M.: Integral majorization type inequalities for the functions in the sense of strong convexity. J. Funct. Spaces **2019**, Article ID 9487823 (2019)
79. Adil Khan, M., Zaheer Ullah, S., Chu, Y.-M.: The concept of coordinate strongly convex functions and related inequalities. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **113**(3), 2235–2251 (2019)
80. Chu, Y.-M., Adil Khan, M., Ali, T., Dragomir, S.S.: Inequalities for α-fractional differentiable functions. J. Inequal. Appl. **2017**, Article ID 93 (2017)
81. Adil Khan, M., Begum, S., Khurshid, Y., Chu, Y.-M.: Ostrowski type inequalities involving conformable fractional integrals. J. Inequal. Appl. **2018**, Article ID 70 (2018)
82. Adil Khan, M., Chu, Y.-M., Kashuri, A., Liko, R., Ali, G.: Conformable fractional integrals versions of Hermite–Hadamard inequalities and their generalizations. J. Funct. Spaces **2018**, Article ID 6928130 (2018)
83. Adil Khan, M., Iqbal, A., Suleman, M., Chu, Y.-M.: Hermite–Hadamard type inequalities for fractional integrals via Green's function. J. Inequal. Appl. **2018**, Article ID 161 (2018)
84. Adil Khan, M., Khurshid, Y., Du, T.-S., Chu, Y.-M.: Generalization of Hermite–Hadamard type inequalities via conformable fractional integrals. J. Funct. Spaces **2018**, Article ID 5357463 (2018)
85. Adik Khan, M., Wu, S.-H., Ullah, H., Chu, Y.-M.: Discrete majorization type inequalities for convex functions on rectangles. J. Inequal. Appl. **2019**, Article ID 16 (2019)
86. Dragomir, S.S.: On the Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane. Taiwan. J. Math. **5**(4), 775–788 (2001)
87. Delavar, M.R., Dragomir, S.S.: On η-convexity. Math. Inequal. Appl. **20**(1), 203–216 (2017)
88. Eshaghi Gordji, M., Rostamian Delavar, M., Dragomir, S.S.: Some inequalities related to η-convex functions. Available at http://www.ajmaa.org/RGMIA/papers/v18/v18a08.pdf