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Weighted norm inequality for bilinear Calderón–Zygmund operators on Herz–Morrey spaces with variable exponents

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Abstract

In this paper, we obtain a weighted norm inequality of bilinear Calderón–Zygmund operators in Herz–Morrey spaces with variable exponents and weight in the variable Muckenhoupt class.

Keywords: Bilinear Calderón–Zygmund operator; Muckenhoupt weight; Variable exponent; Herz–Morrey space

1 Introduction

We denote by $\mathcal{S}(\mathbb{R}^n)$ the space of all Schwartz functions on $\mathbb{R}^n$ and by $\mathcal{S}'(\mathbb{R}^n)$ the space of all tempered distributions on $\mathbb{R}^n$. Let T be a bilinear operator, which is originally defined on the 2-fold of Schwartz function space $\mathcal{S}(\mathbb{R}^n)$, and its value belongs to $\mathcal{S}'(\mathbb{R}^n)$:

$$T:\mathcal{S}(\mathbb{R}^n)\times\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}'(\mathbb{R}^n).$$

T is called bilinear Calderón–Zygmund operator, if it extends to a bounded bilinear operator from $L^{p_1}\times L^{p_2}$ to L^p with $1/p_1+1/p_2=1/p$, and for $f_1,f_2\in L^\infty_C(\mathbb{R}^n)$ (the space of compactly supported bounded functions), $x\notin\operatorname{supp}(f_1)\cap\operatorname{supp}(f_2)$

$$T(f_1,f_2)(x):=\int_{\mathbb{R}^{2n}}K(x,y_1,y_2)f_1(y_1)f_2(y_2)\,dy_1\,dy_2,$$

where the kernel K is a function in $\mathbb{R}^{3n}$ off from the diagonal $x=y_1=y_2$ and there exist positive constants ε, A such that

$$\bigl|K(x,y_1,y_2)\bigr|\le\frac{A}{(|x-y_1|+|x-y_2|+|y_1-y_2|)^{2n}}$$

and

$$\bigl|K(x,y_1,y_2)-K\bigl(x',y_1,y_2\bigr)\bigr|\le\frac{A|x-x'|^\varepsilon}{(|x-y_1|+|x-y_2|+|y_1-y_2|)^{2n+\varepsilon}}$$

whenever $|x-x'|\le\frac{1}{2}\max\{|x-y_1|,|x-y_2|\}$, and the two analogous difference estimates with respect to the variables y_1 and y_2 hold.

Recently, Cruz-Uribe and Guzman proved the boundedness of the bilinear Calderón–Zygmund operator on products of weighted variable Lebesgue spaces in [1]. As a generalization of variable Lebesgue spaces, variable and weighted variable Herz–Morrey (Herz) spaces have been introduced in the last decades; see [2–11]. Motivated by [1], in this paper, we will prove a weighted norm inequality on products of Herz–Morrey spaces with variable exponents and weight in the variable Muckenhoupt class. We only consider the bilinear Calderón–Zygmund operator for simplicity. The analogs of our result for m-linear Calderón–Zygmund operators also hold for $m \geq 3$, because our argument and Lemma 8 in Sect. 3 also hold for m-linear Calderón–Zygmund operators with $m \geq 3$, see Remark 2.7 for [1, Theorem 2.4] in [1]. We mention here that the theory of multilinear Calderón–Zygmund operators started in [12]. After that, the boundedness of multilinear Calderón–Zygmund operators on products of various spaces has been obtained; see [13–19].

The plan of the paper is as follows. In Sect. 2, we collect some notations and state main result. The proof of the main result will be given in Sect. 3.

2 Notations and main result

In this section, we firstly recall some definitions and notations, then we state our results. Let Ω be a positive measurable subset of $\mathbb{R}^n$, given a measurable function $p(\cdot): \Omega \to [1,\infty)$, the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) := \left\{ f \text{ is measurable: } \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty \text{ for some } \lambda > 0 \right\}.$$

The Lebesgue space $L^{p(\cdot)}(\Omega)$ becomes a Banach function space equipped with the norm

$$\|f\|_{L^{p(\cdot)}} := \inf \left\{ \lambda > 0 : \int_{\Omega} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}.$$

The space $L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$ is defined by $L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n) := \{f : f\chi_K \in L^{p(\cdot)}(\mathbb{R}^n)$ for all compact subsets $K \subset \mathbb{R}^n\}$, where and what follows, χ_S denotes the characteristic function of a measurable set $S \subset \mathbb{R}^n$. Let $p(\cdot): \mathbb{R}^n \to (0,\infty)$, we denote $p_- := \operatorname{ess\,inf}_{x\in\mathbb{R}^n} p(x)$, $p_+ := \operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(x)$. The set $\mathcal{P}(\mathbb{R}^n)$ consists of all $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$; $\mathcal{P}_0(\mathbb{R}^n)$ consists of all $p(\cdot)$ satisfying $p_- > 0$ and $p_+ < \infty$. $L^{p(\cdot)}$ can be similarly defined as above for $p(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$. $p'(\cdot)$ means that the conjugate exponent of $p(\cdot)$, that means $1/p(\cdot) + 1/p'(\cdot) = 1$.

Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and w be a weight which is a non-negative measurable function on $\mathbb{R}^n$. Then the weighted variable exponent Lebesgue space $L^{p(\cdot)}(w)$ is the set of all complex-valued measurable function f such that $fw \in L^{p(\cdot)}$. The space $L^{p(\cdot)}(w)$ is a Banach space equipped with the norm

$$\|f\|_{L^{p(\cdot)}(w)} := \|fw\|_{L^{p(\cdot)}}.$$

Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then the standard Hardy–Littlewood maximal function of f is defined by

$$Mf(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|\, dy, \quad \forall x \in \mathbb{R}^n,$$

where the supremum is taken over all balls containing x in $\mathbb{R}^n$. In general, the Hardy–Littlewood maximal operator is not bounded on weighted variable Lebesgue spaces. But if $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and satisfies the following global log-Hölder continuous and $w \in A_{p(\cdot)}$, then M is bounded on $L^{p(\cdot)}(w)$.

Definition 1 Let $\alpha(\cdot)$ be a real-valued measurable function on $\mathbb{R}^n$.

