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Equivalent properties of a reverse half-discrete Hilbert's inequality

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Abstract

By using the weight functions, the idea of introduced parameters and the Euler–Maclaurin summation formula, a reverse half-discrete Hilbert's inequality with the homogeneous kernel and the reverse equivalent forms are given (for $p < 0$, $0 < q < 1$). The equivalent statements of the best possible constant factor related to a few parameters are considered. As applications, two corollaries about the case of the non-homogeneous kernel and some particular cases are obtained.

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1 Introduction

If $0 < \sum_{m=1}^{\infty} a_m^2 < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, then we have the following discrete Hilbert inequality with the best possible constant factor π (cf. [1], Theorem 315):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\left(\sum_{m=1}^{\infty} a_m^2 \sum_{n=1}^{\infty} b_n^2\right)^{1/2}. \tag{1}$$

Assuming that $0 < \int_0^\infty f^2(x)\,dx < \infty$ and $0 < \int_0^\infty g^2(y)\,dy < \infty$, we have the following Hilbert integral inequality (cf. [1], Theorem 316):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \pi\left(\int_0^\infty f^2(x)\,dx \int_0^\infty g^2(y)\,dy\right)^{1/2}, \tag{2}$$

where the constant factor π is the best possible. Inequalities (1) and (2) are important in analysis and its applications (cf. [2–13]).

We still have the following half-discrete Hilbert-type inequality (cf. [1], Theorem 351): If $K(x)$ $(x > 0)$ is a decreasing function, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \phi(s) = \int_0^\infty K(x)x^{s-1}\,dx < \infty$, then

$$\sum_{n=1}^{\infty} n^{p-2}\left(\int_0^\infty K(nx)f(x)\,dx\right)^p < \phi^p\left(\frac{1}{q}\right)\int_0^\infty f^p(x)\,dx. \tag{3}$$

In recent years, some new extensions of (3) were provided by [14–19].

In 2006, using the Euler–Maclaurin summation formula, Krnic et al. [20] gave an extension of (1) with the kernel $\frac{1}{(m+n)^{\lambda}}$ $(0 < \lambda \le 14)$, and, in 2019, according to [20], Adiyasuren et al. [21] considered an extension of (1) involving the partial sums. In 2016–2017, by applying the weight functions, Hong [22, 23] considered some equivalent statements of the extensions of (1) and (2) with a few parameters and conjugate exponents. Some similar work was presented by [24–26].

In this paper, according to [20, 22], by the use of the weight functions, the idea of introduced parameters and the Euler–Maclaurin summation formula, a reverse half-discrete Hilbert inequality with the homogeneous kernel $\frac{1}{(x+n)^{\lambda}}$ $(0 < \lambda \le 28)$ and the reverse equivalent forms are given. The equivalent statements of the best possible constant factor related to a few parameters are considered. As applications, two corollaries about the cases of non-homogeneous kernel and some particular cases are obtained.

2 Some lemmas

In what follows, we assume that $p < 0$ $(0 < q < 1)$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda \in (0, 28]$, $\sigma \in (0, 2] \cap (0, \lambda)$, $\mu \in (0, \lambda)$, $f(x) \ge 0$ $(x \in R_+ = (0, \infty))$, $a_n \ge 0$ $(n \in \mathrm{N} = \{1, 2, \ldots\})$, such that

$$0 < \int_0^\infty x^{p[1-(\frac{\lambda-\sigma}{p}+\frac{\mu}{q})]-1} f^p(x)\,dx < \infty \quad \text{and} \quad 0 < \sum_{n=1}^\infty n^{q[1-(\frac{\sigma}{p}+\frac{\lambda-\mu}{q})]-1} a_n^q < \infty.$$

Lemma 1 *Define the following weight function*:

$$\varpi(\sigma, x) := x^{\lambda-\sigma} \sum_{n=1}^\infty \frac{n^{\sigma-1}}{(x+n)^{\lambda}} \quad (x \in \mathrm{R}_+). \tag{4}$$

We have the following inequality:

$$\varpi(\sigma, x) < B(\sigma, \lambda - \sigma) \quad (x \in \mathrm{R}_+). \tag{5}$$

Proof For fixed $x > 0$, we set the function $g_x(t) := \frac{t^{\sigma-1}}{(x+t)^{\lambda}}$ $(t > 0)$. Using the Euler–Maclaurin summation formula (cf. [20]), for $\rho(t) := t - [t] - \frac{1}{2}$, we have

$$\sum_{n=1}^\infty g_x(n) = \int_1^\infty g_x(t)\,dt + \frac{1}{2} g_x(1) + \int_1^\infty \rho(t) g'_x(t)\,dt = \int_0^\infty g_x(t)\,dt - h(x),$$
$$h(x) := \int_0^1 g_x(t)\,dt - \frac{1}{2} g_x(1) - \int_1^\infty \rho(t) g'_x(t)\,dt.$$

