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On weighted integrability of functions defined by trigonometric series with *p*-bounded variation coefficients

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Abstract

In this paper we introduce new classes of *p*-bounded variation sequences and give a sufficient and necessary condition for weighted integrability of trigonometric series with coefficients belonging to these classes. This is a generalization of the results obtained by the first author [J. Inequal. Appl. 2010:1–19, 2010] and Dyachenko and Tikhonov [Stud. Math. 193(3):285–306, 2009].

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1 Introduction

Let L^s, $1 \leq s < \infty$, be the space of all s-power integrable functions f of period 2π with the norm

$$\|f\|_{L^s} = \left(\int_{-\pi}^{\pi} |f(x)|^s \, dx \right)^{\frac{1}{s}}.$$

Write

$$f(x) = \sum_{k=1}^{\infty} a_k \cos kx, \qquad g(x) = \sum_{k=1}^{\infty} a_k \sin kx$$

for those x, where the above series converge.

Denote by ϕ and λ_n either f or g and either a_n and b_n, respectively.

Let $\triangle_r a_n = a_n - a_{n+r}$ for a sequence of complex numbers (a_n) and $r \in \mathbb{N}$.

Theorem 1 *Let a nonnegative sequence* $(\lambda_n) \in \Re$, $1 < s < \infty$ *and* $1 - s < \alpha < 1$. *Then*

$$x^{-\alpha} |\phi|^s \in L^1 \quad \Longleftrightarrow \quad \sum_{n=1}^{\infty} n^{\alpha+s-2} \lambda_n^s < \infty.$$

This theorem was proved for $\Re = DS$, where DS denotes all decreasing sequences, in [1, 5, 14], and [2]. Later, Theorem 1 was showed in [7] for

$$\Re = \overline{GM}({}_1\beta) := \left\{ (a_n) \subset \mathbb{C} : \sum_{k=n}^{\infty} |\Delta_1 a_k| \le C \cdot {}_1\beta_n \right\},$$

and in [12] for

$$\Re = GM({}_1\beta) := \left\{ (a_n) \subset \mathbb{C} : \sum_{k=n}^{2n-1} |\Delta_1 a_k| \le C \cdot {}_1\beta_n \right\},$$

where ${}_1\beta_n = |a_n|$; C here and throughout the paper denotes a positive constant.

The proof in the case of class

$$\Re = GM({}_2\beta) := \left\{ (a_n) \subset \mathbb{C} : \sum_{k=n}^{2n-1} |\Delta_1 a_k| \le C \cdot {}_2\beta_n \right\},$$

where ${}_2\beta_n = \sum_{k=[n/c]}^{[cn]} \frac{|a_k|}{k}$ for some $c > 1$, is included in [13].

In [3] Dyachenko and Tikhonov extended this theorem to the class

$$\Re = \overline{GM}\big({}_3\beta(\theta)\big) := \left\{ (a_n) \subset \mathbb{C} : \sum_{k=n}^{\infty} |\Delta_1 a_k| \le C \cdot {}_3\beta_n(\theta) \right\},$$

where ${}_3\beta_n(\theta) = n^{\theta-1} \sum_{k=[n/c]}^{\infty} \frac{|a_k|}{k^\theta} < \infty$ for some $c > 1$ and $\theta \in (0, 1]$.

From the articles of Dyachenko and Tikhonov [3] and Leindler [7], it is well known that

$$\begin{aligned} DS &\subsetneq \overline{GM}({}_1\beta) \subsetneq GM({}_1\beta) \subsetneq GM({}_2\beta) \\ &\subsetneq \overline{GM}\big({}_3\beta(1)\big) \subseteq \overline{GM}\big({}_3\beta(\theta_2)\big) \subseteq \overline{GM}\big({}_3\beta(\theta_1)\big), \end{aligned} \tag{1}$$

for $0 < \theta_1 \le \theta_2 \le 1$.

Further, Szal defined a new class of sequences in the following way (see [9]):

Definition 1 Let $\beta := (\beta_n)$ be a nonnegative sequence and r a natural number. The sequence of complex numbers $a := (a_n) \in \overline{GM}(\beta, r)$ if the relation

$$\sum_{k=n}^{\infty} |\triangle_r a_n| \le C\beta_n$$

holds for all $n \in \mathbb{N}$.

Moreover, from [9] we know that

$$\overline{GM}\big({}_3\beta(\theta), r_1\big) \subsetneq \overline{GM}\big({}_3\beta(\theta), r_2\big), \tag{2}$$

where $r_1 < r_2$, $\theta \in (0, 1]$ and $r_1 \mid r_2$.

Let $r \in \mathbb{N}$ and $\alpha \in \mathbb{R}$. We define on the interval $[-\pi, \pi]$ an even function $\omega_{\alpha,r}$, which is given on the interval $[0, \pi]$ by the formula

$$\omega_{\alpha,r}(x) := \begin{cases} (x - \frac{2l\pi}{r})^{-\alpha} & \text{for } x \in (\frac{2l\pi}{r}, \frac{(2l+1)\pi}{r}] \text{ and } l \in U_1, \\ (\frac{2(l+1)\pi}{r} - x)^{-\alpha} & \text{for } x \in (\frac{(2l+1)\pi}{r}, \frac{2(l+1)\pi}{r}) \text{ and } l \in U_2, \\ 0 \quad \text{for } x = \frac{2l\pi}{r} \text{ and } l \in U_3, \end{cases}$$

where $U_1 = \{0, 1, \dots, [r/2]\}$ if r is an odd number and $U_1 = \{0, 1, \dots, [r/2] - 1\}$ if r is an even number, $U_2 = \{0, 1, \dots, [r/2] - 1\}$ for $r \geq 2$, and $U_3 = \{0, 1, \dots, [r/2]\}$ for $r \geq 1$.

Theorem 1 was generalized for the class $\overline{GM}({}_3\beta(\theta), r)$, where $r \in \mathbb{N}$ and $\theta \in (0, 1]$, in [9]. We can formulate this result in the following way.

Theorem 2 ([9, Theorem 5]) *Let a nonnegative sequence* $(\lambda_n) \in \overline{GM}({}_3\beta(\theta), r)$, *where* $r \in \mathbb{N}$, $\theta \in (0, 1]$ *and* $1 \leq s < \infty$. *If*

$$1 - \theta s < \alpha < 1,$$

then $\omega_{\alpha,r}|\phi|^s \in L^1$ *if and only if*

$$\sum_{n=1}^{\infty} n^{\alpha+s-2} |\lambda_n|^s < \infty.$$

Now, we define new classes of sequences.

Definition 2 Let $\beta := (\beta_n)$ be a nonnegative sequence, p a positive real number, $r \in \mathbb{N}$. One says that a sequence $a = (a_n)$ of complex numbers belongs to $GM(p, \beta, r)$ if the relation

$$\left(\sum_{k=n}^{2n-1} |\triangle_r a_k|^p \right)^{\frac{1}{p}} \leq C\beta_n$$

holds for all $n \in \mathbb{N}$.

Moreover, we say that a sequence $(a_n) \in \overline{GM}(p, \beta, r)$ if the relation

$$\left(\sum_{k=n}^{\infty} |\triangle_r a_k|^p \right)^{\frac{1}{p}} \leq C\beta_n$$

holds for all $n \in \mathbb{N}$.

The class $GM(p, \beta, 1)$ was defined by Tikhonov and Liflyand in [8].

In this paper we present some properties of the classes $\overline{GM}(p, {}_3\beta(\theta), r)$ and $GM(p, {}_3\beta(\theta), r)$. Moreover, we will generalize Theorem 2 for the class $GM(p, {}_3\beta(\theta), r)$ with $0 < \theta < \frac{1}{s}$ and $r \in \mathbb{N}$.

We will write $I_1 \ll I_2$ if there exists a positive constant C such that $I_1 \leq CI_2$.