(i) The function $\alpha(\cdot)$ is locally log-Hölder continuous if there exists a constant C_1 such that

$$\left|\alpha(x) - \alpha(y)\right| \leq \frac{C_1}{\log(e + 1/|x-y|)}, \quad x, y \in \mathbb{R}^n, |x-y| < \frac{1}{2}.$$

(ii) The function $\alpha(\cdot)$ is log-Hölder continuous at the origin if there exists a constant C_2 such that

$$\left|\alpha(x) - \alpha(0)\right| \leq \frac{C_2}{\log(e + 1/|x|)}, \quad \forall x \in \mathbb{R}^n.$$

Denote by $\mathcal{P}_0^{\log}(\mathbb{R}^n)$ the set of all log-Hölder continuous functions at the origin.

(iii) The function $\alpha(\cdot)$ is log-Hölder continuous at infinity if there exist $\alpha_\infty \in \mathbb{R}$ and a constant C_3 such that

$$\left|\alpha(x) - \alpha_\infty\right| \leq \frac{C_3}{\log(e + |x|)}, \quad \forall x \in \mathbb{R}^n.$$

Denote by $\mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ the set of all log-Hölder continuous functions at infinity.

(iv) The function $\alpha(\cdot)$ is global log-Hölder continuous if $\alpha(\cdot)$ are both locally log-Hölder continuous and log-Hölder continuous at infinity. Denote by $\mathcal{P}^{\log}(\mathbb{R}^n)$ the set of all global log-Hölder continuous functions.

Definition 2 Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, a positive measurable function w is said to be in $A_{p(\cdot)}$, if exists a positive constant C for all balls B in $\mathbb{R}^n$ such that

$$\sup_B \frac{1}{|B|} \|w\chi_B\|_{L^{p(\cdot)}} \left\|w^{-1}\chi_B\right\|_{L^{p'(\cdot)}} < \infty.$$

Remark 1 In [20], Cruz-Uribe, Fiorenza and Neugebauer found that if $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $w \in A_{p(\cdot)}$, then $w^{-1} \in A_{p'(\cdot)}$.

The Muckenhoupt A_p class with constant exponent $p \in (1, \infty)$ firstly proposed by Muckenhoupt in [21]. The variable Muckenhoupt $A_{p(\cdot)}$ was considered in [20, 22–25].

Lemma 1 (see [20, Theorem 1.5]) *If $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ and $w \in A_{p(\cdot)}$, then there is a positive constant C such that, for each $f \in L^{p(\cdot)}(w)$,*

$$\left\|(Mf)w\right\|_{L^{p(\cdot)}} \leq C\|fw\|_{L^{p(\cdot)}}.$$

To give the definitions of the Herz space and the Herz–Morrey space with variable exponents, we use the following notations. For each $k \in \mathbb{Z}$ we define

$$B_k := \left\{x \in \mathbb{R}^n : |x| \leq 2^k\right\}, \qquad D_k := B_k \backslash B_{k-1},$$

$$\chi_k := \chi_{D_k}, \qquad \widetilde{\chi}_m = \chi_m, \quad m \geq 1, \widetilde{\chi}_0 = \chi_{B_0}.$$

Definition 3 Let $q \in (0,\infty]$, $p(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$, and $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$.

(1) The homogeneous weighted Herz space $\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(w)$ is defined by

$$\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(w) := \bigl\{ f \in L^{p(\cdot)}_{\mathrm{loc}}\bigl(\mathbb{R}^n \backslash \{0\}, w\bigr) : \|f\|_{\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(w)} < \infty \bigr\},$$

where

$$\|f\|_{\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(w)} := \left\{ \sum_{k=-\infty}^{\infty} \bigl\| 2^{k\alpha(\cdot)} f \chi_k \bigr\|^q_{L^{p(\cdot)}(w)} \right\}^{1/q}.$$

(2) The inhomogeneous weighted Herz space $K^{\alpha(\cdot),q}_{p(\cdot)}(w)$ is defined by

$$K^{\alpha(\cdot),q}_{p(\cdot)}(w) := \bigl\{ f \in L^{p(\cdot)}_{\mathrm{loc}}(w) : \|f\|_{K^{\alpha(\cdot),q}_{p(\cdot)}(w)} < \infty \bigr\},$$

where

$$\|f\|_{K^{\alpha(\cdot),q}_{p(\cdot)}(w)} := \left\{ \sum_{m=0}^{\infty} \bigl\| 2^{m\alpha(\cdot)} f \widetilde{\chi}_m \bigr\|^q_{L^{p(\cdot)}(w)} \right\}^{1/q}.$$

Remark 2 If $0 < q_1 \leq q_2 \leq \infty$ and $w \equiv 1$, then $\dot{K}^{\alpha(\cdot),q_1}_{p(\cdot)}(\mathbb{R}^n) \subset \dot{K}^{\alpha(\cdot),q_2}_{p(\cdot)}(\mathbb{R}^n)$. If $w \equiv 1$, $\alpha(\cdot)$ and $p(\cdot)$ are constants, then $\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha,q}_{p}(\mathbb{R}^n)$ is the classical Herz spaces in [26, 27].

To generalize the above spaces to variable exponent $q(\cdot)$, we need the notation of the variable mixed sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$, which is firstly defined by Almeida and Hästö in [28]. Let w be a non-negative measurable function. Given a sequence of functions $\{f_j\}_{j\in\mathbb{Z}}$, define the modular

$$\rho_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))}\bigl((f_j)_j\bigr) := \sum_{j\in\mathbb{Z}} \inf \left\{ \lambda_j : \int_{\mathbb{R}^n} \left(\frac{|f_j(x)w(x)|}{\lambda_j^{\frac{1}{q(x)}}} \right)^{p(x)} dx \leq 1 \right\},$$

where $\lambda^{1/\infty} = 1$. If $q^+ < \infty$ or $q(\cdot) \leq p(\cdot)$, the above can be written as

$$\rho_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))}\bigl((f_j)_j\bigr) = \sum_{j\in\mathbb{Z}} \bigl\| |f_j w|^{q(\cdot)} \bigr\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}.$$

The norm is

$$\bigl\| (f_j)_j \bigr\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))} := \inf \bigl\{ \mu > 0 : \rho_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))}\bigl((f_j/\mu)_j\bigr) \leq 1 \bigr\}.$$

Now, spaces $\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(w)$ and $K^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(w)$ are defined, respectively, by

$$\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(w) := \bigl\{ f \in L^{p(\cdot)}_{\mathrm{loc}}\bigl(\mathbb{R}^n \backslash \{0\}, w\bigr) : \|f\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(w)} < \infty \bigr\},$$

where

$$\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(w)} := \left\| \left(2^{j\alpha(\cdot)} f \chi_j\right)_j \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))}$$

and

$$K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(w) := \left\{ f \in L_{\mathrm{loc}}^{p(\cdot)}(w) : \|f\|_{K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(w)} = \left\| \left(2^{j\alpha(\cdot)} f \chi_j\right)_j \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))} < \infty \right\}.$$

For any quantities A and B, we shall write $A \lesssim B$ to indicate that there exists a constant $C > 0$ such that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$.