We obtain $-\frac{1}{2} g_x(1) = \frac{-1}{2(x+1)^{\lambda}}$,

$$\begin{aligned}
\int_0^1 g_x(t)\,dt &= \int_0^1 \frac{t^{\sigma-1}}{(x+t)^{\lambda}}\,dt = \frac{1}{\sigma}\int_0^1 \frac{dt^{\sigma}}{(x+t)^{\lambda}} = \frac{1}{\sigma}\frac{t^{\sigma}}{(x+t)^{\lambda}}\bigg|_0^1 + \frac{\lambda}{\sigma}\int_0^1 \frac{t^{\sigma}\,dt}{(x+t)^{\lambda+1}} \\
&= \frac{1}{\sigma}\frac{1}{(x+1)^{\lambda}} + \frac{\lambda}{\sigma(\sigma+1)}\int_0^1 \frac{dt^{\sigma+1}}{(x+t)^{\lambda+1}} \\
&> \frac{1}{\sigma}\frac{1}{(x+1)^{\lambda}} + \frac{\lambda}{\sigma(\sigma+1)}\left[\frac{t^{\sigma+1}}{(x+t)^{\lambda+1}}\right]_0^1 + \frac{\lambda(\lambda+1)}{\sigma(\sigma+1)(x+1)^{\lambda+2}}\int_0^1 t^{\sigma+1}\,dt \\
&= \frac{1}{\sigma}\frac{1}{(x+1)^{\lambda}} + \frac{\lambda}{\sigma(\sigma+1)}\frac{1}{(x+1)^{\lambda+1}} + \frac{\lambda(\lambda+1)}{\sigma(\sigma+1)(\sigma+2)}\frac{1}{(x+1)^{\lambda+2}},
\end{aligned}$$

$$-g_x'(t) = -\frac{(\sigma-1)t^{\sigma-2}}{(x+t)^{\lambda}} + \frac{\lambda t^{\sigma-1}}{(x+t)^{\lambda+1}} = \frac{(1-\sigma)t^{\sigma-2}}{(x+t)^{\lambda}} + \frac{\lambda t^{\sigma-2}}{(x+t)^{\lambda}} - \frac{\lambda x t^{\sigma-2}}{(x+t)^{\lambda+1}}$$
$$= \frac{(\lambda+1-\sigma)t^{\sigma-2}}{(x+t)^{\lambda}} - \frac{\lambda x t^{\sigma-2}}{(x+t)^{\lambda+1}}.$$

For $0 < \sigma \le 2$, $\sigma < \lambda \le 28$, we find $\lambda + 1 - \sigma > 0$,

$$(-1)^i \frac{d^i}{dt^i}\left[\frac{t^{\sigma-2}}{(x+t)^{\lambda}}\right] > 0, \qquad (-1)^i \frac{d^i}{dt^i}\left[\frac{t^{\sigma-2}}{(x+t)^{\lambda+1}}\right] > 0 \quad (i = 0,1,2,3),$$

and then, by the Euler–Maclaurin summation formula (cf. [20]), we find

$$(\lambda+1-\sigma)\int_1^{\infty} \rho(t)\frac{t^{\sigma-2}}{(x+t)^{\lambda}}\,dt$$
$$> -\frac{\lambda+1-\sigma}{12(x+1)^{\lambda}},$$
$$-x\lambda\int_1^{\infty} \rho(t)\frac{t^{\sigma-2}}{(x+t)^{\lambda+1}}\,dt > \frac{x\lambda}{12(x+1)^{\lambda+1}} - \frac{x\lambda}{720}\left[\frac{t^{\sigma-2}}{(x+t)^{\lambda+1}}\right]''_{t=1}$$
$$> \frac{(x+1)\lambda - \lambda}{12(x+1)^{\lambda+1}} - \frac{(x+1)\lambda}{720}\left[\frac{(\lambda+1)(\lambda+2)}{(x+1)^{\lambda+3}} + \frac{2(\lambda+1)(2-\sigma)}{(x+1)^{\lambda+2}} + \frac{(2-\sigma)(3-\sigma)}{(x+1)^{\lambda+1}}\right]$$
$$= \frac{\lambda}{12(x+1)^{\lambda}} - \frac{\lambda}{12(x+1)^{\lambda+1}}$$
$$- \frac{\lambda}{720}\left[\frac{(\lambda+1)(\lambda+2)}{(x+1)^{\lambda+2}} + \frac{2(\lambda+1)(2-\sigma)}{(x+1)^{\lambda+1}} + \frac{(2-\sigma)(3-\sigma)}{(x+1)^{\lambda}}\right].$$

Hence, we have $h(x) > \frac{h_1}{(x+1)^{\lambda}} + \frac{\lambda h_2}{(x+1)^{\lambda+1}} + \frac{\lambda(\lambda+1)h_3}{(x+1)^{\lambda+2}}$, where

$$h_1 := \frac{1}{\sigma} - \frac{1}{2} - \frac{1-\sigma}{12} - \frac{\lambda(2-\sigma)(3-\sigma)}{720}, \qquad h_2 := \frac{1}{\sigma(\sigma+1)} - \frac{1}{12} - \frac{(\lambda+1)(2-\sigma)}{720},$$

and $h_3 := \frac{1}{\sigma(\sigma+1)(\sigma+2)} - \frac{\lambda+2}{720}$.