2 Main results

We formulate our results as follows:

Theorem 3 *Let $r \in \mathbb{N}$, $\theta \in (0,1)$, and p be a positive real number. Then*

$$\overline{GM}(p, {}_3\beta(\theta), r) = GM(p, {}_3\beta(\theta), r) \quad \text{and}$$
$$\overline{GM}(p, {}_3\beta(1), r) \subseteq GM(p, {}_3\beta(1), r).$$

Theorem 4 *Let $r \in \mathbb{N}$, $\theta \in (0,1)$, and p_1, p_2 be two positive real numbers such that $0 < p_1 < p_2$. Then*

$$GM(p_1, {}_3\beta(\theta), r) \subsetneq GM(p_2, {}_3\beta(\theta), r).$$

Theorem 5 *Let $r_1, r_2 \in \mathbb{N}$, $r_1 < r_2$, $\theta \in (0,1]$ and $p \geq 1$. If $r_1 | r_2$, then*

$$GM(p, {}_3\beta(\theta), r_1) \subsetneq GM(p, {}_3\beta(\theta), r_2).$$

Theorem 6 *Let $(b_n) \in GM(p, {}_3\beta(\theta), r)$, where $r \in \mathbb{N}$, $p \geq 1$, $0 < \theta < \frac{1}{p}$ and $1 \leq s < \infty$. If*

$$1 - \theta s - s + \frac{s}{p} < \alpha < 1$$

and

$$\sum_{n=1}^{\infty} n^{\alpha - 2 - \frac{s}{p} + 2s} |b_n|^s < \infty$$

then $\omega_{\alpha.r} |\phi|^s \in L^1$.

Theorem 7 *Let a nonnegative sequence (b_n) belong to $GM(p, {}_3\beta(\theta), r)$, where $r \in \mathbb{N}$, $p \geq 1$, $0 < \theta < \frac{1}{p}$ and $1 \leq s < \infty$. If*

$$1 - \theta s < \alpha < 1 + s$$

and $\omega_{\alpha.r} |\phi|^s \in L^1$ then

$$\sum_{n=1}^{\infty} n^{\alpha - 2 + \frac{s}{p}} b_n^s < \infty.$$

Remark 1 If we take $p = 1$, then the result of Szal [9] (Theorem 2) follows from our Theorem 6 and 7. Moreover, by the embedding relations (1) and (2), we can also derive from Theorem 6 and 7 the result of Dyachenko and Tikhonov [3] and all the results mentioned before.

3 Auxiliary results

For $n \in \mathbb{N}$ and $k = 0, 1, 2, \dots$, denote by

$$D_{k,r}(x) = \frac{\sin(k + r/2)x}{2\sin(rx/2)},$$
$$\widetilde{D}_{k,r}(x) = \frac{\cos(k + r/2)x}{2\sin(rx/2)}$$

the Dirichlet-type kernels.

Lemma 1 ([10, Lemma 3.1] and [11, Lemma 17]) *Let $r \in \mathbb{N}$, $l \in \mathbb{Z}$, and $(a_n) \subset \mathbb{C}$. If $x \neq \frac{2l\pi}{r}$, then for all $m \geq n$*

$$\sum_{k=n}^{m} a_k \cos kx = \sum_{k=n}^{m} \triangle_r a_k D_{k,r}(x) - \sum_{k=m+1}^{m+r} a_k D_{k,-r}(x) + \sum_{k=n}^{n+r-1} a_k D_{k,-r}(x),$$
$$\sum_{k=n}^{m} a_k \sin kx = \sum_{k=m+1}^{m+r} a_k \widetilde{D}_{k,-r}(x) - \sum_{k=n}^{n+r-1} a_k \widetilde{D}_{k,-r}(x) - \sum_{k=n}^{m} \triangle_r a_k \widetilde{D}_{k,r}(x).$$

Lemma 2 ([6, Corollary 1]) *Let $p \geq 1$, $\gamma_n > 0$, and $a_n \geq 0$ for $n \in \mathbb{N}$. Then*

$$\sum_{n=1}^{\infty} \gamma_n \left(\sum_{k=1}^{n} a_k\right)^p \leq p^p \sum_{n=1}^{\infty} \gamma_n^{1-p} a_n^p \left(\sum_{k=n}^{\infty} \gamma_k\right)^p,$$
$$\sum_{n=1}^{\infty} \gamma_n \left(\sum_{k=n}^{\infty} a_k\right)^p \leq p^p \sum_{n=1}^{\infty} \gamma_n^{1-p} a_n^p \left(\sum_{k=1}^{n} \gamma_k\right)^p.$$

Lemma 3 ([4, Theorem 19]) *If $a_n \geq 0$ for $n \in \mathbb{N}$ and $0 < p_1 \leq p_2 < \infty$, then*

$$\left(\sum_{n=1}^{\infty} a_n^{p_2}\right)^{\frac{1}{p_2}} \leq \left(\sum_{n=1}^{\infty} a_n^{p_1}\right)^{\frac{1}{p_1}}.$$

Lemma 4 ([4]) *Let $a_k \geq 0$ for $k \in \mathbb{N}$ and $p \geq 1$. Then*

$$\left(\frac{1}{n}\sum_{k=n}^{2n-1} a_k^p\right)^{\frac{1}{p}} \geq \frac{1}{n}\sum_{k=n}^{2n-1} a_k.$$

Lemma 5 *Let $(a_k) \subset \mathbb{C}$, $p \geq 1$, $r, n \in \mathbb{N}$ and $d \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Then*

$$\sum_{k=2^{d+1}(n+1)}^{2^{d+1}(n+1)+r-1} |a_k| \leq \sum_{k=2^d(n+1)}^{2^d(n+1)+r-1} |a_k| + \left[2^d(n+1)\right]^{1-\frac{1}{p}} \left(\sum_{k=2^d(n+1)}^{2^{d+1}(n+1)-1} |\triangle_r a_k|^p\right)^{\frac{1}{p}}.$$

Proof From Lemma 4 we have

$$\left(\sum_{k=2^d(n+1)}^{2^{d+1}(n+1)-1} |\triangle_r a_k|^p\right)^{\frac{1}{p}} = \left[2^d(n+1)\right]^{\frac{1}{p}} \left(\frac{1}{2^d(n+1)} \sum_{k=2^d(n+1)}^{2^{d+1}(n+1)-1} |\triangle_r a_k|^p\right)^{\frac{1}{p}}$$
$$\geq \left[2^d(n+1)\right]^{\frac{1}{p}} \frac{1}{2^d(n+1)} \sum_{k=2^d(n+1)}^{2^{d+1}(n+1)-1} |\triangle_r a_k|$$
$$\geq \left[2^d(n+1)\right]^{\frac{1}{p}-1} \left(\sum_{k=2^{d+1}(n+1)}^{2^{d+1}(n+1)+r-1} |a_k| - \sum_{k=2^d(n+1)}^{2^d(n+1)+r-1} |a_k|\right).$$

Hence

$$\sum_{k=2^{d+1}(n+1)}^{2^{d+1}(n+1)+r-1} |a_k| \leq \sum_{k=2^d(n+1)}^{2^d(n+1)+r-1} |a_k| + \left[2^d(n+1)\right]^{1-\frac{1}{p}} \left(\sum_{k=2^d(n+1)}^{2^{d+1}(n+1)-1} |\triangle_r a_k|^p\right)^{\frac{1}{p}}$$

and this ends our proof. □

Lemma 6 *Let* $(a_k) \in GM(p, {}_3\beta(\theta), r)$, $p \geq 1$, $r \in \mathbb{N}$, $d \in \mathbb{N}_0$, *and* $0 < \theta < \frac{1}{p}$. *Then*

$$\sum_{k=2^d(n+1)}^{2^d(n+1)+r-1} |a_k| \leq C \frac{1}{1-2^{\theta-\frac{1}{p}}} \left(2^d(n+1)\right)^{\theta-\frac{1}{p}} \sum_{k=[\frac{2^d(n+1)}{c}]}^{\infty} \frac{|a_k|}{k^\theta}.$$

Proof We have

$$\sum_{k=2^d(n+1)}^{2^d(n+1)+r-1} |a_k| \leq \sum_{j=0}^{\infty} \sum_{k=2^j 2^d(n+1)}^{2^{j+1}2^d(n+1)-1} |\triangle_r a_k|.$$