The following lemma is a corollary of [29, Theorem 3].

Lemma 2 *Let $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$, $p(\cdot)$, $q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ and w be a weight. If $\alpha(\cdot)$ and $q(\cdot)$ are log-Hölder continuous at infinity, then*

$$K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(w) = K_{p(\cdot)}^{\alpha_\infty,q_\infty}(w).$$

Additionally, if $\alpha(\cdot)$ and $q(\cdot)$ are log-Hölder continuous at the origin, then

$$\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(w)} \approx \left(\sum_{k\leq 0} \left\| 2^{k\alpha(0)} f \chi_k \right\|_{L^{p(\cdot)}(w)}^{q(0)} \right)^{1/q(0)} + \left(\sum_{k>0} \left\| 2^{k\alpha_\infty} f \chi_k \right\|_{L^{p(\cdot)}(w)}^{q_\infty} \right)^{1/q_\infty}.$$

Definition 4 Let $p(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$, $\lambda \in [0,\infty)$. Let $\alpha(\cdot)$ be a bounded real-valued measurable function on $\mathbb{R}^n$. The homogeneous weighted Herz–Morrey space $M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q(\cdot)}(w)$ and non-homogeneous weighted Herz–Morrey space $MK_{p(\cdot),\lambda}^{\alpha(\cdot),q(\cdot)}(w)$ are defined, respectively, by

$$M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q(\cdot)}(w) := \left\{ f \in L_{\mathrm{loc}}^{p(\cdot)}\left(\mathbb{R}^n \setminus \{0\}, w\right) : \|f\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q(\cdot)}(w)} < \infty \right\}$$

and

$$MK_{p(\cdot),\lambda}^{\alpha(\cdot),q(\cdot)}(w) := \left\{ f \in L_{\mathrm{loc}}^{p(\cdot)}\left(\mathbb{R}^n, w\right) : \|f\|_{MK_{p(\cdot),\lambda}^{\alpha(\cdot),q(\cdot)}(w)} < \infty \right\},$$

where

$$\|f\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q(\cdot)}(w)} := \sup_{L\in\mathbb{Z}} 2^{-L\lambda} \left\| \left(2^{\alpha(\cdot)k} f \chi_k\right)_{k\leq L} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))}$$

and

$$\|f\|_{MK_{p(\cdot),\lambda}^{\alpha(\cdot),q(\cdot)}(w)} := \sup_{L\in\mathbb{N}_0} 2^{-L\lambda} \left\| \left(2^{\alpha(\cdot)k} f \tilde{\chi}_k\right)_{k=0}^{L} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))}.$$

Proposition 1 *Let $p(\cdot)$, $q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$, w be a weight, $\lambda \in [0,\infty)$, and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$.*

(i) *If* $\alpha(\cdot), q(\cdot) \in \mathcal{P}_0^{\log}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$, *then, for any* $f \in L_{\mathrm{loc}}^{p(\cdot)}(\mathbb{R}^n\backslash\{0\}, w)$,

$$\begin{aligned}&\|f\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q(\cdot)}(w)}\\&\quad\approx \max\Big\{\sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\big\|\big(2^{k\alpha(0)}f\chi_k\big)_{k\le L}\big\|_{\ell^{q_0}(L^{p(\cdot)}(w))},\\&\qquad\sup_{L>0, L\in\mathbb{Z}}\big[2^{-L\lambda}\big\|\big(2^{k\alpha(0)}f\chi_k\big)_{k<0}\big\|_{\ell^{q_0}(L^{p(\cdot)}(w))} + 2^{-L\lambda}\big\|\big(2^{k\alpha_\infty}f\chi_k\big)_{k=0}^{L}\big\|_{\ell^{q_\infty}(L^{p(\cdot)}(w))}\big]\Big\},\end{aligned}$$

where throughout $q_0 := q(0)$.

(ii) *If* $\alpha(\cdot), q(\cdot) \in \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$, *then*

$$MK_{p(\cdot),\lambda}^{\alpha(\cdot),q(\cdot)}(w) = MK_{p(\cdot),\lambda}^{\alpha_\infty,q_\infty}(w).$$

Proof Obviously,

$$\begin{aligned}\|f\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q(\cdot)}(w)} = \max\Big\{&\sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\big\|\big(2^{k\alpha(\cdot)}f\chi_k\big)_{k\le L}\big\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))},\\&\sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda}\big\|\big(2^{k\alpha(\cdot)}f\chi_k\big)_{k\le L}\big\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))}\Big\}.\end{aligned}$$

When $L \le 0$, from Lemma 2 we know that

$$\big\|\big(2^{k\alpha(\cdot)}f\chi_k\big)_{k\le L}\big\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))} \approx \big\|\big(2^{k\alpha(0)}f\chi_k\big)_{k\le L}\big\|_{\ell^{q_0}(L^{p(\cdot)}(w))}.$$

When $L > 0$, from Lemma 2 again we also obtain

$$\begin{aligned}\big\|\big(2^{k\alpha(\cdot)}f\chi_k\big)_{k<L}\big\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))} \approx{}& \big\|\big(2^{k\alpha(0)}f\chi_k\big)_{k<0}\big\|_{\ell^{q_0}(L^{p(\cdot)}(w))}\\&+ \big\|\big(2^{k\alpha_\infty}f\chi_k\big)_{k=0}^{L}\big\|_{\ell^{q_\infty}(L^{p(\cdot)}(w))}.\end{aligned}$$

Thus we obtain (i). Similarly, we obtain (ii). □

Lemmas 3 and 4 below have been proved by Izuki and Noi in [30, 31].

