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Composite inertial subgradient extragradient methods for variational inequalities and fixed point problems

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Abstract

In this paper, we introduce and investigate composite inertial gradient-based algorithms with a line-search process for solving a variational inequality problem (VIP) with a pseudomonotone and Lipschitz continuous mapping and a common fixed-point problem (CFPP) of finitely many nonexpansive mappings and a strictly pseudocontractive mapping in the framework infinite-dimensional Hilbert spaces. The proposed algorithms are based on an inertial subgradient–extragradient method with the line-search process, hybrid steepest-descent methods, viscosity approximation methods and Mann iteration methods. Under weak conditions, we prove strong convergence of the proposed algorithms to the element in the common solution set of the VIP and CFPP, which solves a certain hierarchical VIP defined on this common solution set.

MSC: 47H05; 47H09

Keywords: Subgradient–extragradient method; Pseudomonotone variational inequality problem; Common fixed-point problem; Strictly pseudocontractive mapping

1 Introduction

Throughout this work, H is assumed to be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. We suppose that C is a convex and closed set in Hilbert space H and P_C is the metric projection from space H onto set C. Let $S : C \to H$ be a nonlinear operator. We denote by Fix(S) the set of fixed points of operator S. A mapping $T : C \to C$ is called strictly pseudocontractive if

$$\|Tx - Ty\|^2 \le \zeta \left\|(T-I)x - (T-I)y\right\|^2 + \|x-y\|^2, \quad \forall x,y \in C, \tag{1.1}$$

where $\zeta \in [0,1)$ is some constant. With $\zeta = 0$, T is called a nonexpansive operator. It is well known that the class of strict pseudocontractions includes the class of nonexpansive mappings and is also continuous. From their direct and indirect applications to other branches of mathematics and other engineering fields, to the the class of nonexpansive operators and the class of strictly pseudocontractive operators are now under the spotlights; see, e.g., [1–8] and the references therein.

Let $A : H \to H$ be a mapping defined on the whole space. The known variational inequality problem (VIP), which find lots of applications in sparse reconstruction, image reconstruction, traffic and transportation systems, is to get $x^* \in C$ with $\langle Ax^*, x - x^* \rangle \geq 0$ $\forall x \in C$. The solution set of the VIP is presented with $\mathrm{VI}(C, A)$. At present, one of the most effective methods for solving the VIP is the extragradient method introduced by Korpelevich [9] in 1976, that is, for any initial $x_0 \in C$, the sequence $\{x_n\}$ is generated by

$$
\begin{cases}
y_n = P_C(I - \tau A)x_n, \\
x_{n+1} = P_C(x_n - \tau A y_n) \quad \forall n \geq 0,
\end{cases}
\tag{1.2}
$$

with τ is such that $L\tau < 1$. If $\mathrm{VI}(C, A) \neq \emptyset$, then the sequence $\{x_n\}$ generated by process (1.2) converges weakly to a point, which lies in $\mathrm{VI}(C, A)$. The literature on the VIP is quite vast and Korpelevich's extragradient method, which is projection-based, has received great attention given by many researchers working on nonlinear programming, who improved it in various ways; see, e.g., [10–17] and the references therein. For the well-known extragradient method, one needs to compute two nearest point projections for every iterative step/process. Without question, the projection onto a feasibility (closed-convex) set is closely related to a minimizer problem (min distance). If feasibility sets are general, this might require a prohibitive amount of computation time; For recent efforts on projection-based methods, see, e.g., [18–22]. In 2011, Censor et al. [21] deeply studied Korpelevich's extragradient method and first introduced the subgradient–extragradient method, in which the second projection onto set C is replaced by a projection onto a half-space:

$$
\begin{cases}
y_n = P_C(I - \tau A)x_n, \\
C_n = \{x \in H : \langle x_n - \tau A x_n - y_n, x - y_n \rangle \leq 0\}, \\
x_{n+1} = P_{C_n}(x_n - \tau A y_n) \quad \forall n \geq 0,
\end{cases}
\tag{1.3}
$$

with τ is a parameter such that $L\tau < 1$. In 2014, Kraikaew and Saejung [23] proposed the Halpern subgradient–extragradient method for solving the VIP

$$
\begin{cases}
y_n = P_C(I - \tau A)x_n, \\
C_n = \{x \in H : \langle x_n - \tau A x_n - y_n, x - y_n \rangle \leq 0\}, \\
z_n = P_{C_n}(x_n - \tau A y_n), \\
x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)z_n \quad \forall n \geq 0,
\end{cases}
\tag{1.4}
$$

where τ is a parameter such that $L\tau < 1$, $\{\alpha_n\}$ is a sequence in $(0, 1)$ with $\lim_{n \to \infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = +\infty$. They proved the strong convergence of the sequence $\{x_n\}$ generated by (1.4) to $P_{\mathrm{VI}(C,A)}x_0$.

In 2018, by virtue of the inertial technique, Thong and Hieu [24] first proposed the inertial subgradient–extragradient method, that is, for any initial $x_0, x_1 \in H$, the sequence $\{x_n\}$

is generated by

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \tau A w_n), \\ C_n = \{x \in H : \langle w_n - \tau A w_n - y_n, x - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{C_n}(w_n - \tau A y_n) \quad \forall n \geq 1, \end{cases} \tag{1.5}$$

with τ is a parameter such that $L\tau < 1$. Under some suitable conditions, they proved the weak convergence of $\{x_n\}$ to an element of $\mathrm{VI}(C, A)$. Very recently, Thong and Hieu [25] introduced two inertial subgradient–extragradient algorithms with linear-search process for solving the VIP with monotone and Lipschitz continuous mapping A and the fixed-point problem (FPP) of a quasi-nonexpansive mapping T with a demiclosedness property in a real Hilbert space.

Algorithm 1.1 (see [25, Algorithm 1])

Initialization: Let $x_0, x_1 \in H$ be arbitrary. Given $\gamma > 0$, $l \in (0, 1)$, $\mu \in (0, 1)$.

Iterative Steps: Compute x_{n+1} as follows:

Step 1. Set $w_n = \alpha_n(x_n - x_{n-1}) + x_n$ and compute $y_n = P_C(w_n - \tau_n A w_n)$, where τ_n is chosen to be the largest $\tau \in \{\gamma, \gamma l, \gamma l^2, \ldots\}$ satisfying $\tau \|A w_n - A y_n\| \leq \mu \|w_n - y_n\|$.

Step 2. Compute $z_n = P_{C_n}(w_n - \tau_n A y_n)$ with $C_n := \{x \in H : \langle w_n - \tau_n A w_n - y_n, x - y_n \rangle \leq 0\}$.

Step 3. Compute $x_{n+1} = (1 - \beta_n)w_n + \beta_n T z_n$. If $w_n = z_n = x_{n+1}$ then $w_n \in \mathrm{Fix}(T) \cap \mathrm{VI}(C, A)$. Set $n := n + 1$ and go to Step 1.

Algorithm 1.2 (see [25, Algorithm 2])

Initialization: Let $x_0, x_1 \in H$ be arbitrary. Given $\gamma > 0$, $l \in (0, 1)$, $\mu \in (0, 1)$.

Iterative Steps: Calculate x_{n+1} as follows:

Step 1. Set $w_n = \alpha_n(x_n - x_{n-1}) + x_n$ and compute $y_n = P_C(w_n - \tau_n A w_n)$, where τ_n is chosen to be the largest $\tau \in \{\gamma, \gamma l, \gamma l^2, \ldots\}$ satisfying $\tau \|A w_n - A y_n\| \leq \mu \|w_n - y_n\|$.

Step 2. Compute $z_n = P_{C_n}(w_n - \tau_n A y_n)$ with $C_n := \{x \in H : \langle w_n - \tau_n A w_n - y_n, x - y_n \rangle \leq 0\}$.

Step 3. Compute $x_{n+1} = (1 - \beta_n)x_n + \beta_n T z_n$. If $w_n = z_n = x_n = x_{n+1}$ then $x_n \in \mathrm{Fix}(T) \cap \mathrm{VI}(C, A)$. Set $n := n + 1$ and go to Step 1.

Under some appropriate mild conditions, they proved the weak convergence of the proposed algorithms to an element of $\mathrm{Fix}(T) \cap \mathrm{VI}(C, A)$. Inspired by the research work of [23–28], we introduce two composite inertial subgradient–extragradient algorithms with a line-search process for solving the VIP with pseudomonotone and Lipschitz continuous mapping and the CFPP of finitely many nonexpansive mappings and a strictly pseudo-contractive mapping in a real Hilbert space. The proposed algorithms are based on an inertial subgradient–extragradient method with a line-search process, hybrid steepest-descent methods, viscosity approximation methods and the Mann iteration method. We prove strong convergence of the proposed algorithms to an element in the common solution set of the VIP and CFPP, which solves a certain hierarchical VIP defined on this common solution set. Finally, our main results are applied to solve the VIP and CFPP in an illustrative example.

This paper is organized as follows: In Sect. 2, we recall some definitions and preliminary results for further use. Section 3 deals with the convergence analysis of the proposed algorithms. Some examples are presented to solve the VIP and CFPP.

2 Preliminaries

Let C be a convex nonempty closed set in a real Hilbert space H. Let $\{x_n\}$ be a vector sequence in H, then we denote by $x_n \to x$ (resp., $x_n \rightharpoonup x$) the strong (resp., weak) convergence of $\{x_n\}$ to x. A single-valued mapping $T : C \to H$ is called

(i) L-Lipschitz continuous (or L-Lipschitzian) if $\exists L > 0$ such that $\|Tx - Ty\| \leq L\|x - y\|$ $\forall x, y \in C$;

(ii) monotone if $\langle Tx - Ty, x - y \rangle \geq 0 \ \forall x, y \in C$;

(iii) pseudomonotone if $\langle Tx, y - x \rangle \geq 0 \Rightarrow \langle Ty, y - x \rangle \geq 0 \ \forall x, y \in C$;

(iv) α-strongly monotone if $\exists \alpha > 0$ such that $\langle Tx - Ty, x - y \rangle \geq \alpha\|x - y\|^2 \ \forall x, y \in C$;

(v) sequentially weakly continuous if $\forall \{x_n\} \subset C$, the relation holds:
$x_n \rightharpoonup x \Rightarrow Tx_n \rightharpoonup Tx$.