For $\lambda \in (0,28]$, $\frac{\lambda}{720} < \frac{1}{24}$, $\sigma \in (0,2]$, it follows that

$$h_1 > \frac{1}{\sigma} - \frac{1}{2} - \frac{1-\sigma}{12} - \frac{(2-\sigma)(3-\sigma)}{24} = \frac{24 - 20\sigma + 7\sigma^2 - \sigma^3}{24\sigma} > 0.$$

In fact, setting $g(\sigma) := 24 - 20\sigma + 7\sigma^2 - \sigma^3$ ($\sigma \in (0,2]$), we obtain

$$g'(\sigma) = -20 + 14\sigma^2 - 3\sigma^2 = -3\left(\sigma - \frac{7}{3}\right)^2 - \frac{11}{3} < 0,$$

and then

$$h_1 > \frac{g(\sigma)}{24\sigma} \ge \frac{g(2)}{24\sigma} = \frac{4}{24\sigma} > 0 \quad \bigl(\sigma \in (0,2]\bigr).$$

We still find that $h_2 > \frac{1}{6} - \frac{1}{12} - \frac{30}{360} = 0$, and $h_3 \geq \frac{1}{24} - \frac{30}{720} = 0$. Hence, we find $h(x) > 0$, and then

$$x^{\lambda-\sigma}\sum_{n=1}^{\infty} g_x(n) < x^{\lambda-\sigma}\int_0^{\infty} g_x(t)\,dt$$
$$= x^{\lambda-\sigma}\int_0^{\infty}\frac{t^{\sigma-1}\,dt}{(x+t)^{\lambda}} = \int_0^{\infty}\frac{u^{\sigma-1}\,du}{(1+u)^{\lambda}} = B(\sigma,\lambda-\sigma),$$

namely, (5) follows. □

Lemma 2 *We have the following reverse inequality*:

$$\begin{aligned} I &= \int_0^{\infty}\sum_{n=1}^{\infty}\frac{f(x)a_n}{(x+n)^{\lambda}}\,dx \\ &> B^{\frac{1}{p}}(\sigma,\lambda-\sigma)B^{\frac{1}{q}}(\mu,\lambda-\mu) \\ &\quad\times\left\{\int_0^{\infty} x^{p[1-(\frac{\lambda-\sigma}{p}+\frac{\mu}{q})]-1}f^p(x)\,dx\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{q[1-(\frac{\sigma}{p}+\frac{\lambda-\mu}{q})]-1}a_n^q\right\}^{\frac{1}{q}}. \end{aligned} \tag{6}$$

Proof For $n \in \mathbf{N}$, setting $x = nu$, we obtain another weight function:

$$\omega(\mu,n) := n^{\lambda-\mu}\int_0^{\infty}\frac{x^{\mu-1}\,dx}{(x+n)^{\lambda}} = \int_0^{\infty}\frac{u^{\mu-1}\,du}{(u+1)^{\lambda}} = B(\mu,\lambda-\mu). \tag{7}$$

For $p < 0$, $0 < q < 1$, by the reverse Hölder inequality (cf. [27]) and Lebesgue term by term integration theorem (cf. [28]), we obtain

$$\begin{aligned} \int_0^{\infty}\sum_{n=1}^{\infty}\frac{f(x)a_n}{(x+n)^{\lambda}}\,dx &= \int_0^{\infty}\sum_{n=1}^{\infty}\frac{1}{(x+n)^{\lambda}}\left[\frac{n^{(\sigma-1)/p}}{x^{(\mu-1)/q}}f(x)\right]\left[\frac{x^{(\mu-1)/q}}{n^{(\sigma-1)/p}}a_n\right]dx \\ &\geq \left\{\int_0^{\infty}\left[\sum_{n=1}^{\infty}\frac{1}{(x+n)^{\lambda}}\frac{n^{\sigma-1}}{x^{(\mu-1)(p-1)}}\right]f^p(x)\,dx\right\}^{\frac{1}{p}} \\ &\quad\times\left\{\sum_{n=1}^{\infty}\left[\int_0^{\infty}\frac{1}{(x+n)^{\lambda}}\frac{x^{\mu-1}}{n^{(\sigma-1)(q-1)}}\,dx\right]a_n^q\right\}^{\frac{1}{q}} \\ &= \left\{\int_0^{\infty}\varpi(\sigma,x)x^{p[1-(\frac{\lambda-\sigma}{p}+\frac{\mu}{q})]-1}f^p(x)\,dx\right\}^{\frac{1}{p}} \\ &\quad\times\left\{\sum_{n=1}^{\infty}\omega(\mu,n)n^{q[1-(\frac{\sigma}{p}+\frac{\lambda-\mu}{q})]-1}a_n^q\right\}^{\frac{1}{q}}. \end{aligned}$$

Then, by (5) and (7), we have (6). □

Remark 1 For $\mu + \sigma = \lambda$, we find

$$\varpi(\sigma,x) = x^{\mu}\sum_{n=1}^{\infty}\frac{n^{\sigma-1}}{(x+n)^{\lambda}} \quad (x \in \mathrm{R}_+),$$

$$0 < \int_0^\infty x^{p(1-\mu)-1} f^p(x)\,dx < \infty \quad \text{and} \quad 0 < \sum_{n=1}^\infty n^{q(1-\sigma)-1} a_n^q < \infty,$$

and then we reduce (6) as follows:

$$\int_0^\infty \sum_{n=1}^\infty \frac{f(x)a_n}{(x+n)^\lambda}\,dx > B(\mu,\sigma)\left[\int_0^\infty x^{p(1-\mu)-1} f^p(x)\,dx\right]^{\frac{1}{p}} \left[\sum_{n=1}^\infty n^{q(1-\sigma)-1} a_n^q\right]^{\frac{1}{q}}. \tag{8}$$

Lemma 3 *The constant factor $B(\mu,\sigma)$ in (8) is the best possible.*

Proof For $0 < \varepsilon < q\sigma$, we set

$$\tilde{f}(x) := \begin{cases} 0, & 0 < x < 1, \\ x^{\mu-\frac{\varepsilon}{p}-1}, & x \geq 1, \end{cases} \qquad \tilde{a}_n := n^{\sigma-\frac{\varepsilon}{q}-1} \quad (n \in \mathrm{N}).$$