Lemma 3 *If* $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ *and* $w \in A_{p(\cdot)}$, *then there exists a constant* $C > 0$ *such that, for all balls* B *in* $\mathbb{R}^n$ *and all measurable subsets* $S \subset B$,

$$\frac{\|\chi_S\|_{L^{p(\cdot)}(w)}}{\|\chi_B\|_{L^{p(\cdot)}(w)}} \le C\frac{|S|}{|B|}. \tag{1}$$

Lemma 4 *If* $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ *and* $w \in A_{p(\cdot)}$, *then there exist constants* $\delta_1, \delta_2 \in (0,1)$ *and* $C > 0$ *such that, for all balls* B *in* $\mathbb{R}^n$ *and all measurable subsets* $S \subset B$,

$$\frac{\|\chi_S\|_{L^{p(\cdot)}(w)}}{\|\chi_B\|_{L^{p(\cdot)}(w)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_1}, \tag{2}$$

$$\frac{\|\chi_S\|_{L^{p'(\cdot)}(w^{-1})}}{\|\chi_B\|_{L^{p'(\cdot)}(w^{-1})}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_2}. \tag{3}$$

Lemma 5 (see [30, Lemma 4]) *If $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ and $w \in A_{p(\cdot)}$, then there exists a positive constant C such that, for all balls B in $\mathbb{R}^n$,*

$$C^{-1} \leq \frac{1}{|B|} \|\chi_B\|_{L^{p(\cdot)}(w)} \|\chi_B\|_{L^{p'(\cdot)}(w^{-1})} \leq C.$$

Our main result is as follows.

Theorem 1 *Assume that T is a bilinear Calderón–Zygmund operator, $p_1(\cdot), p_2(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ satisfying $1/p(x) = 1/p_1(x) + 1/p_2(x)$ for $x \in \mathbb{R}^n$. Let w_1, w_2 be weights, $w = w_1 w_2$, $w_i \in A_{p_i(\cdot)}$, $i = 1,2$. Suppose that $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathcal{P}_0^{\log}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$, $\alpha(0) = \alpha_1(0) + \alpha_2(0)$, $\alpha_\infty = \alpha_{1\infty} + \alpha_{2\infty}$, $q(\cdot) \in \mathcal{P}_0^{\log}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$, $1/q(0) = 1/q_1(0) + 1/q_2(0)$, $1/q_\infty = 1/q_{1\infty} + 1/q_{2\infty}$, $\lambda = \lambda_1 + \lambda_2$, $0 \leq \lambda_i < \infty$, δ_{i1}, $\delta_{i2} \in (0,1)$ are the constants in Lemma 4 for exponents $p_i(\cdot)$ and weights w_i, $i = 1,2$. If $\lambda_i + n\delta_{i2} > \alpha_{i\infty} \geq \alpha_i(0)$ for $i = 1,2$, then*

$$\|T(f_1,f_2)\|_{M\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot),\lambda}(w)} \lesssim \|f_1\|_{M\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot),\lambda_1}(w_1)} \|f_2\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}.$$

From Theorem 1, we obtain the following corollary.

Corollary 1 *Assume that T is a bilinear Calderón–Zygmund operator, $p_1(\cdot), p_2(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ satisfying $1/p(x) = 1/p_1(x) + 1/p_2(x)$ for $x \in \mathbb{R}^n$. Let w_1, w_2 be weights, $w = w_1 w_2$, $w_i \in A_{p_i(\cdot)}$, $i = 1,2$. Suppose that $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathcal{P}_0^{\log}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$, $\alpha(0) = \alpha_1(0) + \alpha_2(0)$, $\alpha_\infty = \alpha_{1\infty} + \alpha_{2\infty}$, $q(\cdot) \in \mathcal{P}_0^{\log}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$, $1/q(0) = 1/q_1(0) + 1/q_2(0)$, $1/q_\infty = 1/q_{1\infty} + 1/q_{2\infty}$, δ_{i1}, $\delta_{i2} \in (0,1)$ are the constants in Lemma 4 for exponents $p_i(\cdot)$ and weights w_i, $i = 1,2$. If $\lambda_i + n\delta_{i2} > \alpha_{i\infty} \geq \alpha_i(0)$ for $i = 1,2$, then*

$$\|T(f_1,f_2)\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(w)} \lesssim \|f_1\|_{\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot)}(w_1)} \|f_2\|_{\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot)}(w_2)}.$$

3 Proof of Theorem 1

To prove Theorem 1, we need a series of lemmas.

Lemma 6 (see [16, Theorem 2.3]) *Let $p(\cdot), p_1(\cdot), p_2(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ such that $1/p(x) = 1/p_1(x) + 1/p_2(x)$ for $x \in \mathbb{R}^n$. Then there exists a constant C_{p,p_1} independent of functions f and g such that*

$$\|fg\|_{L^{p(\cdot)}} \leq C_{p,p_1} \|f\|_{L^{p_1(\cdot)}} \|g\|_{L^{p_2(\cdot)}}$$

holds for every $f \in L^{p_1(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p_2(\cdot)}(\mathbb{R}^n)$. If $p \in \mathcal{P}(\mathbb{R}^n)$, w be weight with $w = w_1 \times w_2$, then

$$\|fg\|_{L^{p(\cdot)}(w)} \leq C_{p,p_1} \|f\|_{L^{p_1(\cdot)}(w_1)} \|g\|_{L^{p_2(\cdot)}(w_2)}.$$

Lemma 7 (see [32, Proposition 1.2]) *Let $0 < p \leq \infty$, $\delta > 0$. Then there is a positive constant C such that*

$$\left(\sum_{j=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} 2^{-|k-j|\delta} a_k\right)^p\right)^{1/p} \leq C \left(\sum_{j=-\infty}^{\infty} a_j^p\right)^{1/p} \tag{4}$$

for non-negative sequences $\{a_j\}_{j=-\infty}^{\infty}$. *Here, when* $p=\infty$, *it is understood that* (4) *stands for*

$$\sup_{j\in\mathbb{Z}}\left(\sum_{k=-\infty}^{\infty}2^{-|k-j|\delta}a_k\right)\le C\sup_{j\in\mathbb{Z}}a_j.$$

The following lemma is a corollary of [1, Theorem 2.8].