It is easy to see that every monotone operator is pseudomonotone but the converse is not true. Also, recall that the mapping $T : C \to C$ is a ζ-strict pseudocontraction for some $\zeta \in [0, 1)$ if and only if the inequality holds $\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2 - \frac{1-\zeta}{2}\|(I - T)x - (I - T)y\|^2$ $\forall x, y \in C$. We know that if T is a ζ-strictly pseudocontractive mapping, then T satisfies Lipschitz condition $\|Tx - Ty\| \leq \frac{1+\zeta}{1-\zeta}\|x - y\| \forall x, y \in C$. For each point $x \in H$, we know that there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ $\forall y \in C$. The mapping P_C is called the metric projection of H onto C. The complementary operator of the metric projection operator is monotone.

The following two lemmas are trivial.

Lemma 2.1 *The following hold*:

(i) $\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2 \ \forall x, y \in H$;

(ii) $\langle P_C x - x, y - P_C x \rangle \geq 0 \ \forall x \in H, y \in C$;

(iii) $\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \ \forall x \in H, y \in C$;

(iv) $\|x - y\|^2 + 2\langle x - y, y \rangle = \|x\|^2 - \|y\|^2 \ \forall x, y \in H$;

(v) $\|\lambda x + \mu y\|^2 + \lambda\mu\|x - y\|^2 = \lambda\|x\|^2 + \mu\|y\|^2 \ \forall x, y \in H, \forall \lambda, \mu \in [0, 1]$ *with* $\lambda + \mu = 1$.

Lemma 2.2 *We have the inequality*

$$\|x + y\|^2 \leq 2\langle y, x + y \rangle + \|x\|^2 \quad \forall x, y \in H.$$

Lemma 2.3 ([21]) *Let* $A : C \to H$ *be pseudomonotone and continuous. Then* $x^* \in C$ *is a solution to the VIP* $\langle Ax^*, x - x^* \rangle \geq 0 \ \forall x \in C$, *if and only if* $\langle Ax, x - x^* \rangle \geq 0 \ \forall x \in C$.

Lemma 2.4 ([29]) *Let* $\{a_n\}$ *be a sequence of nonnegative real numbers satisfying the conditions*: $a_{n+1} \leq \lambda_n \gamma_n + (1 - \lambda_n)a_n \ \forall n \geq 1$, *where* $\{\lambda_n\}$ *and* $\{\gamma_n\}$ *are sequences of real numbers such that* (i) $\{\lambda_n\} \subset [0, 1]$ *and* $\sum_{n=1}^{\infty} \lambda_n = \infty$, *and* (ii) $\limsup_{n \to \infty} \gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\lambda_n \gamma_n| < \infty$. *Then* $\lim_{n \to \infty} a_n = 0$.

Lemma 2.5 ([30]) *Let* $T : C \to C$ *be a* ζ-*strict pseudocontraction. Then* $I - T$ *is demiclosed at zero, i.e., if* $\{x_n\}$ *is a sequence in* C *such that* $x_n \rightharpoonup x \in C$ *and* $(T - I)x_n \to 0$, *then* $(I - T)x = 0$, *where* I *is the identity mapping of* H.

Lemma 2.6 ([31]) *Let* $T : C \to C$ *be a* ζ-*strictly pseudocontractive mapping. Let* γ *and* δ *be two nonnegative real numbers. Then* $\|\gamma(x - y) + \delta(Tx - Ty)\| \leq (\gamma + \delta)\|x - y\| \ \forall x, y \in C$ *provided* $(\gamma + \delta)\zeta \leq \gamma$.

The following lemma is a direct consequence of Yamada [32].

Lemma 2.7 *Let $\lambda \in (0,1]$, $T : C \to H$ be a nonexpansive mapping, and the mapping $T^{\lambda} :$
$C \to H$ be defined by $T^{\lambda}x := Tx - \lambda\mu F(Tx)$ $\forall x \in C$, where $F : H \to H$ is κ-Lipschitzian
and η-strongly monotone. Then T^{λ} is a contraction with the coefficient $1 - \lambda\tau$ provided
$0 < \mu < \frac{2\eta}{\kappa^2}$.*

3 Main results

In this section, we use C to denote the the feasible set and assume always that the following
hold.

$T_i : H \to H$ is a nonexpansive mapping for $i = 1,\dots,N$ and $T : H \to H$ is a ζ-strictly
pseudocontractive mapping.

$A : H \to H$ is Lipschitz continuous with the module L, pseudomonotone on H, and se-
quentially weakly continuous on C, such that $\Omega = \bigcap_{i=0}^{N} \mathrm{Fix}(T_i) \cap \mathrm{VI}(C,A) \neq \emptyset$ with $T_0 := T$.

$f : H \to H$ is a contraction with constant $\delta \in [0,1)$, and $F : H \to H$ is η-strongly mono-
tone and κ-Lipschitzian such that $\delta < \tau := 1 - \sqrt{1 - \rho(2\eta - \rho\kappa^2)}$ for $\rho \in (0, \frac{2\eta}{\kappa^2})$.

$\{\sigma_n\} \subset [0,1]$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset (0,1)$ such that

  (i) $\sup_{n\geq 1} \frac{\sigma_n}{\alpha_n} < \infty$ and $\beta_n + \gamma_n + \delta_n = 1$ $\forall n \geq 1$;
  (ii) $\zeta(\gamma_n + \delta_n) \leq \gamma_n$ $\forall n \geq 1$;
  (iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$;
  (iv) $\limsup_{n\to\infty} \beta_n < 1$, $\liminf_{n\to\infty} \beta_n > 0$ and $\liminf_{n\to\infty} \delta_n > 0$.

In terms of Xu and Kim [33], we write $T_n := T_{n \bmod N}$ for integer $n \geq 1$ with the mod
function taking values in the set $\{1, 2, \dots, N\}$, that is, if $n = jN + q$ for some integers $j \geq 0$
and $0 \leq q < N$, then $T_n = T_N$ if $q = 0$ and $T_n = T_q$ if $0 < q < N$.

Algorithm 3.1

Initialization: Given $l \in (0,1)$, $\gamma > 0$ and $\mu \in (0,1)$. Let $x_0, x_1 \in H$ be arbitrary.

Iterative Steps: Calculate x_{n+1} as follows:

Step 1. Set $w_n = \sigma_n(T_n x_n - T_n x_{n-1}) + T_n x_n$ and compute $y_n = P_C(w_n - \tau_n A w_n)$, where τ_n is
chosen to be the largest $\tau \in \{\gamma, \gamma l, \gamma l^2, \dots\}$ satisfying

$$\tau \|Aw_n - Ay_n\| \leq \mu \|w_n - y_n\|. \tag{3.1}$$

Step 2. Compute $z_n = (I - \alpha_n \rho F)P_{C_n}(w_n - \tau_n A y_n) + \alpha_n f(x_n)$ with $C_n := \{x \in H : \langle w_n -
\tau_n A w_n - y_n, x - y_n \rangle \leq 0\}$.

Step 3. Compute

$$x_{n+1} = \beta_n x_n + \gamma_n z_n + \delta_n T z_n. \tag{3.2}$$

Again set $n := n + 1$ and go to Step 1.

Lemma 3.1 *The Armijo-like search rule* (3.1) *is well defined, and* $\min\{\gamma, \frac{\mu l}{L}\} \leq \tau_n \leq \gamma$.

Proof From the L-Lipschitz continuity of A we get $\mu\|w_n - P_C(w_n - \gamma l^m A w_n)\| \geq \frac{\mu}{L}\|Aw_n -
AP_C(w_n - \gamma l^m A w_n)\|$. Thus, (3.1) holds for all $\gamma l^m \leq \frac{\mu}{L}$ and so τ_n is well defined. Obviously,
$\tau_n \geq \gamma$. If $\tau_n = \gamma$, then the inequality is true. If $\tau_n < \gamma$, then from (3.1) we get $\|Aw_n -

$AP_C(w_n - \frac{\tau_n}{l}Aw_n)\| > \frac{\mu}{\frac{\tau_n}{l}}\|w_n - P_C(w_n - \frac{\tau_n}{l}Aw_n)\|$. Again from the L-Lipschitz continuity of A we obtain $\tau_n > \frac{\mu l}{L}$. Hence the inequality is valid.　　□

Lemma 3.2 *Let $\{w_n\}$, $\{y_n\}$, $\{z_n\}$ be the sequences generated by Algorithm* 3.1. *Then*

$$\|z_n - p\|^2 \le (1 - \alpha_n \tau)\|w_n - p\|^2 - (1 - \alpha_n \tau)(1 - \mu)\big[\|w_n - y_n\|^2 + \alpha_n \delta\|x_n - p\|^2$$
$$+ \|u_n - y_n\|^2\big] + 2\alpha_n\langle(f - \rho F)p, z_n - p\rangle \quad \forall p \in \Omega, \tag{3.3}$$

where $u_n := P_{C_n}(w_n - \tau_n A y_n) \ \forall n \ge 1$.