Using Hölder inequality with $p > 1$, we get

$$\sum_{j=0}^{\infty} \sum_{k=2^j 2^d(n+1)}^{2^{j+1}2^d(n+1)-1} |\triangle_r a_k|$$
$$\leq \sum_{j=0}^{\infty} \left[\left(\sum_{k=2^j 2^d(n+1)}^{2^{j+1}2^d(n+1)-1} |\triangle_r a_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=2^j 2^d(n+1)}^{2^{j+1}2^d(n+1)-1} 1^{\frac{p}{p-1}}\right)^{1-\frac{1}{p}}\right]$$
$$\leq C \sum_{j=0}^{\infty} \left(2^j 2^d(n+1)\right)^{1-\frac{1}{p}} \left(2^j 2^d(n+1)\right)^{\theta-1} \sum_{k=[\frac{2^j 2^d(n+1)}{c}]}^{\infty} \frac{|a_k|}{k^\theta}$$
$$\leq C\left(2^d(n+1)\right)^{\theta-\frac{1}{p}} \sum_{k=[\frac{2^d(n+1)}{c}]}^{\infty} \frac{|a_k|}{k^\theta} \sum_{j=0}^{\infty} \left(2^{\theta-\frac{1}{p}}\right)^j.$$

When $p = 1$, we have

$$\sum_{j=0}^{\infty}\sum_{k=2^j2^d(n+1)}^{2^{j+1}2^d(n+1)-1}|\triangle_r a_k| \le C\sum_{j=0}^{\infty}\bigl(2^j2^d(n+1)\bigr)^{\theta-1}\sum_{k=[\frac{2^j2^d(n+1)}{c}]}^{\infty}\frac{|a_k|}{k^\theta}$$
$$\le C\bigl(2^d(n+1)\bigr)^{\theta-1}\sum_{k=[\frac{2^d(n+1)}{c}]}^{\infty}\frac{|a_k|}{k^\theta}\sum_{j=0}^{\infty}\bigl(2^{\theta-1}\bigr)^j.$$

If $\theta - \frac{1}{p} < 0$, then

$$\sum_{k=2^d(n+1)}^{2^d(n+1)+r-1}|a_k| \le C\frac{1}{1-2^{\theta-\frac{1}{p}}}\bigl(2^d(n+1)\bigr)^{\theta-\frac{1}{p}}\sum_{k=[\frac{2^d(n+1)}{c}]}^{\infty}\frac{|a_k|}{k^\theta}$$

and our proof is complete. □

4 Proofs

4.1 Proof of Theorem 3

Let $(a_n) \in GM(p, {}_3\beta(\theta), r)$, where $p > 0$, $r \in \mathbb{N}$, and $\theta \in (0,1)$. Then

$$\left(\sum_{k=n}^{\infty}|\triangle_r a_k|^p\right)^{\frac{1}{p}} = \left(\sum_{d=0}^{\infty}\sum_{k=2^d n}^{2^{d+1}n-1}|\triangle_r a_k|^p\right)^{\frac{1}{p}}$$
$$\le \left(\sum_{d=0}^{\infty}\left(C\bigl(2^d n\bigr)^{\theta-1}\sum_{k=[\frac{2^d n}{c}]}^{\infty}\frac{|a_k|}{k^\theta}\right)^p\right)^{\frac{1}{p}}$$
$$\le Cn^{\theta-1}\sum_{k=[\frac{n}{c}]}^{\infty}\frac{|a_k|}{k^\theta}\left(\sum_{d=0}^{\infty}\bigl(2^{(\theta-1)p}\bigr)^d\right)^{\frac{1}{p}}.$$

If $0 < \theta < 1$ then $(\theta-1)p < 0$, and we have

$$\left(\sum_{k=n}^{\infty}|\triangle_r a_k|^p\right)^{\frac{1}{p}} \le C\left(\frac{1}{1-2^{(\theta-1)p}}\right)^{\frac{1}{p}}n^{\theta-1}\sum_{k=[\frac{n}{c}]}^{\infty}\frac{|a_k|}{k^\theta}.$$

So $(a_n) \in \overline{GM}(p, {}_3\beta(\theta), r)$.

Now we assume $(a_n) \in \overline{GM}(p, {}_3\beta(1), r)$, $p > 0$, $r \in \mathbb{N}$. We have

$$\left(\sum_{k=n}^{2n-1}|\triangle_r a_k|^p\right)^{\frac{1}{p}} \le \left(\sum_{k=n}^{\infty}|\triangle_r a_k|^p\right)^{\frac{1}{p}} \le Cn^{\theta-1}\sum_{k=[\frac{n}{c}]}^{\infty}\frac{|a_k|}{k^\theta}.$$

This means $(a_n) \in GM(p, {}_3\beta(1), r)$. □

4.2 Proof of Theorem 4

Let $r \in \mathbb{N}$, $\theta \in (0,1]$, $0 < p_1 \leq p_2$, and $(a_n) \in GM(p_1, {}_3\beta(\theta), r)$. We will show that $GM(p_1, {}_3\beta(\theta), r) \subseteq GM(p_2, {}_3\beta(\theta), r)$. Using Lemma 3, we have

$$\left(\sum_{k=n}^{2n-1} |\triangle_r a_k|^{p_2}\right)^{\frac{1}{p_2}} \leq \left(\sum_{k=n}^{2n-1} |\triangle_r a_k|^{p_1}\right)^{\frac{1}{p_1}} \leq cn^{\theta-1} \sum_{k=n}^{\infty} \frac{|a_k|}{k^\theta}.$$

This means that $(a_n) \in GM(p_2, {}_3\beta(\theta), r)$.

Now we will show that $GM(p_1, {}_3\beta(\theta), r) \neq GM(p_2, {}_3\beta(\theta), r)$ for $0 < p_1 < p_2$. Let

$$a_n = \begin{cases} \frac{1}{n^2}, & \text{when } 2r \nmid n, \\ \frac{1}{(n-r)^2} + \frac{1}{n^2 n^{\frac{1}{p_2}}}, & \text{when } 2r | n. \end{cases}$$

We prove that $(a_n) \in GM(p_2, {}_3\beta(\theta), r)$. Suppose

$$A_n = \{k \in \mathbb{N} : n \leq k \leq 2n-1 \text{ and } 2r | k\},$$

$$B_n = \{k \in \mathbb{N} : n \leq k \leq 2n-1, 2r \nmid k \text{ and } 2r \nmid k+r\},$$

$$C_n = \{k \in \mathbb{N} : n \leq k \leq 2n-1, 2r \nmid k \text{ and } 2r | k+r\}.$$

Then

$$\begin{aligned}
&\left(\sum_{k=n}^{2n-1} |a_k - a_{k+r}|^{p_2}\right)^{\frac{1}{p_2}} \\
&= \left(\sum_{k\in A_n} \left|\frac{1}{(k-r)^2} + \frac{1}{k^2 k^{\frac{1}{p_2}}} - \frac{1}{(k+r)^2}\right|^{p_2}\right. \\
&\quad \left. + \sum_{k\in B_n} \left|\frac{1}{k^2} - \frac{1}{(k+r)^2}\right|^{p_2} + \sum_{k\in C_n} \left|\frac{1}{k^2} - \frac{1}{k^2} - \frac{1}{(k+r)^2 (k+r)^{\frac{1}{p_2}}}\right|^{p_2}\right)^{\frac{1}{p_2}} \\
&\leq \left(\sum_{k\in A_n} \left(\frac{4kr}{\frac{1}{4}k^2 k^2} + \frac{1}{k^{2+\frac{1}{p_2}}}\right)^{p_2} + \sum_{k\in B_n} \left(\frac{2kr + r^2}{k^2 (k+r)^2}\right)^{p_2} + \sum_{k\in C_n} \left(\frac{1}{(k+r)^{2+\frac{1}{p_2}}}\right)^{p_2}\right)^{\frac{1}{p_2}} \\
&\leq (16r+1) \left(\sum_{k=n}^{2n-1} \left(\frac{1}{k^{2+\frac{1}{p_2}}}\right)^{p_2}\right)^{\frac{1}{p_2}} \leq \frac{17r}{n^2}.
\end{aligned}$$