Lemma 8 *Let* $p_1(\cdot), p_1(\cdot)\in\mathcal{P}(\mathbb{R}^n)$, $1<(p_i)_-\le(p_i)_+<\infty$ *and* $p_i(\cdot)\in\mathcal{P}^{\log}(\mathbb{R}^n)\cap\mathcal{P}(\mathbb{R}^n)$ *satisfying* $1/p(x)=1/p_1(x)+1/p_2(x)$ *for* $x\in\mathbb{R}^n$, $i=1,2$. *Let* $w_1\in A_{p_1(\cdot)}$, $w_2\in A_{p_2(\cdot)}$ *and* $w=w_1w_2$. *If* T *is a bilinear Calderón–Zygmund operator, then*

$$\|T(f_1,f_2)\|_{L^{p(\cdot)}(w)}\lesssim\|f_1\|_{L^{p_1(\cdot)}(w_1)}\|f_2\|_{L^{p_2(\cdot)}(w_2)}.$$

Proof of Theorem 1 Let f_1 and f_2 be bounded functions with compact support and write

$$f_i=\sum_{l=-\infty}^{\infty}f_i\chi_l=:\sum_{l=-\infty}^{\infty}f_{il},\quad i=1,2.$$

By Proposition 1, we have

$$\begin{aligned}
&\|T(f_1,f_2)\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q(\cdot)}(w)}\\
&\quad\approx\max\Big\{\sup_{L\le0,L\in\mathbb{Z}}2^{-L\lambda}\big\|\big(2^{k\alpha(0)}T(f_1,f_2)\chi_k\big)_{k\le L}\big\|_{l^{q_0}(L^{p(\cdot)}(w))},\\
&\qquad\sup_{L>0,L\in\mathbb{Z}}\Big[2^{-L\lambda}\big\|\big(2^{k\alpha(0)}T(f_1,f_2)\chi_k\big)_{k<0}\big\|_{l^{q_0}(L^{p(\cdot)}(w))}\\
&\qquad+2^{-L\lambda}\big\|\big(2^{k\alpha_\infty}T(f_1,f_2)\chi_k\big)_{k=0}^{L}\big\|_{l^{q_\infty}(L^{p(\cdot)}(w))}\Big]\Big\}\\
&\quad:=\max\{E,F+G\},
\end{aligned}$$

where

$$\begin{aligned}
E&:=\sup_{L\le0,L\in\mathbb{Z}}2^{-L\lambda}\big\|\big(2^{k\alpha(0)}T(f_1,f_2)\chi_k\big)_{k\le L}\big\|_{l^{q_0}(L^{p(\cdot)}(w))},\\
F&:=\sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda}\big\|\big(2^{k\alpha(0)}T(f_1,f_2)\chi_k\big)_{k<0}\big\|_{l^{q_0}(L^{p(\cdot)}(w))},\\
G&:=\sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda}\big\|\big(2^{k\alpha_\infty}T(f_1,f_2)\chi_k\big)_{k=0}^{L}\big\|_{l^{q_\infty}(L^{p(\cdot)}(w))}.
\end{aligned}$$

Since to estimate F is essentially similar to estimate E, so it suffices for us to show that E and G are bounded in weighted Herz–Morrey space with variable exponents. It is easy to see that

$$E\le C\sum_{i=i}^{9}E_i,\qquad G\le C\sum_{i=i}^{9}G_i,$$

where

$$E_1:=\sup_{L\le0,L\in\mathbb{Z}}2^{-L\lambda}\left(\sum_{k=-\infty}^{L}2^{k\alpha(0)q(0)}\left\|\sum_{l=-\infty}^{k-2}\sum_{j=-\infty}^{k-2}T(f_{1l},f_{2j})\chi_k\right\|_{L^{p(\cdot)}(w)}^{q(0)}\right)^{\frac{1}{q(0)}},$$

$$E_2 := \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)}\left\|\sum_{l=-\infty}^{k-2}\sum_{j=k-1}^{k+1} T(f_{1l},f_{2j})\chi_k\right\|_{L^{p(\cdot)}(w)}^{q(0)}\right)^{\frac{1}{q(0)}},$$

$$E_3 := \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)}\left\|\sum_{l=-\infty}^{k-2}\sum_{j=k+2}^{\infty} T(f_{1l},f_{2j})\chi_k\right\|_{L^{p(\cdot)}(w)}^{q(0)}\right)^{\frac{1}{q(0)}},$$

$$E_4 := \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)}\left\|\sum_{l=k-1}^{k+1}\sum_{j=-\infty}^{k-2} T(f_{1l},f_{2j})\chi_k\right\|_{L^{p(\cdot)}(w)}^{q(0)}\right)^{\frac{1}{q(0)}},$$

$$E_5 := \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)}\left\|\sum_{l=k-1}^{k+1}\sum_{j=k-1}^{k+1} T(f_{1l},f_{2j})\chi_k\right\|_{L^{p(\cdot)}(w)}^{q(0)}\right)^{\frac{1}{q(0)}},$$

$$E_6 := \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)}\left\|\sum_{l=k-1}^{k+1}\sum_{j=k+2}^{\infty} T(f_{1l},f_{2j})\chi_k\right\|_{L^{p(\cdot)}(w)}^{q(0)}\right)^{\frac{1}{q(0)}},$$

$$E_7 := \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)}\left\|\sum_{l=k+2}^{\infty}\sum_{j=-\infty}^{k-2} T(f_{1l},f_{2j})\chi_k\right\|_{L^{p(\cdot)}(w)}^{q(0)}\right)^{\frac{1}{q(0)}},$$

$$E_8 := \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)}\left\|\sum_{l=k+2}^{\infty}\sum_{j=k-1}^{k+1} T(f_{1l},f_{2j})\chi_k\right\|_{L^{p(\cdot)}(w)}^{q(0)}\right)^{\frac{1}{q(0)}},$$

$$E_9 := \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)}\left\|\sum_{l=k+2}^{\infty}\sum_{j=k+2}^{\infty} T(f_{1l},f_{2j})\chi_k\right\|_{L^{p(\cdot)}(w)}^{q(0)}\right)^{\frac{1}{q(0)}},$$

$$G_1 := \sup_{L> 0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty}\left\|\sum_{l=-\infty}^{k-2}\sum_{j=-\infty}^{k-2} T(f_{1l},f_{2j})\chi_k\right\|_{L^{p(\cdot)}(w)}^{q_\infty}\right)^{\frac{1}{q_\infty}},$$