Proof First, take an arbitrary $p \in \Omega \subset C \subset C_n$. We note that

$$\|u_n - p\|^2 = \big\|P_{C_n}p - P_{C_n}(w_n - \tau_n A y_n)\big\|^2 \le \langle w_n - \tau_n A y_n - p, u_n - p\rangle$$
$$= \frac{1}{2}\|u_n - p\|^2 + \frac{1}{2}\|w_n - p\|^2 - \frac{1}{2}\|u_n - w_n\|^2 - \langle u_n - p, \tau_n A y_n\rangle.$$

So, it follows that $\|u_n - p\|^2 \le \|w_n - p\|^2 - \|u_n - w_n\|^2 - 2\langle u_n - p, \tau_n A y_n\rangle$, from which, together with (3.1) and the pseudomonotonicity of A, we deduce that $\langle A y_n, p - y_n\rangle \le 0$ and

$$\|u_n - p\|^2 \le 2\tau_n\big(\langle A y_n, p - y_n\rangle + \langle A y_n, y_n - u_n\rangle\big) - \|u_n - w_n\|^2 + \|w_n - p\|^2$$
$$\le 2\tau_n\langle A y_n, y_n - u_n\rangle - \|u_n - w_n\|^2 + \|w_n - p\|^2$$
$$= 2\langle w_n - \tau_n A y_n - y_n, u_n - y_n\rangle - \|y_n - w_n\|^2 - \|u_n - y_n\|^2 + \|w_n - p\|^2. \tag{3.4}$$

Since $u_n = P_{C_n}(w_n - \tau_n A y_n)$ with $C_n := \{x \in H : \langle w_n - \tau_n A w_n - y_n, x - y_n\rangle \le 0\}$, we have $\langle w_n - \tau_n A w_n - y_n, u_n - y_n\rangle \le 0$, which, together with (3.1), implies that

$$2\langle w_n - \tau_n A y_n - y_n, u_n - y_n\rangle = 2\langle w_n - \tau_n A w_n - y_n, u_n - y_n\rangle + 2\tau_n\langle A w_n - A y_n, u_n - y_n\rangle$$
$$\le 2\mu\|w_n - y_n\|\|u_n - y_n\| \le \mu\big(\|w_n - y_n\|^2 + \|u_n - y_n\|^2\big).$$

Therefore, substituting the last inequality for (3.4), we infer that

$$\|u_n - p\|^2 \le \|w_n - p\|^2 - (1 - \mu)\|w_n - y_n\|^2 - (1 - \mu)\|u_n - y_n\|^2 \quad \forall p \in \Omega. \tag{3.5}$$

In addition, from Algorithm 3.1 we have

$$z_n - p = \alpha_n(f - \rho F)p + \alpha_n\big(f(x_n) - f(p)\big) + (I - \alpha_n\rho F)u_n - (I - \alpha_n\rho F)p.$$

Using Lemma 2.2, Lemma 2.7, and the convexity of the function $h(t) = t^2 \ \forall t \in \mathbf{R}$, from (3.5) we get

$$\|z_n - p\|^2$$
$$\le 2\alpha_n\langle p - z_n, (\rho F - f)p\rangle + \big\|(I - \alpha_n\rho F)u_n - (I - \alpha_n\rho F)p + \alpha_n\big(f(x_n) - f(p)\big)\big\|^2$$
$$\le 2\alpha_n\langle p - z_n, (\rho F - f)p\rangle + \big[\alpha_n\delta\|x_n - p\| + (1 - \alpha_n\tau)\|u_n - p\|\big]^2$$
$$\le 2\alpha_n\langle p - z_n, (\rho F - f)p\rangle + \alpha_n\delta\|x_n - p\|^2 + (1 - \alpha_n\tau)\|u_n - p\|^2$$

$$\leq (1 - \alpha_n \tau)\big[\|w_n - p\|^2 - (1 - \mu)\|w_n - y_n\|^2 - (1 - \mu)\|u_n - y_n\|^2\big]$$
$$+ 2\alpha_n\langle p - z_n, (\rho F - f)p\rangle + \alpha_n \delta \|x_n - p\|^2$$
$$= (1 - \alpha_n \tau)\|w_n - p\|^2 - (1 - \alpha_n \tau)(1 - \mu)\big[\|w_n - y_n\|^2 + \|u_n - y_n\|^2\big]$$
$$+ 2\alpha_n\langle p - z_n, (\rho F - f)p\rangle + \alpha_n \delta \|x_n - p\|^2.$$

This completes the proof.　　　　　　　　　　　　　　　　　　　　　　　　　　□

Lemma 3.3 *Let $\{w_n\}$, $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ be bounded sequences generated by Algorithm* 3.1. *If $x_n - x_{n+1} \to 0$, $w_n - x_n \to 0$, $w_n - y_n \to 0$, $w_n - z_n \to 0$ and $\exists \{w_{n_k}\} \subset \{w_n\}$ such that $w_{n_k} \rightharpoonup z \in H$, then $z \in \Omega$.*

Proof From Algorithm 3.1, we get $T_n x_n - x_n + \alpha_n(T_n x_n - T_n x_{n-1}) = w_n - x_n \ \forall n \geq 1$, and hence $\|T_n x_n - x_n\| \leq \|w_n - x_n\| + \alpha_n\|T_n x_n - T_n x_{n-1}\| \leq \|w_n - x_n\| + \|x_n - x_{n-1}\|$. Utilizing the assumptions $x_n - x_{n+1} \to 0$ and $w_n - x_n \to 0$, we obtain

$$\lim_{n \to \infty} \|x_n - T_n x_n\| = 0. \tag{3.6}$$

Combining the assumptions $w_n - x_n \to 0$ and $w_n - z_n \to 0$ implies that, as $n \to \infty$,

$$\|z_n - x_n\| \leq \|w_n - z_n\| + \|w_n - x_n\| \to 0.$$

Moreover, by Algorithm 3.1 we get $z_n - x_n = u_n - x_n - \alpha_n \rho F u_n + \alpha_n f(x_n)$ with $u_n := P_{C_n}(w_n - \tau_n A y_n)$. So it follows from the boundedness of $\{x_n\}$, $\{u_n\}$ that, as $n \to \infty$,

$$\|u_n - x_n\| = \big\|z_n - x_n - \alpha_n f(x_n) + \alpha_n \rho F u_n\big\| \leq \|z_n - x_n\| + \alpha_n\big(\|f(x_n)\| + \|\rho F u_n\|\big) \to 0.$$

Also, by Algorithm 3.1 we get $\beta_n(x_n - z_n) + \delta_n(T z_n - z_n) = x_{n+1} - z_n$, which immediately yields

$$\delta_n \|T z_n - z_n\| = \big\|x_{n+1} - z_n - \beta_n(x_n - z_n)\big\| \leq \|x_{n+1} - x_n\| + 2\|x_n - z_n\|.$$

Since $x_n - x_{n+1} \to 0$, $z_n - x_n \to 0$ and $\liminf_{n \to \infty} \delta_n > 0$, we obtain

$$\lim_{n \to \infty} \|z_n - T z_n\| = 0. \tag{3.7}$$

Noticing $y_n = P_C(I - \tau_n A)w_n$, we have $\langle w_n - \tau_n A w_n - y_n, x - y_n\rangle \leq 0 \ \forall x \in C$, and hence

$$\langle w_n - y_n, x - y_n\rangle + \tau_n\langle A w_n, y_n - w_n\rangle \leq \tau_n\langle A w_n, x - w_n\rangle \quad \forall x \in C. \tag{3.8}$$

Being weakly convergent, $\{w_{n_k}\}$ is bounded. Then, according to the Lipschitz continuity of A, $\{A w_{n_k}\}$ is bounded. Since $w_n - y_n \to 0$, $\{y_{n_k}\}$ is bounded as well. Note that $L\tau_n \geq \min\{L\gamma, \mu l\}$. So, from (3.8) we get $\liminf_{k \to \infty}\langle A w_{n_k}, x - w_{n_k}\rangle \geq 0 \forall x \in C$. Meanwhile, observe that $\langle x - y_n, A y_n\rangle = \langle A y_n - A w_n, x - w_n\rangle + \langle A w_n, x - w_n\rangle + \langle A y_n, w_n - y_n\rangle$. Since $w_n - y_n \to 0$, from L-Lipschitz continuity of A we obtain $A y_n - A w_n \to 0$, which together with (3.8) yields $\liminf_{k \to \infty}\langle A y_{n_k}, x - y_{n_k}\rangle \geq 0 \ \forall x \in C$.

Next we show that $\lim_{n\to\infty}\|x_n - T_r x_n\| = 0$ for $r = 1,\dots,N$. Indeed, note that, for $i = 1,\dots,N$,

$$2\|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\|$$

$$\geq \|x_n - x_{n+i}\| + \|T_{n+i}x_{n+i} - T_{n+i}x_n\| + \|x_{n+i} - T_{n+i}x_{n+i}\|$$

$$\geq \|x_n - T_{n+i}x_n\|.$$

Hence from (3.6) and the assumption $x_n - x_{n+1} \to 0$ we get $\lim_{n\to\infty}\|x_n - T_{n+i}x_n\| = 0$ for $i = 1,\dots,N$. This immediately implies that

$$\lim_{n\to\infty}\|(T_r - I)x_n\| = 0 \quad \text{for } r = 1,\dots,N. \tag{3.9}$$

We now take a sequence $\{\varepsilon_k\} \subset (0,1)$ satisfying $\varepsilon_k \downarrow 0$ as $k \to \infty$. For all $k \geq 1$, we denote by m_k the smallest positive integer such that

$$\varepsilon_k + \langle Ay_{n_j}, x - y_{n_j}\rangle \geq 0 \quad \forall j \geq m_k. \tag{3.10}$$