If there exists a positive constant M ($M \geq B(\mu,\sigma)$), such that (8) is valid when replacing $B(\mu,\sigma)$ by M, then by substitution of $f(x) = \tilde{f}(x)$, $a_n = \tilde{a}_n$, we have

$$\begin{aligned} \tilde{I} &:= \int_0^\infty \sum_{n=1}^\infty \frac{\tilde{f}(x)\tilde{a}_n}{(x+n)^\lambda}\,dx > M\left[\int_0^\infty x^{p(1-\mu)-1} \tilde{f}^p(x)\,dx\right]^{\frac{1}{p}} \left[\sum_{n=1}^\infty n^{q(1-\sigma)-1} \tilde{a}_n^q\right]^{\frac{1}{q}} \\ &= M\left(\int_1^\infty x^{-\varepsilon-1}\,dx\right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty n^{-\varepsilon-1}\right)^{\frac{1}{q}} \\ &\geq M\left(\int_1^\infty x^{-\varepsilon-1}\,dx\right)^{\frac{1}{p}} \left(\int_1^\infty x^{-\varepsilon-1}\,dx\right)^{\frac{1}{q}} = M\int_1^\infty x^{-\varepsilon-1}\,dx = \frac{M}{\varepsilon}. \end{aligned}$$

For $0 < \sigma - \frac{\varepsilon}{q} < 2$ ($0 < q < 1$), by (5), we obtain

$$\begin{aligned} \tilde{I} &= \int_1^\infty \left[x^{(\mu+\frac{\varepsilon}{q})} \sum_{n=1}^\infty \frac{n^{(\sigma-\frac{\varepsilon}{q})-1}}{(x+n)^\lambda}\right] x^{-\varepsilon-1}\,dx = \int_1^\infty \varpi\left(\sigma - \frac{\varepsilon}{q}, x\right) x^{-\varepsilon-1}\,dx \\ &\leq B\left(\mu + \frac{\varepsilon}{q}, \sigma - \frac{\varepsilon}{q}\right) \int_1^\infty x^{-\varepsilon-1}\,dx = \frac{1}{\varepsilon} B\left(\mu + \frac{\varepsilon}{q}, \sigma - \frac{\varepsilon}{q}\right). \end{aligned}$$

Then we have

$$B\left(\mu + \frac{\varepsilon}{q}, \sigma - \frac{\varepsilon}{q}\right) \geq \varepsilon\tilde{I} \geq M.$$

For $\varepsilon \to 0^+$, in view of the continuity of the beta function, it follows that $B(\mu,\sigma) \geq M$. Therefore, $M = B(\mu,\sigma)$ is the best possible constant factor of (8). □

Remark 2 Setting $\hat{\mu} := \frac{\lambda-\sigma}{p} + \frac{\mu}{q}$, $\hat{\sigma} := \frac{\sigma}{p} + \frac{\lambda-\mu}{q}$, we have

$$\hat{\mu} + \hat{\sigma} = \frac{\lambda-\sigma}{p} + \frac{\mu}{q} + \frac{\sigma}{p} + \frac{\lambda-\mu}{q} = \frac{\lambda}{p} + \frac{\lambda}{q} = \lambda.$$

And, for $\lambda - \mu - \sigma \in (-q\sigma, q(\lambda - \sigma))$, we find

$$\hat{\mu} > \frac{\lambda - \sigma}{p} + \frac{(1-q)(\lambda - \sigma)}{q} = 0, \qquad \hat{\mu} < \frac{\lambda - \sigma}{p} + \frac{\lambda - \sigma + q\sigma}{q} = \lambda,$$
$$0 < \hat{\sigma} = \lambda - \hat{\mu} < \lambda, \qquad B(\hat{\mu}, \hat{\sigma}) \in \mathrm{R}_+.$$

We can reduce (6) as follows:

$$\int_0^\infty \sum_{n=1}^\infty \frac{f(x) a_n}{(x+n)^\lambda}\, dx > B^{\frac{1}{p}}(\sigma, \lambda - \sigma) B^{\frac{1}{q}}(\mu, \lambda - \mu) \times \left[\int_0^\infty x^{p(1-\hat{\mu})-1} f^p(x)\, dx\right]^{\frac{1}{p}} \left[\sum_{n=1}^\infty n^{q(1-\hat{\sigma})-1} a_n^q\right]^{\frac{1}{q}}. \tag{9}$$

Lemma 4 *If* $\lambda - \mu - \sigma \in (-q\sigma, q(\lambda - \sigma))$, *the constant factor* $B^{\frac{1}{p}}(\sigma, \lambda - \sigma) B^{\frac{1}{q}}(\mu, \lambda - \mu)$ *in* (9) *is the best possible, then we have* $\lambda - \mu - \sigma = 0$, *namely*, $\mu + \sigma = \lambda$.