Moreover,

$$\frac{17r}{n^2} \leq 2^{2+\theta} 17r \left(n^{\theta-1} \sum_{k=n}^{2n-1} \frac{1}{k^2} \frac{1}{k^\theta}\right) \leq 2^{2+\theta} 17rn^{\theta-1} \sum_{k=[\frac{n}{c}]}^{\infty} \frac{|a_k|}{k^\theta}.$$

This means $(a_n) \in GM(p_2, {}_3\beta(\theta), r)$. We will show that $(a_n) \notin GM(p_1, {}_3\beta(\theta), r)$. We have

$$\left(\sum_{k=n}^{2n-1} |a_k - a_{k+r}|^{p_1}\right)^{\frac{1}{p_1}} \geq \left(\sum_{k\in C_n} \frac{1}{(k+r)^{2p_1 + \frac{p_1}{p_2}}}\right)^{\frac{1}{p_1}} \geq \frac{1}{(4r)^{2+\frac{1}{p_2}+\frac{2}{p_1}}} \frac{n^{\frac{1}{p_1}}}{n^{2+\frac{1}{p_2}}}.$$

Let

$$D_n = \left\{k \in \mathbb{N} : \left[\frac{n}{c}\right] \le k \text{ and } 2r|k\right\},$$
$$E_n = \left\{k \in \mathbb{N} : \left[\frac{n}{c}\right] \le k \text{ and } 2r \nmid k\right\}.$$

On the other hand, we get

$$n^{\theta-1} \sum_{k=[\frac{n}{c}]}^{\infty} \frac{a_k}{k^\theta} = n^{\theta-1}\left(\sum_{k\in D_n} \frac{1}{k^2 k^\theta} + \sum_{k\in E_n}\left(\frac{1}{(k-r)^2} + \frac{1}{k^{2+\frac{1}{p_2}}}\right)\frac{1}{k^\theta}\right)$$
$$\le 5n^{\theta-1} \sum_{k=[\frac{n}{c}]}^{\infty} \frac{1}{k^{2+\theta}} \ll n^{-2}.$$

Therefore the inequality

$$\left(\sum_{k=n}^{2n-1} |\triangle_r a_k|^{p_1}\right)^{\frac{1}{p_1}} \le Cn^{\theta-1} \sum_{k=[\frac{n}{c}]}^{\infty} \frac{a_k}{k^\theta}$$

cannot be satisfied because $n^{\frac{1}{p_1}-\frac{1}{p_2}} \to \infty$ as $n \to \infty$. □

4.3 Proof of Theorem 5

Let $r_1, r_2 \in \mathbb{N}$, $r_1 \le r_2$, $r_1|r_2$, $p \ge 1$ and $(a_n) \in GM(p, {}_3\beta(\theta), r_1)$.

If $r_1|r_2$, then $r_2 = \alpha r_1$, where $\alpha \in \mathbb{N}$. Using Hölder inequality with $p > 1$, we have

$$\begin{aligned}
&\left(\sum_{k=n}^{2n-1} |a_k - a_{k+r_2}|^p\right)^{\frac{1}{p}} \\
&= \left(\sum_{k=n}^{2n-1} \left|\sum_{l=0}^{\alpha-1} (a_{k+lr_1} - a_{k+(l+1)r_1})\right|^p\right)^{\frac{1}{p}} \\
&\le \left(\sum_{k=n}^{2n-1} \left(\sum_{l=0}^{\alpha-1} |a_{k+lr_1} - a_{k+(l+1)r_1}|\right)^p\right)^{\frac{1}{p}} \\
&\le \left(\sum_{k=n}^{2n-1} \left(\left(\sum_{l=0}^{\alpha-1} |a_{k+lr_1} - a_{k+(l+1)r_1}|^p\right)^{\frac{1}{p}} \left(\sum_{l=0}^{\alpha-1} 1^{\frac{p}{p-1}}\right)^{1-\frac{1}{p}}\right)^p\right)^{\frac{1}{p}} \\
&\le \alpha^{1-\frac{1}{p}} \left(\sum_{k=n}^{2n-1} \left(\sum_{l=0}^{\alpha-1} |a_{k+lr_1} - a_{k+(l+1)r_1}|^p\right)\right)^{\frac{1}{p}} \\
&\le \alpha^{1-\frac{1}{p}} \left(\sum_{l=0}^{\alpha-1} \left(C(n+lr_1)^{\theta-1} \sum_{k=[\frac{n+lr_1}{c}]}^{\infty} \frac{|a_k|}{k^\theta}\right)^p\right)^{\frac{1}{p}} \\
&\le \alpha C n^{\theta-1} \sum_{k=[\frac{n}{c}]}^{\infty} \frac{|a_k|}{k^\theta}.
\end{aligned}$$

If $p = 1$ then

$$\sum_{k=n}^{2n} |a_k - a_{k+r_2}| \leq \sum_{k=n}^{2n-1} \sum_{l=0}^{\alpha-1} |a_{k+lr_1} - a_{k+(l-1)r_1}|$$
$$\leq C \sum_{l=0}^{\alpha-1} (n + lr_1)^{\theta-1} \sum_{k=[\frac{n+lr_1}{c}]}^{\infty} \frac{|a_k|}{k^\theta} \leq \alpha C n^{\theta-1} \sum_{k=[\frac{n}{c}]}^{\infty} \frac{|a_k|}{k^\theta}.$$

Hence $(a_n) \in GM(p, {}_3\beta(\theta), r_2)$.

Now, we will show that $GM(p, {}_3\beta(\theta), r_1) \subsetneq GM(p, {}_3\beta(\theta), r_2)$, when $r_1 < r_2$. Let $a_n = \frac{2+\alpha_n}{n^2}$, where $\alpha_n = \begin{cases} -1, & \text{when } r_1 | n, \\ 1, & \text{when } r_1 \nmid n. \end{cases}$

We will prove that $(a_n) \in GM(p, {}_3\beta(\theta), r_2)$ and $(a_n) \notin GM(p, {}_3\beta(\theta), r_1)$. Let

$$A_n := \{k \in \mathbb{N} : n \leq k \leq 2n-1 \text{ and } r_2 | k\},$$
$$B_n := \{k \in \mathbb{N} : n \leq k \leq 2n-1 \text{ and } r_2 \nmid k\}.$$

Then using Lemma 3 for $p \geq 1$, we have

$$\left(\sum_{k=n}^{2n-1} |a_k - a_{k+r_2}|^p\right)^{\frac{1}{p}} = \left(\left(\sum_{k\in A_n} + \sum_{k\in B_n}\right) |a_k - a_{k+r_2}|^p\right)^{\frac{1}{p}}$$
$$= \left(\sum_{k\in A_n} \left|\frac{1}{k^2} - \frac{1}{(k+r_2)^2}\right|^p + \sum_{k\in B_n} \left|\frac{3}{k^2} - \frac{3}{(k+r_2)^2}\right|^p\right)^{\frac{1}{p}}$$
$$\leq \left(3^p \sum_{k=n}^{2n-1} \left|\frac{(k+r_2)^2 - k^2}{(k+r_2)^2 k^2}\right|^p\right)^{\frac{1}{p}}$$
$$= 3\left(\sum_{k=n}^{2n-1} \left|\frac{2r_2 k + r_2^2}{(k+r_2)^2 k^2}\right|^p\right)^{\frac{1}{p}}$$
$$\leq 6r_2 \left(\sum_{k=n}^{2n-1} \left(\frac{1}{k^3}\right)^p\right)^{\frac{1}{p}} \leq 6r_2 \sum_{k=n}^{2n-1} \frac{1}{k^3} \leq \frac{6r_2}{n} \sum_{k=n}^{2n-1} \frac{1}{k^2}.$$