Since $\{\varepsilon_k\}$ is decreasing, it is clear that $\{m_k\}$ is increasing. Noticing that $\{y_{m_k}\} \subset C$ guarantees $Ay_{m_k} \neq 0 \ \forall k \geq 1$, we set $u_{m_k} = \frac{Ay_{m_k}}{\|Ay_{m_k}\|^2}$, we get $\langle Ay_{m_k}, u_{m_k}\rangle = 1 \ \forall k \geq 1$. So, from (3.10) we get $\langle Ay_{m_k}, x + \varepsilon_k u_{m_k} - y_{m_k}\rangle \geq 0 \ \forall k \geq 1$. Again from the pseudomonotonicity of A we have $\langle A(x + \varepsilon_k u_{m_k}), x + \varepsilon_k u_{m_k} - y_{m_k}\rangle \geq 0 \ \forall k \geq 1$. This immediately leads to

$$\langle Ax, x - y_{m_k}\rangle \geq \langle Ax - A(x + \varepsilon_k u_{m_k}), x + \varepsilon_k u_{m_k} - y_{m_k}\rangle - \varepsilon_k \langle Ax, u_{m_k}\rangle \quad \forall k \geq 1. \tag{3.11}$$

We claim $\lim_{k\to\infty}\varepsilon_k u_{m_k} = 0$. As fact, from $w_{n_k} \rightharpoonup z$ and $w_n - y_n \to 0$, we obtain $y_{n_k} \rightharpoonup z$. So, $\{y_n\} \subset C$ guarantees $z \in C$. Again from the sequentially weak continuity of A, one knows $Ay_{n_k} \rightharpoonup Az$. Thus, one has $Az \neq 0$ (otherwise, z is a solution). Taking into account the sequentially weak lower semicontinuity of the norm $\|\cdot\|$, one gets $0 < \|Az\| \leq \liminf_{k\to\infty}\|Ay_{n_k}\|$. Note that $\{y_{m_k}\} \subset \{y_{n_k}\}$ and $\varepsilon_k \downarrow 0$ as $k \to \infty$. So it follows that $0 \leq \limsup_{k\to\infty}\|\varepsilon_k u_{m_k}\| = \limsup_{k\to\infty}\frac{\varepsilon_k}{\|Ay_{m_k}\|} \leq \frac{\limsup_{k\to\infty}\varepsilon_k}{\liminf_{k\to\infty}\|Ay_{n_k}\|} = 0$. Hence $\varepsilon_k u_{m_k} \to 0$.

Next one claims $z \in \Omega$. Indeed, from $w_n - x_n \to 0$ and $w_{n_k} \rightharpoonup z$, we get $x_{n_k} \rightharpoonup z$. From (3.9) we have $(x_{n_k} - T_r x_{n_k}) \to 0$ for $r = 1,\dots,N$. Note that Lemma 2.5 guarantees the demiclosedness of $I - T_r$ at zero for $r = 1,\dots,N$. Thus $z \in \text{Fix}(T_r)$. Since r is an arbitrary element in the finite set $\{1,\dots,N\}$, we get $z \in \bigcap_{r=1}^N \text{Fix}(T_r)$. Meanwhile, from $(w_n - z_n) \to 0$ and $w_{n_k} \rightharpoonup z$, we get $z_{n_k} \rightharpoonup z$. From (3.7) we have $(z_{n_k} - Tz_{n_k}) \to 0$. From Lemma 2.5 it follows that $I - T$ is demiclosed at zero, and hence we get $(I - T)z = 0$, i.e., $z \in \text{Fix}(T)$. On the other hand, letting $k \to \infty$, we deduce that the right hand side of (3.11) tends to zero by the uniform continuity of A, the boundedness of $\{w_{m_k}\}$, $\{u_{m_k}\}$ and the limit $\lim_{k\to\infty}\varepsilon_k u_{m_k} = 0$. Thus, we get $\langle Ax, x - z\rangle = \liminf_{k\to\infty}\langle Ax, x - y_{m_k}\rangle \geq 0 \ \forall x \in C$. By Lemma 2.3 we have $z \in \text{VI}(C,A)$. Therefore, $z \in \bigcap_{i=0}^N \text{Fix}(T_i) \cap \text{VI}(C,A) = \Omega$. This completes the proof. $\square$

Theorem 3.1 *Let the sequence $\{x_n\}$ be generated by Algorithm 3.1. Then*

$$x_n \to x^* \in \Omega \quad \Leftrightarrow \quad \begin{cases} x_n - x_{n+1} \to 0, \\ x_n - y_n \to 0, \end{cases}$$

where $x^* \in \Omega$ is a unique solution to the VIP: $\langle(\rho F - f)x^*, p - x^*\rangle \geq 0 \ \forall p \in \Omega$.

Proof First of all, since $\limsup_{n\to\infty} \beta_n < 1$ and $\liminf_{n\to\infty} \beta_n > 0$, we may assume, without loss of generality, that $\{\beta_n\} \subset [a, b] \subset (0, 1)$. one asserts that $P_\Omega(f + I - \rho F)$ is a contraction. This sends to the situation that $P_\Omega(f + I - \rho F)$ is a contraction. Then $P_\Omega(f + I - \rho F)$ has a unique fixed point. Say $x^* \in H$, that is, $x^* = P_\Omega(f + I - \rho F)x^*$. Thus, there is a unique solution $x^* \in \Omega = \bigcap_{i=0}^{N} \mathrm{Fix}(T_i) \cap \mathrm{VI}(C, A)$ to the VIP

$$\langle (\rho F - f)x^*, p - x^* \rangle \geq 0 \quad \forall p \in \Omega. \tag{3.12}$$

It is now easy to see that the necessity of the theorem is valid. Indeed, if $x_n \to x^* \in \Omega = \bigcap_{i=0}^{N} \mathrm{Fix}(T_i) \cap \mathrm{VI}(C, A)$, then $x^* = T_i x^*$ for $i = 0, 1, \ldots, N$ and $x^* = P_C(x^* - \tau_n Ax^*)$, which, together with Algorithm 3.1, imply that

$$\begin{aligned}
\left\| w_n - x^* \right\| &= \left\| T_n x_n - x^* + \alpha_n(T_n x_n - T_n x_{n-1}) \right\| \\
&\leq \left\| x_n - x^* \right\| + \alpha_n \| x_n - x_{n-1} \| \to 0 \quad (n \to \infty),
\end{aligned}$$

and hence

$$\begin{aligned}
\| y_n - x_n \| &\leq \left\| y_n - x^* \right\| + \left\| x_n - x^* \right\| \\
&= \left\| P_C(w_n - \tau_n Aw_n) - P_C\left(x^* - \tau_n Ax^*\right) \right\| + \left\| x_n - x^* \right\| \\
&\leq \left\| w_n - x^* \right\| + \tau_n \left\| Aw_n - Ax^* \right\| + \left\| x_n - x^* \right\| \\
&\leq (1 + \gamma L)\left\| w_n - x^* \right\| + \left\| x_n - x^* \right\| \to 0 \quad (n \to \infty).
\end{aligned}$$

In addition, it is clear that

$$\| x_n - x_{n+1} \| \leq \left\| x_n - x^* \right\| + \left\| x_{n+1} - x^* \right\| \to 0 \quad (n \to \infty).$$

Next we show the sufficiency of the theorem. To this aim, we assume $\lim_{n\to\infty}(\| x_n - x_{n+1} \| + \| x_n - y_n \|) = 0$ and divide the proof of the sufficiency into several steps.

Step 1. We show that $\{x_n\}$ is bounded. Indeed, take an arbitrary $p \in \Omega = \bigcap_{i=0}^{N} \mathrm{Fix}(T_i) \cap \mathrm{VI}(C, A)$. Then $Tp = p$, $T_n p = p$ $\forall n \geq 1$, and (3.5) holds, i.e.,

$$\| u_n - p \|^2 + (1 - \mu)\| w_n - y_n \|^2 + (1 - \mu)\| u_n - y_n \|^2 \leq \| w_n - p \|^2. \tag{3.13}$$

This immediately implies that

$$\| u_n - p \| \leq \| w_n - p \| \quad \forall n \geq 1. \tag{3.14}$$

From the definition of w_n, we get

$$\| w_n - p \| \leq \| T_n x_n - p \| + \sigma_n \| T_n x_n - T_n x_{n-1} \| \leq \| x_n - p \| + \alpha_n \cdot \frac{\sigma_n}{\alpha_n} \| x_n - x_{n-1} \|. \tag{3.15}$$

Since $\sup_{n\geq 1} \frac{\sigma_n}{\alpha_n} < \infty$ and $\sup_{n\geq 1} \| x_n - x_{n-1} \| < \infty$, $\sup_{n\geq 1} \frac{\sigma_n}{\alpha_n} \| x_n - x_{n-1} \| < \infty$, which hence guarantees that there is $M_1 > 0$ such that

$$\frac{\sigma_n}{\alpha_n} \| x_n - x_{n-1} \| \leq M_1 \quad \forall n \geq 1. \tag{3.16}$$

Combining (3.14), (3.15) and (3.16), we obtain

$$\|u_n - p\| \le \|w_n - p\| \le \alpha_n M_1 + \|x_n - p\| \quad \forall n \ge 1. \tag{3.17}$$