Proof If the constant factor $B^{\frac{1}{p}}(\sigma, \lambda - \sigma) B^{\frac{1}{q}}(\mu, \lambda - \mu)$ in (9) is the best possible, then, by (8), the unique best possible constant factor must be $B(\hat{\mu}, \hat{\sigma})$ ($\in \mathrm{R}_+$), namely,

$$B(\hat{\mu}, \hat{\sigma}) = B^{\frac{1}{p}}(\sigma, \lambda - \sigma) B^{\frac{1}{q}}(\mu, \lambda - \mu).$$

By the reverse Hölder inequality (cf. [27]), we find

$$\begin{aligned} B(\hat{\mu}, \hat{\sigma}) &= \int_0^\infty \frac{t^{\hat{\mu}-1}}{(1+t)^\lambda}\, dt = \int_0^\infty \frac{t^{\frac{\lambda-\sigma}{p} + \frac{\mu}{q} - 1}}{(1+t)^\lambda}\, dt = \int_0^\infty \frac{1}{(1+t)^\lambda} \big(t^{\frac{\lambda-\sigma-1}{p}}\big)\big(t^{\frac{\mu-1}{q}}\big)\, dt \\ &\geq \left[\int_0^\infty \frac{1}{(1+t)^\lambda} t^{\lambda-\sigma-1}\, dt\right]^{\frac{1}{p}} \left[\int_0^\infty \frac{1}{(1+t)^\lambda} t^{\mu-1}\, dt\right]^{\frac{1}{q}} \\ &= B^{\frac{1}{p}}(\sigma, \lambda - \sigma) B^{\frac{1}{q}}(\mu, \lambda - \mu). \end{aligned} \tag{10}$$

We observe that (10) keeps the form of an equality if and only if there exist constants A, B, such that they are not all zero and $At^{\lambda-\sigma-1} = Bt^{\mu-1}$ *a.e. in* R_+. Suppose that $A \neq 0$. We find that

$$t^{\lambda-\mu-\sigma} = \frac{B}{A} \quad a.e. in\ \mathrm{R}_+,$$

and then $\lambda - \mu - \sigma = 0$, namely, $\mu + \sigma = \lambda$. □

3 Main results

Theorem 1 *Inequality* (6) *is equivalent to the following inequalities*:

$$J_1 := \left\{\sum_{n=1}^\infty n^{p(\frac{\sigma}{p} + \frac{\lambda-\mu}{q})-1} \left[\int_0^\infty \frac{f(x)}{(x+n)^\lambda}\, dx\right]^p\right\}^{\frac{1}{p}} > B^{\frac{1}{p}}(\sigma, \lambda - \sigma) B^{\frac{1}{q}}(\mu, \lambda - \mu) \left\{\int_0^\infty x^{p[1-(\frac{\lambda-\sigma}{p} + \frac{\mu}{q})]-1} f^p(x)\, dx\right\}^{\frac{1}{p}}, \tag{11}$$

$$J_2 := \left\{\int_0^\infty x^{q(\frac{\lambda-\sigma}{p}+\frac{\mu}{q})-1}\left[\sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda}\right]^q dx\right\}^{\frac{1}{q}}$$
$$> B^{\frac{1}{p}}(\sigma,\lambda-\sigma)B^{\frac{1}{q}}(\mu,\lambda-\mu)\left\{\sum_{n=1}^\infty n^{q[1-(\frac{\sigma}{p}+\frac{\lambda-\mu}{q})]-1}a_n^q\right\}^{\frac{1}{q}}. \quad (12)$$

If the constant factor $B^{\frac{1}{p}}(\sigma,\lambda-\sigma)B^{\frac{1}{q}}(\mu,\lambda-\mu)$ *in* (6) *is the best possible, then so is the constant factor in* (11) *and* (12).

In particular, for $\mu+\sigma=\lambda$ *in* (6), (11) *and* (12), *we have* (8) *and the following equivalent reverse inequalities with the best possible constant factor* $B(\mu,\sigma)$:

$$\left\{\sum_{n=1}^\infty n^{p\sigma-1}\left[\int_0^\infty \frac{f(x)}{(x+n)^\lambda}dx\right]^p\right\}^{\frac{1}{p}} > B(\mu,\sigma)\left[\int_0^\infty x^{p(1-\mu)-1}f^p(x)\,dx\right]^{\frac{1}{p}}, \quad (13)$$

$$\left\{\int_0^\infty x^{q\mu-1}\left[\sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda}\right]^q dx\right\}^{\frac{1}{q}} > B(\mu,\sigma)\left[\sum_{n=1}^\infty n^{q(1-\sigma)-1}a_n^q\right]^{\frac{1}{q}}. \quad (14)$$

Proof Suppose that (11) is valid. By Lebesgue term by term integration theorem and the reverse Hölder inequality (cf. [27, 28]), we have

$$I = \sum_{n=1}^\infty \int_0^\infty \frac{f(x)a_n}{(x+n)^\lambda}dx = \sum_{n=1}^\infty\left[n^{\frac{-1}{p}+(\frac{\sigma}{p}+\frac{\lambda-\mu}{q})}\int_0^\infty \frac{f(x)}{(x+n)^\lambda}dx\right]\left[n^{\frac{1}{p}-(\frac{\sigma}{p}+\frac{\lambda-\mu}{q})}a_n\right]$$
$$\geq J_1\left\{\sum_{n=1}^\infty n^{q[1-(\frac{\sigma}{p}+\frac{\lambda-\mu}{q})]-1}a_n^q\right\}^{\frac{1}{q}}. \quad (15)$$

Then, by (11), we have (6). On the other hand, assuming that (6) is valid, we set

$$a_n := n^{p(\frac{\sigma}{p}+\frac{\lambda-\mu}{q})-1}\left[\int_0^\infty \frac{f(x)}{(x+n)^\lambda}dx\right]^{p-1}, \quad n\in\mathrm{N}.$$