$$G_2 := \sup_{L> 0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty}\left\|\sum_{l=-\infty}^{k-2}\sum_{j=k-1}^{k+1} T(f_{1l},f_{2j})\chi_k\right\|_{L^{p(\cdot)}(w)}^{q_\infty}\right)^{\frac{1}{q_\infty}},$$

$$G_3 := \sup_{L> 0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty}\left\|\sum_{l=-\infty}^{k-2}\sum_{j=k+2}^{\infty} T(f_{1l},f_{2j})\chi_k\right\|_{L^{p(\cdot)}(w)}^{q_\infty}\right)^{\frac{1}{q_\infty}},$$

$$G_4 := \sup_{L> 0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty}\left\|\sum_{l=k-1}^{k+1}\sum_{j=-\infty}^{k-2} T(f_{1l},f_{2j})\chi_k\right\|_{L^{p(\cdot)}(w)}^{q_\infty}\right)^{\frac{1}{q_\infty}},$$

$$G_5 := \sup_{L> 0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty}\left\|\sum_{l=k-1}^{k+1}\sum_{j=k-1}^{k+1} T(f_{1l},f_{2j})\chi_k\right\|_{L^{p(\cdot)}(w)}^{q_\infty}\right)^{\frac{1}{q_\infty}},$$

$$G_6 := \sup_{L> 0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty}\left\|\sum_{l=k-1}^{k+1}\sum_{j=k+2}^{\infty} T(f_{1l},f_{2j})\chi_k\right\|_{L^{p(\cdot)}(w)}^{q_\infty}\right)^{\frac{1}{q_\infty}},$$

$$G_7 := \sup_{L> 0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty}\left\|\sum_{l=k+2}^{\infty}\sum_{j=-\infty}^{k-2} T(f_{1l},f_{2j})\chi_k\right\|_{L^{p(\cdot)}(w)}^{q_\infty}\right)^{\frac{1}{q_\infty}},$$

$$G_8 := \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty}\left\|\sum_{l=k+2}^{\infty}\sum_{j=k-1}^{k+1} T(f_{1l}, f_{2j})\chi_k\right\|_{L^{p(\cdot)}(w)}^{q_\infty}\right)^{\frac{1}{q_\infty}},$$

$$G_9 := \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty}\left\|\sum_{l=k+2}^{\infty}\sum_{j=k+2}^{\infty} T(f_{1l}, f_{2j})\chi_k\right\|_{L^{p(\cdot)}(w)}^{q_\infty}\right)^{\frac{1}{q_\infty}}.$$

We shall use the following estimates. If $l \le k-1$, then, by Hölder's inequality and Lemmas 4 and 5, we have

$$\begin{aligned}
&\left\|2^{-kn}\int_{\mathbb{R}^n} f_{il}\,dy_i\chi_k\right\|_{L^{p_i(\cdot)}(w_i)} \\
&\quad\le C2^{-kn}\|\chi_{B_k}\|_{L^{p_i(\cdot)}(w_i)}\|f_{il}w_i\chi_l\|_{L^{p_i(\cdot)}}\|\chi_l w_i^{-1}\|_{L^{p_i'(\cdot)}} \\
&\quad\le C2^{-kn}|B_k|\|\chi_{B_k}\|^{-1}_{L^{p_i'(\cdot)}(w_i^{-1})}\|\chi_{B_l}\|_{L^{p_i'(\cdot)}(w_i^{-1})}\|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)} \\
&\quad\le C2^{(l-k)n\delta_{2i}}\|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)}.
\end{aligned} \tag{5}$$

If $l = k$, then

$$\begin{aligned}
&\left\|2^{-kn}\int_{\mathbb{R}^n} f_{il}\,dy_i\chi_k\right\|_{L^{p_i(\cdot)}(w_i)} \\
&\quad\le C2^{-kn}\|\chi_{B_k}\|_{L^{p_i(\cdot)}(w_i)}\|f_{il}w_i\chi_l\|_{L^{p_i(\cdot)}}\|\chi_l w_i^{-1}\|_{L^{p_i'(\cdot)}} \\
&\quad\le C2^{-kn}\|\chi_{B_k}\|_{L^{p_i(\cdot)}(w_i)}\|\chi_{B_l}\|_{L^{p_i'(\cdot)}(w_i^{-1})}\|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)} \\
&\quad\le \|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)}.
\end{aligned} \tag{6}$$

If $l \ge k+1$, then

$$\begin{aligned}
&\left\|2^{-kn}\int_{\mathbb{R}^n} f_{il}\,dy_i\chi_k\right\|_{L^{p_i(\cdot)}(w_i)} \\
&\quad\le C2^{-kn}\|\chi_{B_k}\|_{L^{p_i(\cdot)}(w_i)}\|f_{il}w_i\chi_l\|_{L^{p_i(\cdot)}}\|\chi_l w_i^{-1}\|_{L^{p_i'(\cdot)}} \\
&\quad\le C2^{-kn}\|\chi_{B_k}\|_{L^{p_i(\cdot)}(w_i)}\|\chi_{B_l}\|_{L^{p_i(\cdot)}(w_i)}\|\chi_{B_l}\|^{-1}_{L^{p_i(\cdot)}(w_i)} \\
&\qquad\times\|\chi_{B_l}\|_{L^{p_i'(\cdot)}(w_i^{-1})}\|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)} \\
&\quad\le C2^{(l-k)n(1-\delta_{1i})}\|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)}.
\end{aligned} \tag{7}$$

By the symmetry of f_1 and f_2, it is only necessary to estimate E_1, E_2, E_3, E_5, E_6, and E_9.