So, from Algorithm 3.1, Lemma 2.7 and (3.17) it follows that

$$
\begin{aligned}
\|z_n - p\| &\le (1 - \alpha_n \tau)\|u_n - p\| + \alpha_n \|(f - \rho F)p\| + \alpha_n \delta \|x_n - p\| \\
&\le (1 - \alpha_n \tau)\big(\|x_n - p\| + \alpha_n M_1\big) + \alpha_n \|(f - \rho F)p\| + \alpha_n \delta \|x_n - p\| \\
&\le \big(1 - \alpha_n(\tau - \delta)\big)\|x_n - p\| + \alpha_n\big(M_1 + \|(f - \rho F)p\|\big),
\end{aligned}
$$

which, together with Lemma 2.6 and $(\gamma_n + \delta_n)\zeta \le \gamma_n$, yields

$$
\begin{aligned}
\|x_{n+1} - p\| &\le (1 - \beta_n)\left\| \frac{1}{1 - \beta_n}\big[\gamma_n(z_n - p) + \delta_n(Tz_n - p)\big] \right\| + \beta_n \|x_n - p\| \\
&\le (1 - \beta_n)\|z_n - p\| + \beta_n \|x_n - p\| \\
&\le (1 - \beta_n)\big[\big(1 - \alpha_n(\tau - \delta)\big)\|x_n - p\| + \alpha_n\big(M_1 + \|(f - \rho F)p\|\big)\big] + \beta_n \|x_n - p\| \\
&= \alpha_n(1 - \beta_n)\big(M_1 + \|(f - \rho F)p\|\big) + \big[1 - \alpha_n(1 - \beta_n)(\tau - \delta)\big]\|x_n - p\| \\
&= \alpha_n(1 - \beta_n)(\tau - \delta)\frac{M_1 + \|(f - \rho F)p\|}{\tau - \delta} + \big[1 - \alpha_n(1 - \beta_n)(\tau - \delta)\big]\|x_n - p\| \\
&\le \max\left\{ \frac{M_1 + \|(f - \rho F)p\|}{\tau - \delta}, \|x_n - p\| \right\}.
\end{aligned}
$$

By induction, we obtain $\|x_n - p\| \le \max\{\|x_1 - p\|, \frac{M_1 + \|(\rho F - f)p\|}{\tau - \delta}\}$ $\forall n \ge 1$. Thus, $\{x_n\}$ is a bounded point sequence, and so are the sequences $\{u_n\}$, $\{w_n\}$, $\{y_n\}$, $\{z_n\}$, $\{f(x_n)\}$, $\{Tz_n\}$, $\{Fu_n\}$, $\{T_n x_n\}$.

  Step 2. We show that

$$(1 - \alpha_n \tau)(1 - \beta_n)(1 - \mu)\big[\|w_n - y_n\|^2 + \|u_n - y_n\|^2\big] \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n M_4,$$

for some $M_4 > 0$. Indeed, utilizing Lemma 2.6, Lemma 3.2 and the convexity of $\|\cdot\|^2$, from $(\gamma_n + \delta_n)\zeta \le \gamma_n$ we get

$$
\begin{aligned}
\|x_{n+1} &- p\|^2 \\
&\le (1 - \beta_n)\left\| \frac{1}{1 - \beta_n}\big[\gamma_n(z_n - p) + \delta_n(Tz_n - p)\big] \right\|^2 + \beta_n \|x_n - p\|^2 \\
&\le (1 - \beta_n)\|z_n - p\|^2 + \beta_n \|x_n - p\|^2 \\
&\le (1 - \beta_n)\big\{\alpha_n \delta \|x_n - p\|^2 + (1 - \alpha_n \tau)\|w_n - p\|^2 + \beta_n \|x_n - p\|^2 \\
&\quad - (1 - \alpha_n \tau)(1 - \mu)\big[\|w_n - y_n\|^2 + \|u_n - y_n\|^2\big] + 2\alpha_n \langle (f - \rho F)p, z_n - p \rangle\big\} \\
&\le (1 - \beta_n)\big\{\alpha_n \delta \|x_n - p\|^2 + (1 - \alpha_n \tau)\|w_n - p\|^2 \\
&\quad - (1 - \alpha_n \tau)(1 - \mu)\big[\|w_n - y_n\|^2 + \|u_n - y_n\|^2\big] + \alpha_n M_2\big\} + \beta_n \|x_n - p\|^2, \tag{3.18}
\end{aligned}
$$

where $\sup_{n\geq 1} 2\|(f - \rho F)p\|\|z_n - p\| \leq M_2$ for some $M_2 > 0$. Also, from (3.17) we have

$$
\begin{aligned}
\|w_n - p\|^2 &\leq \left(\|x_n - p\| + \alpha_n M_1\right)^2 \\
&= \alpha_n\left(2M_1\|x_n - p\| + \alpha_n M_1^2\right) + \|x_n - p\|^2 \\
&\leq \alpha_n M_3 + \|x_n - p\|^2,
\end{aligned}
\tag{3.19}
$$

where $\sup_{n\geq 1}(2M_1\|x_n - p\| + \beta_n M_1^2) \leq M_3$ for some $M_3 > 0$. Substituting (3.19) for (3.18), we obtain

$$
\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad\leq (1 - \beta_n)\big\{\alpha_n\delta\|x_n - p\|^2 + (1 - \alpha_n\tau)\big[\|x_n - p\|^2 + \alpha_n M_3\big] + \beta_n\|x_n - p\|^2 \\
&\qquad - (1 - \alpha_n\tau)(1 - \mu)\big[\|w_n - y_n\|^2 + \|u_n - y_n\|^2\big] + \alpha_n M_2\big\} \\
&\quad= (1 - \beta_n)\big\{\big(1 - \alpha_n(\tau - \delta)\big)\|x_n - p\|^2 + (1 - \alpha_n\tau)\alpha_n M_3 + \beta_n\|x_n - p\|^2 \\
&\qquad - (1 - \alpha_n\tau)(1 - \mu)\big[\|w_n - y_n\|^2 + \|u_n - y_n\|^2\big] + \alpha_n M_2\big\} \\
&\quad= \big[1 - \alpha_n(1 - \beta_n)(\tau - \delta)\big]\|x_n - p\|^2 + (1 - \beta_n)(1 - \alpha_n\tau)\alpha_n M_3 \\
&\qquad - (1 - \alpha_n\tau)(1 - \beta_n)(1 - \mu)\big[\|w_n - y_n\|^2 + \|u_n - y_n\|^2\big] + (1 - \beta_n)\alpha_n M_2 \\
&\quad\leq \|x_n - p\|^2 - (1 - \alpha_n\tau)(1 - \beta_n)(1 - \mu)\big[\|w_n - y_n\|^2 + \|u_n - y_n\|^2\big] + \alpha_n M_4,
\end{aligned}
\tag{3.20}
$$

where $M_4 := M_2 + M_3$. This immediately implies that

$$
\begin{aligned}
&(1 - \alpha_n\tau)(1 - \beta_n)(1 - \mu)\big[\|w_n - y_n\|^2 + \|u_n - y_n\|^2\big] \\
&\quad\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n M_4.
\end{aligned}
\tag{3.21}
$$

Step 3. We show that

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \big[1 - \alpha_n(1 - \beta_n)(\tau - \delta)\big]\|x_n - p\|^2 + \alpha_n(1 - \beta_n)(\tau - \delta) \\
&\quad \times \left[\frac{2}{\tau - \delta}\langle(f - \rho F)p, z_n - p\rangle + \frac{M}{\tau - \delta}\cdot\frac{\sigma_n}{\alpha_n}\cdot 3\|x_n - x_{n-1}\|\right]
\end{aligned}
$$

for some $M > 0$. Indeed, we have

$$
\begin{aligned}
\|w_n - p\|^2 &\leq \left(\|x_n - p\| + \sigma_n\|x_n - x_{n-1}\|\right)^2 \\
&= \|x_n - p\|^2 + \sigma_n\|x_n - x_{n-1}\|\left(2\|x_n - p\| + \sigma_n\|x_n - x_{n-1}\|\right).
\end{aligned}
\tag{3.22}
$$

Combining (3.18) and (3.22), we have

$$
\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad\leq \beta_n\|x_n - p\|^2 + (1 - \beta_n)\big[\alpha_n\delta\|x_n - p\|^2 \\
&\qquad + (1 - \alpha_n\tau)\|w_n - p\|^2 + 2\alpha_n\langle(f - \rho F)p, z_n - p\rangle\big] \\
&\quad\leq \beta_n\|x_n - p\|^2 + (1 - \beta_n)\big\{\alpha_n\delta\|x_n - p\|^2 + (1 - \alpha_n\tau)\big[\|x_n - p\|^2
\end{aligned}
$$

$$+ \sigma_n \|x_n - x_{n-1}\| \big(2\|x_n - p\| + \sigma_n \|x_n - x_{n-1}\| \big) \big] + 2\alpha_n \langle (f - \rho F)p, z_n - p \rangle \big\}$$

$$\leq \beta_n \|x_n - p\|^2 + (1 - \beta_n) \big\{ \big(1 - \alpha_n(\tau - \delta) \big) \|x_n - p\|^2$$

$$+ \sigma_n \|x_n - x_{n-1}\| \big(2\|x_n - p\| + \sigma_n \|x_n - x_{n-1}\| \big) + 2\alpha_n \langle (f - \rho F)p, z_n - p \rangle \big\}$$

$$= \big[1 - \alpha_n(1 - \beta_n)(\tau - \delta) \big] \|x_n - p\|^2 + (1 - \beta_n)\sigma_n \|x_n - x_{n-1}\|$$

$$\times \big(2\|x_n - p\| + \sigma_n \|x_n - x_{n-1}\| \big) + 2\alpha_n(1 - \beta_n)\langle (f - \rho F)p, z_n - p \rangle$$

$$\leq \big[1 - \alpha_n(1 - \beta_n)(\tau - \delta) \big] \|x_n - p\|^2 + \alpha_n(1 - \beta_n)(\tau - \delta) \bigg[\frac{2}{\tau - \delta} \langle (f - \rho F)p, z_n - p \rangle$$

$$+ \frac{M}{\tau - \delta} \cdot \frac{\sigma_n}{\alpha_n} \cdot 3\|x_n - x_{n-1}\| \bigg], \tag{3.23}$$

where $\sup_{n \geq 1}\{\|x_n - p\|, \alpha_n\|x_n - x_{n-1}\|\} \leq M$ for some $M > 0$.