If $J_1=\infty$, then (11) is naturally valid; if $J_1=0$, then it is impossible to make (11) valid, namely $J_1>0$. Suppose that $0<J_1<\infty$. By (6), we have

$$\sum_{n=1}^\infty n^{q[1-(\frac{\sigma}{p}+\frac{\lambda-\mu}{q})]-1}a_n^q = J_1^p = I$$
$$> B^{\frac{1}{p}}(\sigma,\lambda-\sigma)B^{\frac{1}{q}}(\mu,\lambda-\mu)$$
$$\times\left\{\int_0^\infty x^{p[1-(\frac{\lambda-\sigma}{p}+\frac{\mu}{q})]-1}f^p(x)\,dx\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^\infty n^{q[1-(\frac{\sigma}{p}+\frac{\lambda-\mu}{q})]-1}a_n^q\right\}^{\frac{1}{q}},$$

$$\left\{\sum_{n=1}^\infty n^{q[1-(\frac{\sigma}{p}+\frac{\lambda-\mu}{q})]-1}a_n^q\right\}^{\frac{1}{p}} = J_1 > B^{\frac{1}{p}}(\sigma,\lambda-\sigma)B^{\frac{1}{q}}(\mu,\lambda-\mu)$$
$$\times\left\{\int_0^\infty x^{p[1-(\frac{\lambda-\sigma}{p}+\frac{\mu}{q})]-1}f^p(x)\,dx\right\}^{\frac{1}{p}},$$

namely, (11) follows, which is equivalent to (6).

Suppose that (12) is valid. By the reverse Hölder inequality, we have

$$\begin{aligned} I &= \int_0^\infty \left[x^{\frac{1}{q}-(\frac{\lambda-\sigma}{p}+\frac{\mu}{q})} f(x)\right]\left[x^{\frac{-1}{q}+(\frac{\lambda-\sigma}{p}+\frac{\mu}{q})} \sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda}\right] dx \\ &\geq \left\{\int_0^\infty x^{p\{1-(\frac{\lambda-\sigma}{p}+\frac{\mu}{q})]-1} f^p(x)\,dx\right\}^{\frac{1}{p}} J_2. \end{aligned} \tag{16}$$

Then, by (12), we have (6). On the other hand, assuming that (6) is valid, we set

$$f(x) := x^{q(\frac{\lambda-\sigma}{p}+\frac{\mu}{q})-1}\left[\sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda}\right]^{q-1}, \quad x \in \mathrm{R}_+.$$

If $J_2 = \infty$, then (12) is naturally valid; if $J_2 = 0$, then it is impossible to make (12) valid, namely $J_2 > 0$. Suppose that $0 < J_2 < \infty$. By (6), we have

$$\begin{aligned} &\int_0^\infty x^{p[1-(\frac{\lambda-\sigma}{p}+\frac{\mu}{q})]-1} f^p(x)\,dx \\ &\quad = J_2^q = I \\ &\quad > B^{\frac{1}{p}}(\sigma,\lambda-\sigma)B^{\frac{1}{q}}(\mu,\lambda-\mu) \\ &\qquad \times \left\{\int_0^\infty x^{p[1-(\frac{\lambda-\sigma}{p}+\frac{\mu}{q})]-1} f^p(x)\,dx\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^\infty n^{q[1-(\frac{\sigma}{p}+\frac{\lambda-\mu}{q})]-1} a_n^q\right\}^{\frac{1}{q}}, \end{aligned}$$

$$\begin{aligned} \left\{\int_0^\infty x^{p[1-(\frac{\lambda-\sigma}{p}+\frac{\mu}{q})]-1} f^p(x)\,dx\right\}^{\frac{1}{q}} &= J_2 > B^{\frac{1}{p}}(\sigma,\lambda-\sigma)B^{\frac{1}{q}}(\mu,\lambda-\mu) \\ &\quad \times \left\{\sum_{n=1}^\infty n^{q[1-(\frac{\sigma}{p}+\frac{\lambda-\mu}{q})]-1} a_n^q\right\}^{\frac{1}{q}}, \end{aligned}$$

namely, (12) follows, which is equivalent to (6).

Hence, inequalities (6), (11) and (12) are equivalent.

If the constant factor $B^{\frac{1}{p}}(\sigma,\lambda-\sigma)B^{\frac{1}{q}}(\mu,\lambda-\mu)$ in (6) is the best possible, then so is the constant factor in (11) and (12). Otherwise, by (15) (or (16)), we would reach the contradiction that the constant factor in (6) is not the best possible. □

Theorem 2 *The following statements* (i), (ii), (iii) *and* (iv) *are equivalent*:

(i) $B^{\frac{1}{p}}(\sigma,\lambda-\sigma)B^{\frac{1}{q}}(\mu,\lambda-\mu)$ *is independent of* p, q;

(ii) $B^{\frac{1}{p}}(\sigma,\lambda-\sigma)B^{\frac{1}{q}}(\mu,\lambda-\mu)$ *is expressible as a single integral*;

(iii) $B^{\frac{1}{p}}(\sigma,\lambda-\sigma)B^{\frac{1}{q}}(\mu,\lambda-\mu)$ *is the best possible of* (6);

(iv) *If* $\lambda-\mu-\sigma \in (-q\sigma, q(\lambda-\sigma))$, *then* $\mu+\sigma=\lambda$.

Proof (i)⇒(ii). In view of $B^{\frac{1}{p}}(\sigma,\lambda-\sigma)B^{\frac{1}{q}}(\mu,\lambda-\mu)$ is independent of p, q, we find

$$\begin{aligned} &B^{\frac{1}{p}}(\sigma,\lambda-\sigma)B^{\frac{1}{q}}(\mu,\lambda-\mu) \\ &\quad = \lim_{\substack{p\to-\infty,\\ q\to1^+}} B^{\frac{1}{p}}(\sigma,\lambda-\sigma)B^{\frac{1}{q}}(\mu,\lambda-\mu) = B(\mu,\lambda-\mu), \end{aligned}$$

which is a single integral $\int_0^\infty \frac{t^{\mu-1}}{(1+t)^\lambda}\,dt$.