To estimate E_1, since $l, j \le k-2$, we deduce that, for $i = 1, 2$,

$$|x-y_i| \ge |x| - |y_i| > 2^{k-1} - 2^{\min\{l,j\}} \ge 2^{k-2}, \quad x\in D_k, y_1\in D_l, y_2\in D_j.$$

Therefore, for $x \in D_k$, we have

$$\bigl|K(x, y_1, y_2)\bigr| \le C\bigl(|x-y_1| + |x-y_2|\bigr)^{-2n} \le C2^{-2kn}.$$

Thus, $\forall x \in D_k$ and $l, j \leq k-2$, we have

$$\begin{aligned} \big|T(f_{1l},f_{2j})(x)\big| &\lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f_{1l}(y_1)||f_{2j}(y_2)|}{(|x-y_1|+|x-y_2|)^{2n}}\,dy_1\,dy_2 \\ &\lesssim 2^{-2kn}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\big|f_{1l}(y_1)\big|\big|f_{2j}(y_2)\big|\,dy_1\,dy_2. \end{aligned}$$

Therefore, by Hölder's inequality, we obtain

$$\begin{aligned} &\Bigg\|\sum_{l=-\infty}^{k-2}\sum_{j=-\infty}^{k-2} T(f_{1l},f_{2j})\chi_k\Bigg\|_{L^{p(\cdot)}(w)} \\ &\quad\lesssim 2^{-2kn}\Bigg\|\sum_{l=-\infty}^{k-2}\sum_{j=-\infty}^{k-2}\int_{\mathbb{R}^n}\big|f_{1l}(y_1)\big|\,dy_1\int_{\mathbb{R}^n}\big|f_{2j}(y_2)\big|\,dy_2\,\chi_k\Bigg\|_{L^{p(\cdot)}(w)} \\ &\quad\lesssim \Bigg\|\sum_{l=-\infty}^{k-2} 2^{-kn}\int_{\mathbb{R}^n}\big|f_{1l}(y_1)\big|\,dy_1\,\chi_k\Bigg\|_{L^{p_1(\cdot)}(w_1)} \\ &\quad\quad\times\Bigg\|\sum_{j=-\infty}^{k-2} 2^{-kn}\int_{\mathbb{R}^n}\big|f_{2j}(y_2)\big|\,dy_2\,\chi_k\Bigg\|_{L^{p_2(\cdot)}(w_2)}. \end{aligned} \tag{8}$$

Since $1/q(0) = 1/q_1(0) + 1/q_2(0)$, $\lambda = \lambda_1 + \lambda_2$, by Hölder's inequality, we have

$$\begin{aligned} E_1 &\lesssim \sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda}\Bigg(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)}\Bigg\|\sum_{l=-\infty}^{k-2} 2^{-kn}\int_{\mathbb{R}^n}\big|f_{1l}(y_1)\big|\,dy_1\,\chi_k\Bigg\|_{L^{p_1(\cdot)}(w_1)}^{q(0)} \\ &\quad\times\Bigg\|\sum_{j=-\infty}^{k-2} 2^{-kn}\int_{\mathbb{R}^n}\big|f_{2j}(y_2)\big|\,dy_2\,\chi_k\Bigg\|_{L^{p_2(\cdot)}(w_2)}^{q(0)}\Bigg)^{\frac{1}{q(0)}} \\ &\lesssim \sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda_1} \\ &\quad\times\Bigg(\sum_{k=-\infty}^{L} 2^{k\alpha_1(0)q_1(0)}\Bigg\|\sum_{l=-\infty}^{k-2} 2^{-kn}\int_{\mathbb{R}^n}\big|f_{1l}(y_1)\big|\,dy_1\,\chi_k\Bigg\|_{L^{p_1(\cdot)}(w_1)}^{q_1(0)}\Bigg)^{\frac{1}{q_1(0)}} \\ &\quad\times\sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda_2} \\ &\quad\times\Bigg(\sum_{k=-\infty}^{L} 2^{k\alpha_2(0)q_2(0)}\Bigg\|\sum_{j=-\infty}^{k-2} 2^{-kn}\int_{\mathbb{R}^n}\big|f_{2j}(y_2)\big|\,dy_2\,\chi_k\Bigg\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)}\Bigg)^{\frac{1}{q_2(0)}} \\ &:= E_{1,1}\times E_{1,2}, \end{aligned}$$

where

$$\begin{aligned} E_{1,i} &:= \sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda_i} \\ &\quad\times\Bigg\{\sum_{k=-\infty}^{L} 2^{k\alpha_i(0)q_i(0)}\Bigg\|\sum_{l=-\infty}^{k-2} 2^{-kn}\int_{\mathbb{R}^n}\big|f_{il}(y_i)\big|\,dy_i\,\chi_k\Bigg\|_{L^{p_i(\cdot)}(w_i)}^{q_i(0)}\Bigg\}^{\frac{1}{q_i(0)}}. \end{aligned}$$

Since $n\delta_{2i} - \alpha_i(0) > 0$, by (5) and Lemma 7 we obtain

$$\begin{aligned}
E_{1,i} &\lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_i} \left\{ \sum_{k=-\infty}^{L} 2^{k\alpha_i(0)q_i(0)} \left(\sum_{l=-\infty}^{k-2} 2^{(l-k)n\delta_{2i}} \|f_{il}\|_{L^{p_i(\cdot)}(w_i)} \right)^{q_i(0)} \right\}^{\frac{1}{q_i(0)}} \\
&= \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_i} \\
&\quad \times \left\{ \sum_{k=-\infty}^{L} \left(\sum_{l=-\infty}^{k-2} 2^{l\alpha_i(0)} \|f_{il}\|_{L^{p_i(\cdot)}(w_i)} 2^{(l-k)(n\delta_{2i}-\alpha_i(0))} \right)^{q_i(0)} \right\}^{\frac{1}{q_i(0)}} \\
&\lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_i} \left(\sum_{l=-\infty}^{L-2} 2^{l\alpha_i(0)q_i(0)} \|f_{il}\|_{L^{p_i(\cdot)}(w_i)}^{q_i(0)} \right)^{\frac{1}{q_i(0)}} \\
&\lesssim \|f_i\|_{M\dot{K}^{\alpha_i(\cdot),q_i(\cdot)}_{p_i(\cdot),\lambda_i}(w_i)},
\end{aligned}$$

where we wrote $2^{-|k-l|(n\delta_{2i}-\alpha_i(0))} \lesssim 2^{-|k-l|\varepsilon_i}$ for some $\varepsilon_i \in (0, n\delta_{2i} - \alpha_i(0))$. Thus, we obtain

$$E_1 \lesssim \|f_1\|_{M\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot),\lambda_1}(w_1)} \|f_2\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}.$$