Step 4. We show that $\{x_n\}$ converges strongly to a unique solution $x^* \in \Omega$ to the VIP (3.12). Indeed, putting $p = x^*$, we deduce from (3.23) that

$$\|x_{n+1} - x^*\|^2 \leq \big[1 - \alpha_n(1 - \beta_n)(\tau - \delta) \big] \|x_n - x^*\|^2$$

$$+ \alpha_n(1 - \beta_n)(\tau - \delta) \bigg[\frac{2}{\tau - \delta} \langle (f - \rho F)x^*, z_n - x^* \rangle + \frac{M}{\tau - \delta} \cdot \frac{\sigma_n}{\alpha_n} \cdot 3\|x_n - x_{n-1}\| \bigg]. \tag{3.24}$$

By Lemma 2.4, it suffices to show that $\limsup_{n \to \infty}\langle (f - \rho F)x^*, z_n - x^* \rangle \leq 0$. From (3.21), $x_n - x_{n+1} \to 0$, $\alpha_n \to 0$ and $\{\beta_n\} \subset [a, b] \subset (0, 1)$, we obtain

$$\limsup_{n \to \infty}(1 - \alpha_n\tau)(1 - b)(1 - \mu)\big[\|w_n - y_n\|^2 + \|u_n - y_n\|^2 \big]$$

$$\leq \limsup_{n \to \infty}(1 - \alpha_n\tau)(1 - \beta_n)(1 - \mu)\big[\|w_n - y_n\|^2 + \|u_n - y_n\|^2 \big]$$

$$\leq \limsup_{n \to \infty}\big[\|x_n - p\|^2 + \alpha_n M_4 - \|x_{n+1} - p\|^2 \big]$$

$$\leq \limsup_{n \to \infty}\big[\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \big] + \limsup_{n \to \infty}\alpha_n M_4$$

$$\leq \limsup_{n \to \infty}\big(\|x_n - p\| + \|x_{n+1} - p\| \big)\|x_n - x_{n+1}\| = 0.$$

This immediately implies that

$$\lim_{n \to \infty}\|w_n - y_n\| = 0 \quad \text{and} \quad \lim_{n \to \infty}\|u_n - y_n\| = 0. \tag{3.25}$$

Since $z_n = (I - \alpha_n\rho F)u_n + \alpha_n f(x_n)$ with $u_n := P_{C_n}(w_n - \tau_n A y_n)$, from (3.25) and the boundedness of $\{x_n\}$, $\{u_n\}$, we get

$$\|z_n - y_n\| = \big\| \alpha_n f(x_n) - \alpha_n\rho F u_n + u_n - y_n \big\|$$

$$\leq \|u_n - y_n\| + \alpha_n\big(\|f(x_n)\| + \|\rho F u_n\| \big) \to 0 \quad (n \to \infty), \tag{3.26}$$

and hence

$$\|z_n - x_n\| \leq \|z_n - y_n\| + \|y_n - x_n\| \to 0 \quad (n \to \infty) \tag{3.27}$$

(due to the assumption $\|x_n - y_n\| \to 0$). Obviously, the assumption $\|x_n - y_n\| \to 0$ together with (3.25) and (3.26), guarantees that, as $n \to \infty$, $\|w_n - x_n\| \leq \|w_n - y_n\| + \|y_n - x_n\| \to 0$ and $\|w_n - z_n\| \leq \|w_n - y_n\| + \|y_n - z_n\| \to 0$. From the property of $\{z_n\}$, it follows that there is a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that

$$\limsup_{n \to \infty} \langle (f - \rho F)x^*, z_n - x^* \rangle = \lim_{k \to \infty} \langle (f - \rho F)x^*, z_{n_k} - x^* \rangle. \tag{3.28}$$

Since H is reflexive and $\{z_n\}$ is bounded, it is guaranteed that $z_{n_k} \rightharpoonup \tilde{z}$. Hence from (3.28) one gets

$$\limsup_{n \to \infty} \langle (f - \rho F)x^*, z_n - x^* \rangle = \lim_{k \to \infty} \langle (f - \rho F)x^*, z_{n_k} - x^* \rangle = \langle (f - \rho F)x^*, \tilde{z} - x^* \rangle. \tag{3.29}$$

It is easy to see from $w_n - z_n \to 0$ and $z_{n_k} \rightharpoonup \tilde{z}$ that $w_{n_k} \rightharpoonup \tilde{z}$. Since $x_n - x_{n+1} \to 0$, $w_n - x_n \to 0$, $w_n - y_n \to 0$, $w_n - z_n \to 0$ and $w_{n_k} \rightharpoonup \tilde{z}$, by Lemma 3.3 we infer that $\tilde{z} \in \Omega$. Therefore, from (3.12) and (3.29) we conclude that

$$\limsup_{n \to \infty} \langle (f - \rho F)x^*, z_n - x^* \rangle = \langle (f - \rho F)x^*, \tilde{z} - x^* \rangle \leq 0. \tag{3.30}$$

Note that $\{\beta_n\} \subset [a, b] \subset (0, 1)$, $\{\alpha_n(1 - \beta_n)(\tau - \delta)\} \subset [0, 1]$, $\sum_{n=1}^{\infty} \alpha_n(1 - \beta_n)(\tau - \delta) = \infty$, and

$$\limsup_{n \to \infty} \left[\frac{2}{\tau - \delta} \langle (f - \rho F)x^*, z_n - x^* \rangle + \frac{M}{\tau - \delta} \cdot \frac{\sigma_n}{\alpha_n} \cdot 3\|x_n - x_{n-1}\| \right] \leq 0. \tag{3.31}$$

Consequently, applying Lemma 2.4 to (3.24), we have $\lim_{n \to 0} \|x_n - x^*\| = 0$. This completes the proof. $\qquad\square$

Next, we introduce another composite inertial subgradient–extragradient algorithms with a line-search process.

Algorithm 3.2

Initialization: Given $\gamma > 0$, $l \in (0, 1)$, $\mu \in (0, 1)$. Let $x_0, x_1 \in H$ be arbitrary.

Iterative Steps: Calculate x_{n+1} as follows:

Step 1. Set $w_n = \sigma_n(T_n x_n - T_n x_{n-1}) + T_n x_n$ and compute $y_n = P_C(w_n - \tau_n A w_n)$, where τ_n is chosen to be the largest $\tau \in \{\gamma, \gamma l, \gamma l^2, \ldots\}$ satisfying

$$\tau \|A w_n - A y_n\| \leq \mu \|w_n - y_n\|. \tag{3.32}$$

Step 2. Compute $z_n = (I - \alpha_n \rho F)P_{C_n}(w_n - \tau_n A y_n) + \alpha_n f(x_n)$ with $C_n := \{x \in H : \langle w_n - \tau_n A w_n - y_n, x - y_n \rangle \leq 0\}$.

Step 3. Compute

$$x_{n+1} = \beta_n w_n + \gamma_n z_n + \delta_n T z_n. \tag{3.33}$$

Again set $n := n + 1$ and go to Step 1.

It is worth pointing out that Lemmas 3.1, 3.2 and 3.3 are still valid for Algorithm 3.2.

Theorem 3.2 *Let the sequence $\{x_n\}$ be generated by Algorithm 3.2. Then*

$$x_n \to x^* \in \Omega \quad \Leftrightarrow \quad \begin{cases} x_n - x_{n+1} \to 0, \\ x_n - y_n \to 0, \end{cases}$$

where $x^* \in \Omega$ is a unique solution to the VIP: $\langle (\rho F - f)x^*, p - x^* \rangle \geq 0 \; \forall p \in \Omega$.

Proof Utilizing the same arguments as in the proof of Theorem 3.1, we deduce that there exists a unique solution $x^* \in \Omega = \bigcap_{i=0}^{N} \operatorname{Fix}(T_i) \cap \operatorname{VI}(C, A)$ to the VIP (3.12), and that the necessity of the theorem is valid.

Next we show the sufficiency of the theorem. To the aim, we assume $\lim_{n \to \infty} (\|x_n - x_{n+1}\| + \|x_n - y_n\|) = 0$ and divide the proof of the sufficiency into several steps.

Step 1. We show that $\{x_n\}$ is bounded. Indeed, utilizing the same arguments as in Step 1 of the proof of Theorem 3.1, we obtain inequalities (3.13)–(3.17). So, from Algorithm 3.2, Lemma 2.7 and (3.17) it follows that

$$
\begin{aligned}
\|z_n - p\| &= \left\| (I - \alpha_n \rho F)u_n - (I - \alpha_n \rho F)p + \alpha_n (f - \rho F)p + \alpha_n \big(f(x_n) - f(p)\big) \right\| \\
&\leq \alpha_n \big(\|(f - \rho F)p\| + M_1 \big) + \big(1 - \alpha_n(\tau - \delta)\big)\|x_n - p\|,
\end{aligned}
$$

which, together with Lemma 2.6 and $(\gamma_n + \delta_n)\zeta \leq \gamma_n$, yields

$$
\begin{aligned}
\|x_{n+1} - p\| &= \left\| \gamma_n(z_n - p) + \delta_n(Tz_n - p) + \beta_n(w_n - p) \right\| \\
&\leq (1 - \beta_n) \left\| \frac{1}{1 - \beta_n} \big[\gamma_n(z_n - p) + \delta_n(Tz_n - p) \big] \right\| + \beta_n \|w_n - p\| \\
&\leq (1 - \beta_n)\|z_n - p\| + \beta_n \big(\alpha_n M_1 + \|x_n - p\| \big) \\
&\leq (1 - \beta_n) \big[\big(1 - \alpha_n(\tau - \delta)\big)\|x_n - p\| + \alpha_n \big(M_1 + \|(f - \rho F)p\| \big) \big] \\
&\quad + \beta_n \big(\alpha_n M_1 + \|x_n - p\| \big) \\
&= \alpha_n(1 - \beta_n) \left(\frac{M_1}{1 - \beta_n} + \|(f - \rho F)p\| \right) + \big[1 - \alpha_n(1 - \beta_n)(\tau - \delta)\big]\|x_n - p\| \\
&\leq \alpha_n(1 - \beta_n)(\tau - \delta) \cdot \frac{\frac{M_1}{1 - b} + \|(f - \rho F)p\|}{\tau - \delta} + \big[1 - \alpha_n(1 - \beta_n)(\tau - \delta)\big]\|x_n - p\| \\
&\leq \max \left\{ \frac{\frac{M_1}{1 - b} + \|(f - \rho F)p\|}{\tau - \delta}, \|x_n - p\| \right\}.
\end{aligned}
$$

By induction, we obtain $\|x_n - p\| \leq \max\{\frac{\frac{M_1}{1-b} + \|(f - \rho F)p\|}{\tau - \delta}, \|x_1 - p\|\} \forall n \geq 1$. Thus, $\{x_n\}$ is bounded point sequence. Hence the sequences $\{u_n\}$, $\{w_n\}$, $\{y_n\}$, $\{z_n\}$, $\{f(x_n)\}$, $\{Tz_n\}$, $\{Fu_n\}$, $\{T_n x_n\}$ are also bounded sequences.