(ii)⇒(iv). Suppose that $B^{\frac{1}{p}}(\sigma,\lambda-\sigma)B^{\frac{1}{q}}(\mu,\lambda-\mu)$ is expressible as a single integral $\int_0^\infty \frac{t^{\frac{\lambda-\sigma}{p}+\frac{\mu}{q}-1}}{(1+t)^\lambda}\,dt$. Then (10) keeps the form of equality. By the proof of Lemma 4, for $\lambda-\mu-\sigma\in(-q\sigma,q(\lambda-\sigma))$, we have $\mu+\sigma=\lambda$.

(iv)⇒(i). If $\mu+\sigma=\lambda$, then

$$B^{\frac{1}{p}}(\sigma,\lambda-\sigma)B^{\frac{1}{q}}(\mu,\lambda-\mu)=B(\mu,\sigma),$$

which is independent of p, q.

Hence, we have (i)⇔(ii)⇔(iv).

(i)⇒(iii). By Lemma 3, for $\mu+\sigma=\lambda$, $B^{\frac{1}{p}}(\sigma,\lambda-\sigma)B^{\frac{1}{q}}(\mu,\lambda-\mu)$ is the best possible of (6).

(iii)⇒(iv). By Lemma 4, we have $\mu+\sigma=\lambda$.

Therefore, we show that (iv)⇔(iii), and then the statements (i), (ii), (iii) and (iv) are equivalent. □

4 Two corollaries and some particular inequalities

Replacing x by $\frac{1}{x}$, and then setting $F(x)=x^{\lambda-2}f(\frac{1}{x})$ in Theorem 1 and then Theorem 2, we have the following.

Corollary 1 *If $F(x)$, $a_n\geq 0$, such that*

$$0<\int_0^\infty x^{p[1-(\frac{\sigma}{p}+\frac{\lambda-\mu}{q})]-1}F^p(x)\,dx<\infty \quad \text{and} \quad 0<\sum_{n=1}^\infty n^{q[1-(\frac{\sigma}{p}+\frac{\lambda-\mu}{q})]-1}a_n^q<\infty,$$

then the following inequalities are equivalent:

$$\begin{aligned}&\int_0^\infty\sum_{n=1}^\infty\frac{F(x)a_n}{(1+xn)^\lambda}\,dx\\&\quad>B^{\frac{1}{p}}(\sigma,\lambda-\sigma)B^{\frac{1}{q}}(\mu,\lambda-\mu)\\&\qquad\times\left\{\int_0^\infty x^{p[1-(\frac{\sigma}{p}+\frac{\lambda-\mu}{q})]-1}F^p(x)\,dx\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^\infty n^{q[1-(\frac{\sigma}{p}+\frac{\lambda-\mu}{q})]-1}a_n^q\right\}^{\frac{1}{q}},\end{aligned} \tag{17}$$

$$\begin{aligned}&\left\{\sum_{n=1}^\infty n^{p(\frac{\sigma}{p}+\frac{\lambda-\mu}{q})-1}\left[\int_0^\infty\frac{F(x)}{(1+xn)^\lambda}\,dx\right]^p\right\}^{\frac{1}{p}}\\&\quad>B^{\frac{1}{p}}(\sigma,\lambda-\sigma)B^{\frac{1}{q}}(\mu,\lambda-\mu)\left\{\int_0^\infty x^{p[1-(\frac{\sigma}{p}+\frac{\lambda-\mu}{q})]-1}F^p(x)\,dx\right\}^{\frac{1}{p}},\end{aligned} \tag{18}$$

$$\begin{aligned}&\left\{\int_0^\infty x^{q(\frac{\sigma}{p}+\frac{\lambda-\mu}{q})-1}\left[\sum_{n=1}^\infty\frac{a_n}{(1+xn)^\lambda}\right]^q dx\right\}^{\frac{1}{q}}\\&\quad>B^{\frac{1}{p}}(\sigma,\lambda-\sigma)B^{\frac{1}{q}}(\mu,\lambda-\mu)\left\{\sum_{n=1}^\infty n^{q[1-(\frac{\sigma}{p}+\frac{\lambda-\mu}{q})]-1}a_n^q\right\}^{\frac{1}{q}}.\end{aligned} \tag{19}$$

If the constant factor $B^{\frac{1}{p}}(\sigma,\lambda-\sigma)B^{\frac{1}{q}}(\mu,\lambda-\mu)$ in (17) *is the best possible, then so is the constant factor in* (18) *and* (19). *In particular, for $\mu=\lambda-\sigma$ in* (17), (18) *and* (19), *we have* *the following equivalent inequalities with the best possible constant factor* $B(\lambda-\sigma,\sigma)$:

$$\int_0^\infty \sum_{n=1}^\infty \frac{F(x)a_n}{(1+xn)^\lambda}\,dx > B(\lambda-\sigma,\sigma) \times \left[\int_0^\infty x^{p(1-\sigma)-1}F^p(x)\,dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^\infty n^{q(1-\sigma)-1}a_n^q\right]^{\frac{1}{q}}, \tag{20}$$

$$\left\{\sum_{n=1}^\infty n^{p\sigma-1}\left[\int_0^\infty \frac{F(x)}{(1+xn)^\lambda}\,dx\right]^p\right\}^{\frac{1}{p}} > B(\lambda-\sigma,\sigma)\left[\int_0^\infty x^{p(1-\sigma)-1}F^p(x)\,dx\right]^{\frac{1}{p}}, \tag{21}$$

$$\left\{\int_0^\infty x^{q\sigma-1}\left[\sum_{n=1}^\infty \frac{a_n}{(1+xn)^\lambda}\right]^q dx\right\}^{\frac{1}{q}} > B(\lambda-\sigma,\sigma)\left[\sum_{n=1}^\infty n^{q(1-\sigma)-1}a_n^q\right]^{\frac{1}{q}}. \tag{22}$$