Moreover,

$$\frac{6r_2}{n} \sum_{k=n}^{2n-1} \frac{1}{k^2} = 6r_2 n^{\theta-1} \frac{1}{(2n)^\theta} 2^\theta \sum_{k=n}^{2n-1} \frac{1}{k^2}$$
$$\leq 6r_2 2^\theta n^{\theta-1} \sum_{k=n}^{2n-1} \frac{1}{k^2} \frac{1}{k^\theta}$$
$$\leq 6r_2 2^\theta n^{\theta-1} \sum_{k=n}^{2n-1} \frac{a_k}{k^\theta} \leq 6r_2 2^\theta n^{\theta-1} \sum_{k=[\frac{n}{c}]}^{\infty} \frac{a_k}{k^\theta}.$$

It means that $(a_n) \in GM(p, {}_3\beta(\theta), r_2)$. Furthermore,

$$\left(\sum_{k=n}^{2n-1} |a_k - a_{k+r_1}|^p\right)^{\frac{1}{p}} \geq \left(\sum_{k\in A_n} |a_k - a_{k+r_1}|^p\right)^{\frac{1}{p}} \geq \left(\sum_{k\in A_n} \left|\frac{1}{k^3} - \frac{3}{(k+r_1)^2}\right|^p\right)^{\frac{1}{p}}$$
$$= \left(\sum_{k\in A_n} \left|\frac{(k+r_1)^2 - 3k^2}{(k+r_1)^2k^2}\right|^p\right)^{\frac{1}{p}} = \left(\sum_{k\in A_n} \left|\frac{-2k^2 + 2kr_1 + r_1^2}{(k+r_1)^2k^2}\right|^p\right)^{\frac{1}{p}}.$$

If $n \geq 5r_1$, then $2n^2 - 2nr_1 - r_1^2 \geq (n+r_1)^2$. Whence for $n \geq 5r_1$,

$$\left(\sum_{k=n}^{2n-1} |a_k - a_{k+r_1}|^p\right)^{\frac{1}{p}} \geq \left(\sum_{k\in A_n} \left(\frac{2k^2 - 2kr_1 - r_1^2}{(k+r_1)^2k^2}\right)^p\right)^{\frac{1}{p}}$$
$$\geq \frac{1}{(2n)^2}\left(\frac{n}{2r_1}\right)^{\frac{1}{p}} = \frac{1}{2^{2+\frac{1}{p}} r_1} n^{-2+\frac{1}{p}}.$$

On the other hand,

$$n^{\theta-1} \sum_{k=[\frac{n}{c}]}^{\infty} \frac{a_k}{k^\theta} \leq n^{\theta-1} \sum_{k=[\frac{n}{c}]}^{\infty} \frac{3}{k^2}\frac{1}{k^\theta} \leq 3n^{\theta-1} \sum_{k=[\frac{n}{c}]}^{\infty} \frac{1}{k^{2+\theta}} \ll n^{-2}.$$

Therefore, the inequality

$$\left(\sum_{k=n}^{2n-1} |\triangle_{r_1} a_k|^p\right)^{\frac{1}{p}} \leq C n^{\theta-1} \sum_{k=[\frac{n}{c}]}^{\infty} \frac{a_k}{k^\theta}$$

cannot be satisfied because $n^{\frac{1}{p}} \to \infty$ as $n \to \infty$. □

4.4 Proof of Theorem 6

We prove the theorem for the case when $\phi(x) = g(x)$. We have

$$\left\|\omega_{\alpha,r}|g|^s\right\|_{L^1} = 2\int_0^\pi \omega_{\alpha,r}(x)\left|g(x)\right|^s dx.$$

For an odd r,

$$\int_0^\pi \omega_{\alpha,r}(x)\left|g(x)\right|^s dx = \sum_{l=0}^{[r/2]} \int_{\frac{2l\pi}{r}}^{\frac{2l\pi}{r}+\frac{\pi}{r}} \omega_{\alpha,r}(x)\left|\sum_{k=1}^{\infty} b_k \sin kx\right|^s dx$$
$$+ \sum_{l=0}^{[r/2]-1} \int_{\frac{2l\pi}{r}+\frac{\pi}{r}}^{\frac{2(l+1)\pi}{r}} \omega_{\alpha,r}(x)\left|\sum_{k=1}^{\infty} b_k \sin kx\right|^s dx$$

(for $r = 1$ the last sum should be omitted), and for an even r,

$$\int_0^\pi \omega_{\alpha,r}(x)\left|g(x)\right|^s dx = \sum_{l=0}^{[r/2]} \left(\int_{\frac{2l\pi}{r}}^{\frac{2l\pi}{r}+\frac{\pi}{r}} + \int_{\frac{2l\pi}{r}+\frac{\pi}{r}}^{\frac{2(l+1)\pi}{r}}\right) \omega_{\alpha,r}(x)\left|\sum_{k=1}^{\infty} b_k \sin kx\right|^s dx.$$

Now, we estimate the following integral:

$$\int_{\frac{2l\pi}{r}}^{\frac{2l\pi}{r}+\frac{\pi}{r}} \omega_{\alpha,r}(x)\left|\sum_{k=1}^{\infty} b_k \sin kx\right|^s dx \ll \left(\int_{\frac{2l\pi}{r}}^{\frac{2l\pi}{r}+\frac{\pi}{r}} \omega_{\alpha,r}(x)\left|\sum_{k=1}^{n} b_k \sin kx\right|^s dx + \int_{\frac{2l\pi}{r}}^{\frac{2l\pi}{r}+\frac{\pi}{r}} \omega_{\alpha,r}(x)\left|\sum_{k=n+1}^{\infty} b_k \sin kx\right|^s dx\right) := I_1 + I_2.$$

By Lemma 2, for $\alpha < 1$, we have

$$\begin{aligned} I_1 &= \sum_{n=r}^{\infty} \int_{\frac{2l\pi}{r}+\frac{\pi}{n+1}}^{\frac{2l\pi}{r}+\frac{\pi}{n}} \left(x - \frac{2l\pi}{r}\right)^{-\alpha} \left|\sum_{k=1}^{n} b_k \sin kx\right|^s dx \\ &\ll \sum_{n=r}^{\infty} n^{\alpha-2}\left(\sum_{k=1}^{n} |b_k|\right)^s \\ &\le \sum_{n=1}^{\infty} n^{\alpha-2-\frac{s}{p}+2s}|b_n|^s. \end{aligned} \tag{3}$$

Using Lemma 1 when $m \to \infty$ and the inequality

$$\frac{r}{\pi}x - 2l \le \left|\sin \frac{rx}{2}\right| \quad \text{for } x \in \left(\frac{2l\pi}{r}, \frac{2l\pi}{r} + \frac{\pi}{r}\right),$$

we get

$$\begin{aligned} I_2 &= \sum_{n=r}^{\infty} \int_{2l\pi/r+\pi/(n+1)}^{2l\pi/r+\pi/n} \left(x - \frac{2l\pi}{r}\right)^{-\alpha} \left|\sum_{k=n+1}^{\infty} b_k \sin kx\right|^s dx \\ &\ll \sum_{n=r}^{\infty} n^{\alpha} \int_{2l\pi/r+\pi/(n+1)}^{2l\pi/r+\pi/n} \left|\sum_{d=0}^{\infty}\left(\sum_{k=2^{d+1}(n+1)}^{2^{d+1}(n+1)-1+r} b_k \widetilde{D}_{k,-r}(x)\right.\right. \\ &\quad \left.\left. - \sum_{k=2^d(n+1)}^{2^d(n+1)+r-1} b_k \widetilde{D}_{k,-r}(x) - \sum_{k=2^d(n+1)}^{2^{d+1}(n+1)-1} \triangle_r b_k \widetilde{D}_{k,r}(x)\right)\right|^s dx \\ &\le \sum_{n=r}^{\infty} n^{\alpha} \int_{2l\pi/r+\pi/(n+1)}^{2l\pi/r+\pi/n} \frac{1}{(rx/\pi - 2l)^s} \\ &\quad \times \left(\sum_{d=0}^{\infty}\left(\sum_{k=2^{d+1}(n+1)}^{2^{d+1}(n+1)-1+r} |b_k| + \sum_{k=2^d(n+1)}^{2^d(n+1)+r-1} |b_k| + \sum_{k=2^d(n+1)}^{2^{d+1}(n+1)-1} |\triangle_r b_k|\right)\right)^s dx \\ &\ll \sum_{n=r}^{\infty} n^{\alpha+s-2}\left(\sum_{d=0}^{\infty}\left(\sum_{k=2^{d+1}(n+1)}^{2^{d+1}(n+1)-1+r} |b_k| + \sum_{k=2^d(n+1)}^{2^d(n+1)+r-1} |b_k| + \sum_{k=2^d(n+1)}^{2^{d+1}(n+1)-1} |\triangle_r b_k|\right)\right)^s. \end{aligned}$$