Step 2. We show that

$$
\begin{aligned}
&(1 - \alpha_n \tau)(1 - \beta_n)(1 - \mu)\big[\|w_n - y_n\|^2 + \|u_n - y_n\|^2 \big] \\
&\quad \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n M_4,
\end{aligned}
$$

for some $M_4 > 0$. Indeed, utilizing Lemma 2.6, Lemma 3.2 and the convexity of $\| \cdot \|^2$, from $(\gamma_n + \delta_n)\zeta \leq \gamma_n$ we get

$$
\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad = \big\| \gamma_n(z_n - p) + \delta_n(Tz_n - p) + \beta_n(w_n - p) \big\|^2 \\
&\quad \leq (1 - \beta_n) \bigg\| \frac{1}{1 - \beta_n} \big[\gamma_n(z_n - p) + \delta_n(Tz_n - p) \big] \bigg\|^2 + \beta_n \|w_n - p\|^2 \\
&\quad \leq (1 - \beta_n)\|z_n - p\|^2 + \beta_n \|w_n - p\|^2 \\
&\quad \leq (1 - \beta_n)\big\{ \alpha_n\delta\|x_n - p\|^2 + (1 - \alpha_n\tau)\|w_n - p\|^2 + \beta_n\|w_n - p\|^2 \\
&\qquad - (1 - \alpha_n\tau)(1 - \mu)\big[\|w_n - y_n\|^2 + \|u_n - y_n\|^2 \big] + 2\alpha_n\langle(f - \rho F)p, z_n - p\rangle \big\} \\
&\quad \leq (1 - \beta_n)\big\{ \alpha_n\delta\|x_n - p\|^2 + (1 - \alpha_n\tau)\|w_n - p\|^2 + \beta_n\|w_n - p\|^2 \\
&\qquad - (1 - \alpha_n\tau)(1 - \mu)\big[\|w_n - y_n\|^2 + \|u_n - y_n\|^2 \big] + \alpha_n M_2 \big\},
\end{aligned}
\tag{3.34}
$$

where $\sup_{n\geq 1} 2\|(f - \rho F)p\|\|z_n - p\| \leq M_2$ for some $M_2 > 0$. Also, from (3.17) we have

$$
\begin{aligned}
\|w_n - p\|^2 &\leq \big(\|x_n - p\| + \alpha_n M_1 \big)^2 \\
&= \alpha_n\big(2M_1\|x_n - p\| + \alpha_n M_1^2 \big) + \|x_n - p\|^2 \\
&\leq \alpha_n M_3 + \|x_n - p\|^2,
\end{aligned}
\tag{3.35}
$$

where $\sup_{n\geq 1}(2M_1\|x_n - p\| + \beta_n M_1^2) \leq M_3$ for some $M_3 > 0$. Substituting (3.35) for (3.34) guarantees

$$
\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad \leq \beta_n\big[\|x_n - p\|^2 + \alpha_n M_3 \big] + (1 - \beta_n)\big\{ \alpha_n\delta\|x_n - p\|^2 + (1 - \alpha_n\tau)\big[\|x_n - p\|^2 \\
&\qquad + \alpha_n M_3 \big] - (1 - \alpha_n\tau)(1 - \mu)\big[\|w_n - y_n\|^2 + \|u_n - y_n\|^2 \big] + \alpha_n M_2 \big\} \\
&\quad = \beta_n\|x_n - p\|^2 + \beta_n\alpha_n M_3 + (1 - \beta_n)\big\{ \big(1 - \alpha_n(\tau - \delta)\big)\|x_n - p\|^2 \\
&\qquad + (1 - \alpha_n\tau)\alpha_n M_3 - (1 - \alpha_n\tau)(1 - \mu)\big[\|w_n - y_n\|^2 + \|u_n - y_n\|^2 \big] + \alpha_n M_2 \big\} \\
&\quad = \big[1 - \alpha_n(1 - \beta_n)(\tau - \delta) \big]\|x_n - p\|^2 + \beta_n\alpha_n M_3 + (1 - \beta_n)(1 - \alpha_n\tau)\alpha_n M_3 \\
&\qquad - (1 - \alpha_n\tau)(1 - \beta_n)(1 - \mu)\big[\|w_n - y_n\|^2 + \|u_n - y_n\|^2 \big] + (1 - \beta_n)\alpha_n M_2 \\
&\quad \leq \|x_n - p\|^2 - (1 - \alpha_n\tau)(1 - \beta_n)(1 - \mu)\big[\|w_n - y_n\|^2 + \|u_n - y_n\|^2 \big] + \alpha_n M_4,
\end{aligned}
\tag{3.36}
$$

where $M_4 := M_2 + M_3$. This immediately implies that

$$
\begin{aligned}
&(1 - \alpha_n\tau)(1 - \mu)(1 - \beta_n)\big[\|w_n - y_n\|^2 + \|u_n - y_n\|^2 \big] \\
&\quad \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n M_4.
\end{aligned}
\tag{3.37}
$$

Step 3. We show that

$$\|x_{n+1} - p\|^2 \leq \left[1 - \alpha_n(1 - \beta_n)(\tau - \delta)\right]\|x_n - p\|^2 + \alpha_n(1 - \beta_n)(\tau - \delta)$$
$$\times \left[\frac{2}{\tau - \delta}\langle(f - \rho F)p, z_n - p\rangle + \frac{M}{\tau - \delta} \cdot \frac{\sigma_n}{\alpha_n} \cdot \frac{3}{1 - b}\|x_n - x_{n-1}\|\right]$$

for some $M > 0$. Indeed, we have

$$\|w_n - p\|^2 \leq \left(\sigma_n\|x_n - x_{n-1}\| + \|x_n - p\|\right)^2$$
$$= \sigma_n\|x_n - x_{n-1}\|\left(2\|x_n - p\| + \sigma_n\|x_n - x_{n-1}\|\right) + \|x_n - p\|^2. \tag{3.38}$$

Combining (3.34) and (3.38), we have

$$\|x_{n+1} - p\|^2$$
$$\leq (1 - \beta_n)\left[\alpha_n\delta\|x_n - p\|^2 + (1 - \alpha_n\tau)\|w_n - p\|^2\right.$$
$$\left. + 2\alpha_n\langle(f - \rho F)p, z_n - p\rangle\right] + \beta_n\|w_n - p\|^2$$
$$\leq \beta_n\left[\|x_n - p\|^2 + \sigma_n\|x_n - x_{n-1}\|\left(2\|x_n - p\| + \sigma_n\|x_n - x_{n-1}\|\right)\right]$$
$$+ (1 - \beta_n)\left\{\alpha_n\delta\|x_n - p\|^2\right.$$
$$+ (1 - \alpha_n\tau)\left[\|x_n - p\|^2 + \sigma_n\|x_n - x_{n-1}\|\left(2\|x_n - p\| + \sigma_n\|x_n - x_{n-1}\|\right)\right]$$
$$\left. + 2\alpha_n\langle(f - \rho F)p, z_n - p\rangle\right\}$$
$$\leq \beta_n\sigma_n\|x_n - x_{n-1}\|\left(2\|x_n - p\| + \sigma_n\|x_n - x_{n-1}\|\right) + \beta_n\|x_n - p\|^2$$
$$+ (1 - \beta_n)\left\{\left(1 - \alpha_n(\tau - \delta)\right)\|x_n - p\|^2 + \sigma_n\|x_n - x_{n-1}\|\left(2\|x_n - p\| + \sigma_n\|x_n - x_{n-1}\|\right)\right.$$
$$\left. + 2\alpha_n\langle(f - \rho F)p, z_n - p\rangle\right\}$$
$$= \left[1 - \alpha_n(1 - \beta_n)(\tau - \delta)\right]\|x_n - p\|^2 + \sigma_n\|x_n - x_{n-1}\|$$
$$\times \left(2\|x_n - p\| + \sigma_n\|x_n - x_{n-1}\|\right) + 2\alpha_n(1 - \beta_n)\langle(f - \rho F)p, z_n - p\rangle$$
$$\leq \left[1 - \alpha_n(1 - \beta_n)(\tau - \delta)\right]\|x_n - p\|^2 + \alpha_n(1 - \beta_n)(\tau - \delta)\left[\frac{2}{\tau - \delta}\langle p - z_n, (\rho F - f)p\rangle\right.$$
$$\left. + \frac{M}{(1 - \beta_n)(\tau - \delta)} \cdot \frac{\sigma_n}{\alpha_n} \cdot 3\|x_n - x_{n-1}\|\right]$$
$$\leq \left[1 - \alpha_n(1 - \beta_n)(\tau - \delta)\right]\|x_n - p\|^2 + \alpha_n(1 - \beta_n)(\tau - \delta)\left[\frac{2}{\tau - \delta}\langle p - z_n, (\rho F - f)p\rangle\right.$$
$$\left. + \frac{M}{\tau - \delta} \cdot \frac{\sigma_n}{\alpha_n} \cdot \frac{3}{1 - b}\|x_n - x_{n-1}\|\right], \tag{3.39}$$

where $\sup_{n \geq 1}\{\|x_n - p\|, \alpha_n\|x_n - x_{n-1}\|\} \leq M$ for some $M > 0$.