Corollary 2 *The following statements* (I), (II), (III) *and* (IV) *are equivalent*:

(I) $B^{\frac{1}{p}}(\sigma,\lambda-\sigma)B^{\frac{1}{q}}(\mu,\lambda-\mu)$ *is independent of* p, q;

(II) $B^{\frac{1}{p}}(\sigma,\lambda-\sigma)B^{\frac{1}{q}}(\mu,\lambda-\mu)$ *is expressible as a single integral*;

(III) $B^{\frac{1}{p}}(\sigma,\lambda-\sigma)B^{\frac{1}{q}}(\mu,\lambda-\mu)$ *is the best possible of* (17);

(IV) *If* $\lambda-\mu-\sigma\in(-q\sigma,q(\lambda-\sigma))$, *then we have* $\mu=\lambda-\sigma$.

Remark 3 (i) For $\sigma=2<\lambda\ (\le 28)$, $\mu=\lambda-2$ in (8), (13) and (14), since

$$B(\lambda-2,2)=\frac{\Gamma(\lambda-2)\Gamma(2)}{\Gamma(\lambda)}=\frac{\Gamma(\lambda-2)}{(\lambda-1)(\lambda-2)\Gamma(\lambda-2)}=\frac{1}{(\lambda-1)(\lambda-2)},$$

we have the following equivalent reverse inequalities with the best possible constant factor $\frac{1}{(\lambda-1)(\lambda-2)}$:

$$\int_0^\infty \sum_{n=1}^\infty \frac{f(x)a_n}{(x+n)^\lambda}\,dx > \frac{1}{(\lambda-1)(\lambda-2)}\left[\int_0^\infty x^{p(3-\lambda)-1}f^p(x)\,dx\right]^{\frac{1}{p}}\left(\sum_{n=1}^\infty n^{-q-1}a_n^q\right)^{\frac{1}{q}}, \tag{23}$$

$$\left\{\sum_{n=1}^\infty n^{2p-1}\left[\int_0^\infty \frac{f(x)}{(x+n)^\lambda}\,dx\right]^p\right\}^{\frac{1}{p}} > \frac{1}{(\lambda-1)(\lambda-2)}\left[\int_0^\infty x^{p(3-\lambda)-1}f^p(x)\,dx\right]^{\frac{1}{p}}, \tag{24}$$

$$\left\{\int_0^\infty x^{q(\lambda-2)-1}\left[\sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda}\right]^q dx\right\}^{\frac{1}{q}} > \frac{1}{(\lambda-1)(\lambda-2)}\left(\sum_{n=1}^\infty n^{-q-1}a_n^q\right)^{\frac{1}{q}}. \tag{25}$$

(ii) For $\sigma=2<\lambda\ (\le 28)$, $\mu=\lambda-2$ in (20), (21) and (22), we have the following equivalent reverse inequalities with the best possible constant factor $\frac{1}{(\lambda-1)(\lambda-2)}$:

$$\int_0^\infty \sum_{n=1}^\infty \frac{F(x)a_n}{(1+xn)^\lambda}\,dx > \frac{1}{(\lambda-1)(\lambda-2)}\left(\int_0^\infty x^{-p-1}F^p(x)\,dx\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty n^{-q-1}a_n^q\right)^{\frac{1}{q}}, \tag{26}$$

$$\left\{\sum_{n=1}^\infty n^{2p-1}\left[\int_0^\infty \frac{F(x)}{(1+xn)^\lambda}\,dx\right]^p\right\}^{\frac{1}{p}} > \frac{1}{(\lambda-1)(\lambda-2)}\left(\int_0^\infty x^{-p-1}F^p(x)\,dx\right)^{\frac{1}{p}}, \tag{27}$$

$$\left\{\int_0^\infty x^{2q-1}\left[\sum_{n=1}^\infty \frac{a_n}{(1+xn)^\lambda}\right]^q dx\right\}^{\frac{1}{q}} > \frac{1}{(\lambda-1)(\lambda-2)}\left(\sum_{n=1}^\infty n^{-q-1}a_n^q\right)^{\frac{1}{q}}. \tag{28}$$

5 Conclusions

In this paper, according to [20, 22], by applying the weight functions, the idea of introduced parameters and Euler–Maclaurin summation formula, a reverse half-discrete Hilbert inequality with the homogeneous kernel and the reverse equivalent forms are given in Lemma 2 and Theorem 1 (for $p < 0$, $0 < q < 1$). The equivalent statements of the best possible constant factor related to some parameters are proved in Theorem 2. As applications, two corollaries about the reverse cases of the non-homogeneous kernel and some particular cases are considered in Corollary 1, Corollary 2 and Remark 3. The lemmas and theorems provide an extensive account of this type of inequalities.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
BY carried out the mathematical studies, participated in the sequence alignment and drafted the manuscript. AW and QC participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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