Further by Hölder inequality with $p > 1$, we get

$$I_2 \ll \sum_{n=r}^{\infty} n^{\alpha+s-2}\left(\sum_{d=0}^{\infty}\left[\left(\sum_{k=2^d(n+1)}^{2^{d+1}(n+1)-1} |\triangle_r b_k|^p\right)^{\frac{1}{p}}\left(\sum_{k=2^d(n+1)}^{2^{d+1}(n+1)-1} 1\right)^{1-\frac{1}{p}} + \sum_{k=2^{d+1}(n+1)}^{2^{d+1}(n+1)-1+r} |b_k| + \sum_{k=2^d(n+1)}^{2^d(n+1)+r-1} |b_k|\right]\right)^s$$

$$\leq \sum_{n=r}^{\infty} n^{\alpha+s-2}\left(\sum_{d=0}^{\infty}\left[\left(\sum_{k=2^d(n+1)}^{2^{d+1}(n+1)-1} |\triangle_r b_k|^p\right)^{\frac{1}{p}}\left(2^d(n+1)\right)^{1-\frac{1}{p}} + \sum_{k=2^{d+1}(n+1)}^{2^{d+1}(n+1)-1+r} |b_k| + \sum_{k=2^d(n+1)}^{2^d(n+1)+r-1} |b_k|\right]\right)^s.$$

Applying Lemma 5, we have

$$I_2 \ll \sum_{n=r}^{\infty} n^{\alpha+s-2}\left(\sum_{d=0}^{\infty}\left[\left(2^d(n+1)\right)^{1-\frac{1}{p}}\left(2^d(n+1)\right)^{\theta-1}\sum_{k=[\frac{2^d(n+1)}{c}]}^{\infty}\frac{|b_k|}{k^\theta} + \sum_{k=2^d(n+1)}^{2^d(n+1)+r-1} |b_k|\right]\right)^s.$$

From Lemma 6, we get

$$I_2 \ll \sum_{n=r}^{\infty} n^{\alpha+s-2}\left(\sum_{d=0}^{\infty}\left[\left(2^d(n+1)\right)^{\theta-\frac{1}{p}}\sum_{k=[\frac{2^d(n+1)}{c}]}^{\infty}\frac{|b_k|}{k^\theta} + \frac{1}{1-2^{\theta-\frac{1}{p}}}\left(2^d(n+1)\right)^{\theta-\frac{1}{p}}\sum_{k=[\frac{2^d(n+1)}{c}]}^{\infty}\frac{|b_k|}{k^\theta}\right]\right)^s$$

$$\ll \sum_{n=r}^{\infty} n^{\alpha+s-2+\theta s-\frac{s}{p}}\left(\sum_{d=0}^{\infty}\left(2^d\right)^{\theta-\frac{1}{p}}\sum_{k=[\frac{2^d(n+1)}{c}]}^{\infty}\frac{|b_k|}{k^\theta}\right)^s.$$

If $\theta - \frac{1}{p} < 0$, then

$$I_2 \ll \sum_{n=r}^{\infty} n^{\alpha+s-2+\theta s-\frac{s}{p}}\left(\sum_{k=[\frac{n+1}{c}]}^{\infty}\frac{|b_k|}{k^\theta}\right)^s$$

$$\ll \sum_{n=r}^{\infty} n^{\alpha+s-2-\frac{s}{p}+\theta s}\left(\sum_{k=[\frac{n}{c}]}^{n}\frac{|b_k|}{k^\theta}\right)^s + \sum_{n=r}^{\infty} n^{\alpha+s-2-\frac{s}{p}+\theta s}\left(\sum_{k=n}^{\infty}\frac{|b_k|}{k^\theta}\right)^s$$

$$\leq \sum_{n=1}^{\infty} n^{\alpha-2-\frac{s}{p}}\left(\sum_{k=1}^{n} k|b_k|\right)^s + \sum_{n=1}^{\infty} n^{\alpha+s-2-\frac{s}{p}+\theta s}\left(\sum_{k=n}^{\infty}\frac{|b_k|}{k^\theta}\right)^s.$$

Now, we use Lemma 2 and get

$$
\begin{aligned}
I_2 &\ll \sum_{n=1}^{\infty} \bigl(n^{\alpha-2-\frac{s}{p}}\bigr)^{1-s} \bigl(n|b_n|\bigr)^s \Biggl(\sum_{k=n}^{\infty} k^{\alpha-2-\frac{s}{p}}\Biggr)^s \\
&\quad + \sum_{n=1}^{\infty} \bigl(n^{\alpha+s-2-\frac{s}{p}+\theta s}\bigr)^{1-s} \biggl(\frac{|b_n|}{n^{\theta}}\biggr)^s \Biggl(\sum_{k=1}^{n} k^{\alpha+s-2-\frac{s}{p}+\theta s}\Biggr)^s.
\end{aligned}
$$

For $1+\frac{s}{p}-\theta s-s<\alpha<1+\frac{s}{p}$, we have

$$
I_2 \ll \sum_{n=1}^{\infty} n^{\alpha-2-\frac{s}{p}+2s}|b_n|^s. \tag{4}
$$

Now, we estimate the following integral:

$$
\begin{aligned}
\int_{\frac{2l\pi}{r}+\frac{\pi}{r}}^{\frac{2(l+1)\pi}{r}} \omega_{\alpha,r}(x)\Biggl|\sum_{k=1}^{\infty} b_k \sin kx\Biggr|^s dx &\ll \int_{\frac{2(l+1)\pi}{r}-\frac{\pi}{r}}^{\frac{2(l+1)\pi}{r}} \omega_{\alpha,r}(x)\Biggl|\sum_{k=1}^{n} b_k \sin kx\Biggr|^s dx \\
&\quad + \int_{\frac{2(l+1)\pi}{r}-\frac{\pi}{r}}^{\frac{2(l+1)\pi}{r}} \omega_{\alpha,r}(x)\Biggl|\sum_{k=n+1}^{\infty} b_k \sin kx\Biggr|^s dx \\
&:= I_3 + I_4.
\end{aligned}
$$

By Lemma 2, for $\alpha<1$, we have

$$
\begin{aligned}
I_3 &= \sum_{n=r}^{\infty} \int_{\frac{2(l+1)\pi}{r}-\frac{\pi}{n}}^{\frac{2(l+1)\pi}{r}-\frac{\pi}{n+1}} \biggl(\frac{2(l+1)\pi}{r}-x\biggr)^{-\alpha}\Biggl|\sum_{k=1}^{n} b_k \sin kx\Biggr|^s dx \\
&\ll \sum_{n=1}^{\infty} n^{\alpha-2}\Biggl(\sum_{k=1}^{n}|b_k|\Biggr)^s \\
&\ll \sum_{n=1}^{\infty} n^{\alpha+s-2}|b_n|^s \le \sum_{n=1}^{\infty} n^{\alpha-2-\frac{s}{p}+2s}|b_n|^s.
\end{aligned} \tag{5}
$$

Using Lemma 1 with $m\to\infty$ and the inequality

$$
2(l+1)-\frac{r}{\pi}x \le \biggl|\sin\frac{rx}{2}\biggr| \quad \text{for } x\in\biggl(\frac{(2l+1)\pi}{r},\frac{2(l+1)\pi}{r}\biggr),
$$

we have

$$
I_4 = \sum_{n=r}^{\infty} \int_{\frac{2(l+1)\pi}{r}-\frac{\pi}{n}}^{\frac{2(l+1)\pi}{r}-\frac{\pi}{n+1}} \biggl(\frac{2(l+1)\pi}{r}-x\biggr)^{-\alpha}\Biggl|\sum_{k=n+1}^{\infty} b_k \sin kx\Biggr|^s dx,
$$

and similarly as in the case I_2 we obtain

$$
I_4 \ll \sum_{n=1}^{\infty} n^{\alpha-2-\frac{s}{p}+2s}|b_n|^s. \tag{6}
$$

Finally, combining (3)–(6), we obtain that

$$\int_{-\pi}^{\pi} \omega_{\alpha,r}(x)\left|g(x)\right|^{s} dx \le C\sum_{n=1}^{\infty} n^{\alpha-2-\frac{s}{p}+2s}|b_n|^{s}.$$