Step 4. We show that $\{x_n\}$ converges strongly to a unique solution $x^* \in \Omega$ to the VIP (3.12). Indeed, utilizing the same argument as in Step 4 of the proof of Theorem 3.1, we obtain the desired assertion. This completes the proof. □

Remark 3.1 Compared with the corresponding results in Kraikaew and Saejung [23], Thong and Hieu [24, 25], our results improve and extend them in the following aspects.

(i) The problem of finding an element of $\mathrm{VI}(C,A)$ in [23] is extended to develop our problem of finding an element of $\bigcap_{i=0}^{N} \mathrm{Fix}(T_i) \cap \mathrm{VI}(C,A)$ where T_i is nonexpansive for $i = 1,\ldots,N$ and $T_0 = T$ is strictly pseudocontractive. The Halpern subgradient–extragradient method for solving the VIP in [23] is extended to develop our composite inertial subgradient–extragradient method with line-search process for solving the VIP and CFPP, which is based on an inertial subgradient–extragradient method with a line-search process, a hybrid steepest-descent method, a viscosity approximation method and a Mann iteration method.

(ii) The problem of finding an element of $\mathrm{VI}(C,A)$ in [24] is extended to develop our problem of finding an element of $\bigcap_{i=0}^{N} \mathrm{Fix}(T_i) \cap \mathrm{VI}(C,A)$ where T_i is nonexpansive for $i = 1,\ldots,N$ and $T_0 = T$ is strictly pseudocontractive. The inertial subgradient–extragradient method with weak convergence for solving the VIP in [24] is extended to develop our composite inertial subgradient–extragradient method with line-search process (which is strongly convergent) for solving the VIP and CFPP, which is based on an inertial subgradient–extragradient method with line-search process, a hybrid steepest-descent method, a viscosity approximation method and a Mann iteration method.

(iii) The problem of finding an element of $\mathrm{VI}(C,A) \cap \mathrm{Fix}(T)$ (where A is monotone and T is quasi-nonexpansive) in [23] is extended to develop our problem of finding an element of $\bigcap_{i=0}^{N} \mathrm{Fix}(T_i) \cap \mathrm{VI}(C,A)$ where A is pseudomonotone, T_i is nonexpansive for $i = 1,\ldots,N$ and $T_0 = T$ is strictly pseudocontractive. The inertial subgradient–extragradient method with line-search (which is weakly convergent) for solving the VIP and FPP in [24] is extended to develop our composite inertial subgradient–extragradient method with line-search process (which is strongly convergent) for solving the VIP and CFPP, which is based on an inertial subgradient–extragradient method with a line-search process, a hybrid steepest-descent method, a viscosity approximation method and a Mann iteration method. It is worth pointing out that the inertial subgradient–extragradient method with a line-search process in [23] combines the inertial subgradient–extragradient method [24] with the Mann iteration method.

4 An example

In this section, our main results are applied to solve the VIP and CFPP in an illustrated example. The initial point $x_0 = x_1$ is randomly chosen in **R**. Take $f(x) = F(x) = \frac{1}{2}x$, $\gamma = l = \mu = \frac{1}{2}$, $\sigma_n = \alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{1}{3}$, $\gamma_n = \frac{1}{2}$, $\delta_n = \frac{1}{6}$ and $\rho = 2$. Then we know that $\delta = \kappa = \eta = \frac{1}{2}$, and

$$\tau = 1 - \sqrt{1 - \rho\left(2\eta - \rho\kappa^2\right)} = 1 - \sqrt{1 - 2\left(2 \cdot \frac{1}{2} - 2\left(\frac{1}{2}\right)^2\right)} = 1 \in (0,1].$$

We first provide an example of Lipschitz continuous and pseudomonotone mapping A, strictly pseudocontractive mapping T and nonexpansive mapping T_1 with $\Omega = \mathrm{Fix}(T_1) \cap \mathrm{Fix}(T) \cap \mathrm{VI}(C,A) \neq \emptyset$. Let $C = [-1,0]$ and $H = \mathbf{R}$ with the inner product $\langle a,b \rangle = ab$ and induced norm $\|\cdot\| = |\cdot|$. Let $A, T, T_1 : H \to H$ be defined as $Ax := \frac{1}{1+|\sin x|} - \frac{1}{1+|x|}$, $Tx := \frac{1}{2}x + \frac{3}{8}\sin x$ and $T_1 x := \sin x$ for all $x \in H$. Now, we first show that A is pseudomonotone

and Lipschitz continuous with $L = 2$. Indeed, for all $x, y \in H$ we have

$$
\begin{aligned}
\|Ax - Ay\| &= \left| \frac{1}{1 + \|\sin x\|} - \frac{1}{1 + \|x\|} - \frac{1}{1 + \|\sin y\|} + \frac{1}{1 + \|y\|} \right| \\
&\le \left| \frac{1}{1 + \|x\|} - \frac{1}{1 + \|y\|} \right| + \left| \frac{1}{1 + \|\sin x\|} - \frac{1}{1 + \|\sin y\|} \right| \\
&= \left| \frac{1 + \|y\| - 1 - \|x\|}{(1 + \|x\|)(1 + \|y\|)} \right| + \left| \frac{1 + \|\sin y\| - 1 - \|\sin x\|}{(1 + \|\sin x\|)(1 + \|\sin y\|)} \right| \\
&= \left| \frac{\|y\| - \|x\|}{(1 + \|x\|)(1 + \|y\|)} \right| + \left| \frac{\|\sin y\| - \|\sin x\|}{(1 + \|\sin x\|)(1 + \|\sin y\|)} \right| \\
&\le \frac{\|x - y\|}{(1 + \|x\|)(1 + \|y\|)} + \frac{\|\sin x - \sin y\|}{(1 + \|\sin x\|)(1 + \|\sin y\|)} \\
&\le \|x - y\| + \|\sin x - \sin y\| \\
&\le 2\|x - y\|.
\end{aligned}
$$

This implies that A is Lipschitz continuous with $L = 2$. Next, we show that A is pseudomonotone. For any given $x, y \in H$, it is clear that

$$
\begin{aligned}
\langle Ax, y - x \rangle &= \left(\frac{1}{1 + |\sin x|} - \frac{1}{1 + |x|} \right)(y - x) \ge 0 \\
\Rightarrow \quad \langle Ay, y - x \rangle &= \left(\frac{1}{1 + |\sin y|} - \frac{1}{1 + |y|} \right)(y - x) \ge 0.
\end{aligned}
$$

Furthermore, it is easy to see that T is strictly pseudocontractive with constant $\zeta = \frac{1}{2}$. Indeed, we observe that, for all $x, y \in H$,

$$
\|Tx - Ty\| \le \frac{1}{2}\|x - y\| + \frac{3}{8}\|\sin x - \sin y\| \le \|x - y\| + \frac{1}{2}\|(I - T)x - (I - T)y\|.
$$

It is clear that $(\gamma_n + \delta_n)\zeta = (\frac{1}{2} + \frac{1}{6}) \cdot \frac{1}{2} \le \frac{1}{2} = \gamma_n$ for all $n \ge 1$. In addition, it is clear that T_1 is nonexpansive and $\text{Fix}(T_1) = \{0\}$. Therefore, $\Omega = \text{Fix}(T_1) \cap \text{Fix}(T) \cap \text{VI}(C, A) = \{0\} \ne \emptyset$. In this case, Algorithm 3.1 can be rewritten as follows:

$$
\begin{cases}
w_n = T_1 x_n + \frac{1}{n+1}(T_1 x_n - T_1 x_{n-1}), \\
y_n = P_C(w_n - \tau_n A w_n), \\
z_n = \frac{1}{n+1} \cdot \frac{1}{2} x_n + \frac{n}{n+1} P_{C_n}(w_n - \tau_n A y_n), \\
x_{n+1} = \frac{1}{3} x_n + \frac{1}{2} z_n + \frac{1}{6} T z_n \quad \forall n \ge 1,
\end{cases}
\tag{4.1}
$$

where for each $n \ge 1$, C_n and τ_n are chosen as in Algorithm 3.2. Then, by Theorem 3.2, we know that $\{x_n\}$ converges to $0 \in \Omega = \text{Fix}(T_1) \cap \text{Fix}(T) \cap \text{VI}(C, A)$ if and only if $|x_n - x_{n+1}| + |x_n - y_n| \to 0$ as $n \to \infty$.

On the other hand, Algorithm 3.2 can be rewritten as follows:

$$
\begin{cases}
w_n = T_1 x_n + \frac{1}{n+1}(T_1 x_n - T_1 x_{n-1}), \\
y_n = P_C(w_n - \tau_n A w_n), \\
z_n = \frac{1}{n+1} \cdot \frac{1}{2} x_n + \frac{n}{n+1} P_{C_n}(w_n - \tau_n A y_n), \\
x_{n+1} = \frac{1}{3} w_n + \frac{1}{2} z_n + \frac{1}{6} T z_n \quad \forall n \geq 1,
\end{cases}
\tag{4.2}
$$

where for each $n \geq 1$, C_n and τ_n are chosen as in Algorithm 3.2. Then, by Theorem 3.2, we know that $\{x_n\}$ converges to $0 \in \Omega = \mathrm{Fix}(T_1) \cap \mathrm{Fix}(T) \cap \mathrm{VI}(C, A)$ if and only if $|x_n - x_{n+1}| + |x_n - y_n| \to 0$ as $n \to \infty$.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The two authors contributed equally. Both authors read and approved the final version of the manuscript.

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