The case when $\phi(x) = \sum_{k=1}^{\infty} b_k \cos kx$ can be proved similarly. □

4.5 Proof of Theorem 7

We prove the theorem for the case where $\phi(x) = \sum_{k=1}^{\infty} b_k \sin kx$. We follow the method adopted by Tikhonov [9]. Note that if $1-\theta s < \alpha < 1+s$, then $\phi \in L^1$. Namely, if $s > 1$ then using Hölder inequality, we have

$$\begin{aligned}\int_0^{\pi} \left|\phi(x)\right| dx &= \int_0^{\pi} \left(\omega_{\alpha,r}(x)\right)^{\frac{1}{s}} \left|\phi(x)\right| \left(\frac{1}{\omega_{\alpha,r}(x)}\right)^{\frac{1}{s}} dx \\ &\le \left(\int_0^{\pi} \omega_{\alpha,r}(x)\left|\phi(x)\right|^{s} dx\right)^{\frac{1}{s}} \left(\int_0^{\pi} \left(\omega_{\alpha,r}(x)^{-\frac{1}{s}}\right)^{\frac{1}{1-\frac{1}{s}}} dx\right)^{1-\frac{1}{s}}.\end{aligned}$$

We will show that $\int_0^{\pi} (\omega_{\alpha,r}(x))^{-\frac{1}{s-1}} dx < \infty$. We can write

$$\int_0^{\pi} \left(\omega_{\alpha,r}(x)\right)^{-\frac{1}{s-1}} dx = \sum_{l=0}^{[\frac{r}{2}]} \left(\int_{\frac{2l\pi}{r}}^{\frac{(2l+1)\pi}{r}} \left(x-\frac{2l\pi}{r}\right)^{\frac{\alpha}{s-1}} dx + \int_{\frac{(2l+1)\pi}{r}}^{\frac{2(l+1)\pi}{r}} \left(\frac{2(l+1)\pi}{r}-x\right)^{\frac{\alpha}{s-1}} dx\right),$$

when r is an even number, and

$$\begin{aligned}&\int_0^{\pi} \left(\omega_{\alpha,r}(x)\right)^{-\frac{1}{s-1}} dx \\ &\quad= \sum_{l=0}^{[\frac{r}{2}]} \int_{\frac{2l\pi}{r}}^{\frac{(2l+1)\pi}{r}} \left(x-\frac{2l\pi}{r}\right)^{\frac{\alpha}{s-1}} dx + \sum_{l=0}^{[\frac{r}{2}]-1} \int_{\frac{(2l+1)\pi}{r}}^{\frac{2(l+1)\pi}{r}} \left(\frac{2(l+1)\pi}{r}-x\right)^{\frac{\alpha}{s-1}} dx,\end{aligned}$$

when r is an odd number.

Using integration by substitution, we get

$$\begin{aligned}\int_0^{\pi} \left(\omega_{\alpha,r}(x)\right)^{-\frac{1}{s-1}} dx &= \sum_{l=0}^{[\frac{r}{2}]} \left(\int_0^{\frac{\pi}{r}} y^{\frac{\alpha}{s-1}} dy + \int_0^{\frac{\pi}{r}} y^{\frac{\alpha}{s-1}} dy\right) \\ &= 2\left(\left[\frac{r}{2}\right]+1\right)\frac{s-1}{\alpha+s-1}\left(\frac{\pi}{r}\right)^{\frac{\alpha+s-1}{s-1}},\end{aligned}$$

when r is an even number, and

$$\begin{aligned}\int_0^{\pi} \left(\omega_{\alpha,r}(x)\right)^{-\frac{1}{s-1}} dx &= \sum_{l=0}^{[\frac{r}{2}]} \int_0^{\frac{\pi}{r}} y^{\frac{\alpha}{s-1}} dy + \sum_{l=0}^{[\frac{r}{2}]-1} \int_0^{\frac{\pi}{r}} y^{\frac{\alpha}{s-1}} dx \\ &= \left(2\left[\frac{r}{2}\right]+1\right)\frac{s-1}{\alpha+s-1}\left(\frac{\pi}{r}\right)^{\frac{\alpha+s-1}{s-1}},\end{aligned}$$

when r is an odd number.

If $s=1$ then $\alpha>0$ and

$$\int_0^\pi |\phi(x)|\,dx = \int_0^\pi \omega_{\alpha,r}(x)|\phi(x)|\frac{1}{\omega_{\alpha,r}(x)}\,dx$$
$$\le \sup_x \frac{1}{\omega_{\alpha,r}(x)}\int_0^\pi \omega_{\alpha,r}(x)|\phi(x)|\,dx = \left(\frac{\pi}{r}\right)^\alpha \int_0^\pi \omega_{\alpha,r}(x)|\phi(x)|\,dx.$$

Further, integrating ϕ, we have

$$F(x) := \int_0^x \phi(t)\,dt = \sum_{n=1}^\infty \frac{b_n}{n}(1-\cos nx) = 2\sum_{n=1}^\infty \frac{b_n}{n}\sin^2\frac{nx}{2},$$

and consequently,

$$F\left(\frac{\pi}{k}\right) \ge \sum_{n=[k/2]}^{k} \frac{b_n}{n}. \tag{7}$$

Since $(b_n) \in GM(p, {}_3\beta(\theta), r)$ and using Lemma 4, we get for $\theta - \frac{1}{p} < 0$ that

$$b_\nu \le \sum_{k=\nu}^{\nu+r-1} b_l = \sum_{d=0}^\infty \sum_{k=2^d\nu}^{2^{d+1}\nu-1} |\triangle_r b_k| \le \sum_{d=0}^\infty 2^d\nu\left[\frac{1}{2^d\nu}\sum_{k=2^d\nu}^{2^{d+1}\nu-1}|\triangle_r b_k|^p\right]^{\frac{1}{p}}$$
$$\le C\sum_{d=0}^\infty (2^d\nu)^{\theta-\frac{1}{p}} \sum_{k=[\frac{2^d\nu}{c}]}^\infty \frac{b_k}{k^\theta} \le C\nu^{\theta-\frac{1}{p}}\sum_{d=0}^\infty (2^{\theta-\frac{1}{p}})^d \sum_{k=[\frac{\nu}{c}]}^\infty \frac{b_k}{k^\theta}$$
$$\le \frac{1}{1-2^{\theta-\frac{1}{p}}}C\nu^{\theta-\frac{1}{p}}\sum_{k=[\frac{\nu}{c}]}^\infty \frac{b_k}{k^\theta} \ll \nu^{\theta-\frac{1}{p}}\sum_{k=[\frac{\nu}{c}]}^\infty \frac{b_k}{k^\theta} \le C\nu^{\theta-\frac{1}{p}}\sum_{d=0}^\infty\left(2^{d+1}\left[\frac{\nu}{c}\right]\right)^{1-\theta}\sum_{k=2^d[\frac{\nu}{c}]}^{2^{d+1}[\frac{\nu}{c}]}\frac{b_k}{k}.$$

Using (7) yields

$$b_\nu \ll \nu^{\theta-\frac{1}{p}}\sum_{d=0}^\infty\left(2^d\left[\frac{\nu}{c}\right]\right)^{1-\theta}F\left(\frac{\pi}{2^{d+1}[\frac{\nu}{c}]}\right) \ll \nu^{\theta-\frac{1}{p}}\sum_{d=0}^\infty\left(2^d\left[\frac{\nu}{c}\right]\right)^{-\theta}\sum_{k=2^d[\frac{\nu}{c}]}^{2^{d+1}[\frac{\nu}{c}]-1}F\left(\frac{\pi}{k}\right)$$
$$\ll \nu^{\theta-\frac{1}{p}}\sum_{k=[\frac{\nu}{c}]}^\infty \frac{1}{k^\theta}F\left(\frac{\pi}{k}\right).$$

Elementary calculations give

$$\sum_{k=1}^\infty k^{\alpha-2+\frac{s}{p}}b_k^s \ll \sum_{k=1}^\infty k^{\alpha-2+\frac{s}{p}+(\theta-\frac{1}{p})s}\left(\sum_{\nu=[\frac{k}{c}]}^\infty \frac{1}{\nu^\theta}F\left(\frac{\pi}{\nu}\right)\right)^s$$
$$\ll \sum_{k=1}^\infty k^{\alpha-2}\left(\sum_{\nu=[\frac{k}{c}]}^k F\left(\frac{\pi}{\nu}\right)\right)^s + \sum_{k=1}^\infty k^{\alpha-2+\theta s}\left(\sum_{\nu=k}^\infty \frac{1}{\nu^\theta}F\left(\frac{\pi}{\nu}\right)\right)^s$$
$$\ll \sum_{k=1}^\infty k^{\alpha-2-s}\left(\sum_{\nu=[\frac{k}{c}]}^k \nu F\left(\frac{\pi}{\nu}\right)\right)^s + \sum_{k=1}^\infty k^{\alpha-2+\theta s}\left(\sum_{\nu=k}^\infty \frac{1}{\nu^\theta}F\left(\frac{\pi}{\nu}\right)\right)^s.$$

Using Lemma 2, for $1 - \theta s < \alpha < 1 + s$, we have

$$\sum_{k=1}^{\infty} k^{\alpha-2-s} \left(\sum_{\nu=[\frac{k}{c}]}^{k} \nu F\left(\frac{\pi}{\nu}\right) \right)^{s} \ll \sum_{k=1}^{\infty} k^{(\alpha-2-s)(1-s)} \left(kF\left(\frac{\pi}{k}\right) \right)^{s} \left(\sum_{\nu=k}^{\infty} \nu^{\alpha-2-s} \right)^{s}$$

and

$$\sum_{k=1}^{\infty} k^{\alpha-2+\theta s} \left(\sum_{\nu=k}^{\infty} \frac{1}{\nu^{\theta}} F\left(\frac{\pi}{\nu}\right) \right)^{s} \ll \sum_{k=1}^{\infty} k^{(\alpha-2+\theta s)(1-s)} \left(\frac{1}{k^{\theta}} F\left(\frac{\pi}{k}\right) \right)^{s} \left(\sum_{\nu=1}^{k} \nu^{\alpha-2+\theta s} \right)^{s}.$$

Therefore, for $1 - \theta s < \alpha < 1 + s$, we get

$$\sum_{k=1}^{\infty} k^{\alpha-2+\frac{s}{p}} b_k^{s} \ll \sum_{k=1}^{\infty} k^{\alpha-2+s} \left(F\left(\frac{\pi}{k}\right) \right)^{s}.$$

Denoting by $d_\nu := \int_{\frac{\pi}{\nu+1}}^{\frac{\pi}{\nu}} |\phi(x)|\,dx$, we get

$$\sum_{k=1}^{\infty} k^{\alpha-2+\frac{s}{p}} b_k^{s} \ll \sum_{k=1}^{\infty} k^{\alpha-2+s} \left(\sum_{\nu=k}^{\infty} d_\nu \right)^{s}.$$

By Lemma 2, for $\alpha > 1 - s$, we obtain

$$\begin{aligned} \sum_{k=1}^{\infty} k^{\alpha-2+s} \left(\sum_{\nu=k}^{\infty} d_\nu \right)^{s} &\ll \sum_{k=1}^{\infty} k^{(\alpha-2+s)(1-s)} d_k^{s} \left(\sum_{\nu=1}^{k} \nu^{\alpha-2+s} \right)^{s} \\ &\ll \sum_{k=1}^{\infty} k^{(\alpha-2+s)(1-s)} k^{(\alpha-2+s+1)s} d_k^{s} = \sum_{k=1}^{\infty} k^{\alpha-2+2s} d_k^{s}. \end{aligned}$$

Applying Hölder inequality when $s > 1$, we have

$$d_k^{s} \ll \frac{1}{k^{2(s-1)}} \int_{\frac{\pi}{(k+1)}}^{\frac{\pi}{k}} |\phi(x)|^{s}\,dx.$$

Finally, using the latter estimate, we get

$$\begin{aligned} \sum_{k=1}^{\infty} k^{\alpha-2+\frac{s}{p}} b_k^{s} &\ll \sum_{k=1}^{\infty} k^{\alpha-2+2s} d_k^{s} \\ &\le \sum_{k=1}^{r} k^{\alpha-2+2s} \left(\int_{\frac{\pi}{(k+1)}}^{\frac{\pi}{k}} |\phi(x)|\,dx \right)^{s} + \sum_{k=r}^{\infty} k^{\alpha} \int_{\frac{\pi}{(k+1)}}^{\frac{\pi}{k}} |\phi(x)|^{s}\,dx \\ &\ll \left(\int_0^{\pi} |\phi(x)|\,dx \right)^{s} + \sum_{k=r}^{\infty} \int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}} x^{-\alpha} |\phi(x)|^{s}\,dx \\ &\le \left(\int_0^{\pi} |\phi(x)|\,dx \right)^{s} + \int_0^{\pi} \omega_{\alpha,r}(x) |\phi(x)|^{p}\,dx < \infty. \end{aligned}$$

The case when $\phi(x) = \sum_{k=1}^{\infty} b_k \cos kx$ can by proved similarly. □

5 Conclusions

We have introduced two new classes of p-bounded variation sequences, $\overline{GM}(p,\beta,r)$ and $GM(p,\beta,r)$, where $\beta := (\beta_n)$ is a nonnegative sequence, p a positive real number, $r \in \mathbb{N}$, $\theta \in (0,1]$. Moreover, we have studied properties of such classes and obtained a sufficient and necessary condition for weighted integrability of functions defined by trigonometric series with coefficients belonging to these classes. In particular, from our theorems we derive all related earlier results.

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References

1. Boas, R.P. Jr.: Integrability Theorems for Trigonometric Transforms. Springer, New York (1967)
2. Chen, Y.M.: On the integrability of functions defined by trigonometrical series. Math. Z. **66**, 9–12 (1956)
3. Dyachenko, M., Tikhonov, S.: Integrability and continuity of functions represented by trigonometric series: coefficients criteria. Stud. Math. **193**(3), 285–306 (2009)
4. Hardy, G.H., Littlewood, J.E., Pólya, G.: Inequalities. Cambridge University Press, Cambridge (1934)
5. Heywood, P.: On the integrability of functions defined by trigonometric series. Quart. J. Math. **5**, 71–76 (1954)
6. Leindler, L.: Further sharpening of inequalities of Hardy and Littlewood. Acta Sci. Math. **54**, 285–289 (1990)
7. Leindler, L.: A new class of numerical sequences and its applications to sine and cosine series. Anal. Math. **28**, 279–286 (2002)
8. Liflyand, E., Tikhonov, S.: A concept of general monotonicity and applications. Math. Nachr. **284**(8–9), 1083–1098 (2011)
9. Szal, B.: On weighted L^p integrability of functions defined by trigonometric series. J. Inequal. Appl. **2010**, 1 (2010)
10. Szal, B.: A new class of numerical sequences and its applications to uniform convergence of sine series. Math. Nachr. **284**(14–15), 1985–2002 (2011)
11. Szal, B.: On L-convergence of trigonometric series. J. Math. Anal. Appl. **373**, 449–463 (2011)
12. Tikhonov, S.: Trigonometric series with general monotone coefficients. J. Math. Anal. Appl. **326**, 721–735 (2007)
13. You, D.S., Zhou, P., Zhou, S.P.: On L^p integrability and convergence of trigonometric series. Stud. Math. **182**(3), 215–226 (2007)
14. Zygmund, A.: Trigonometric Series. Cambridge University Press, Cambridge (1977)