To estimate E_2, since $l \le k-2$, $k-1 \le j \le k+1$ for $i = 1, 2$, we have

$$|x - y_1| \ge |x| - |y_1| \ge 2^{k-2}, \quad x \in D_k, y_1 \in D_l.$$

Therefore, by Hölder's inequality, we obtain

$$\begin{aligned}
&\left\| \sum_{l=-\infty}^{k-2} \sum_{j=k-1}^{k+1} T(f_{1l}, f_{2j})\chi_k \right\|_{L^{p(\cdot)}(w)} \\
&\quad \lesssim 2^{-2kn} \left\| \sum_{l=-\infty}^{k-2} \sum_{j=k-1}^{k+1} \int_{\mathbb{R}^n} |f_{1l}(y_1)|\, dy_1 \int_{\mathbb{R}^n} |f_{2j}(y_2)|\, dy_2 \chi_k \right\|_{L^{p(\cdot)}(w)} \\
&\quad \lesssim \left\| \sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |f_{1l}(y_1)|\, dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)} \\
&\qquad \times \left\| \sum_{j=k-1}^{k+1} 2^{-kn} \int_{\mathbb{R}^n} |f_{2j}(y_2)|\, dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}.
\end{aligned} \tag{9}$$

Since $1/q(0) = 1/q_1(0) + 1/q_2(0)$, $\lambda = \lambda_1 + \lambda_2$, by Hölder's inequality, we have

$$\begin{aligned}
E_2 &\lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda} \left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)} \left\| \sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |f_{1l}(y_1)|\, dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q(0)} \right. \\
&\quad \left. \times \left\| \sum_{j=k-1}^{k+1} 2^{-kn} \int_{\mathbb{R}^n} |f_{2j}(y_2)|\, dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q(0)} \right)^{\frac{1}{q(0)}} \\
&\lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_1}
\end{aligned}$$

$$\times \left(\sum_{k=-\infty}^{L} 2^{k\alpha_1(0)q_1(0)} \left\| \sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |f_{1l}(y_1)| \, dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q_1(0)} \right)^{\frac{1}{q_1(0)}}$$
$$\times \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_2}$$
$$\times \left(\sum_{k=-\infty}^{L} 2^{k\alpha_2(0)q_2(0)} \left\| \sum_{j=k-1}^{k+1} 2^{-kn} \int_{\mathbb{R}^n} |f_{2j}(y_2)| \, dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)} \right)^{\frac{1}{q_2(0)}}$$
$$:= E_{2,1} \times E_{2,2}.$$

It is obvious that

$$E_{2,1} = E_{1,1} \lesssim \|f_1\|_{M\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot),\lambda_1}(w_1)}.$$

Now we estimate $E_{2,2}$. Taking (5), (6) and (7) together, we have

$$E_{2,2} \lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_2} \left(\sum_{k=-\infty}^{L} 2^{k\alpha_2(0)q_2(0)} \left\| \sum_{j=k-1}^{k+1} 2^{(j-k)n} f_{2j} \right\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)} \right)^{\frac{1}{q_2(0)}}$$
$$\lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_2} \left(\sum_{k=-\infty}^{L} 2^{k\alpha_2(0)q_2(0)} \|f_{2j}\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)} \right)^{\frac{1}{q_2(0)}}$$
$$\lesssim \|f_2\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)},$$

where we used $2^{-n\delta_{22}} < 1$ and $2^{(j-k)n(1-\delta_{12})} < 2^{(j-k)n}$, $j \in \{k-1, k, k+1\}$ for (5) and (7), respectively. Thus, we obtain

$$E_2 \lesssim \|f_1\|_{M\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot),\lambda_1}(w_1)} \|f_2\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}.$$

To estimate E_3, since $l \le k-2$, $j \ge k+2$, then we have

$$|x-y_1| \ge |x| - |y_1| \ge 2^{k-2}, \qquad |x-y_2| \ge |y_2| - |x| > 2^{j-2}, \quad x \in D_k, y_1 \in D_l, y_2 \in D_j.$$

Therefore, $\forall x \in D_k$, $l \le k-2$, $j \ge k+2$, we get

$$\left| T(f_{1l}, f_{2j})(x) \right| \lesssim \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f_{1l}(y_1)||f_{2j}(y_2)|}{(|x-y_1| + |x-y_2|)^{2n}} \, dy_1 \, dy_2$$
$$\lesssim 2^{-kn} 2^{-jn} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f_{1l}(y_1)| |f_{2j}(y_2)| \, dy_1 \, dy_2.$$

Thus, by Hölder's inequality, we have

$$\left\| \sum_{l=-\infty}^{k-2} \sum_{j=k+2}^{\infty} T(f_{1l}, f_{2j}) \chi_k \right\|_{L^{p(\cdot)}(w)}$$
$$\lesssim 2^{-kn} 2^{-jn} \left\| \sum_{l=-\infty}^{k-2} \int_{\mathbb{R}^n} |f_{1l}(y_1)| \, dy_1 \sum_{j=k+2}^{\infty} \int_{\mathbb{R}^n} |f_{2j}(y_2)| \, dy_2 \chi_k \right\|_{L^{p(\cdot)}(w)}$$

$$
\begin{aligned}
&\lesssim \left\| \sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} \left| f_{1l}(y_1) \right| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)} \\
&\quad \times \left\| \sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} \left| f_{2j}(y_2) \right| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}.
\end{aligned} \tag{10}
$$

Since $1/q(0) = 1/q_1(0) + 1/q_2(0)$, $\lambda = \lambda_1 + \lambda_2$, by Hölder's inequality, we have

$$
\begin{aligned}
E_3 &\lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda} \left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)} \left\| \sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} \left| f_{1l}(y_1) \right| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q(0)} \right. \\
&\quad \left. \times \left\| \sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} \left| f_{2j}(y_2) \right| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q(0)} \right)^{\frac{1}{q(0)}} \\
&\lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_1} \\
&\quad \times \left(\sum_{k=-\infty}^{L} 2^{k\alpha_1(0)q_1(0)} \left\| \sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} \left| f_{1l}(y_1) \right| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q_1(0)} \right)^{\frac{1}{q_1(0)}} \\
&\quad \times \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_2} \\
&\quad \times \left(\sum_{k=-\infty}^{L} 2^{k\alpha_2(0)q_2(0)} \left\| \sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} \left| f_{2j}(y_2) \right| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)} \right)^{\frac{1}{q_2(0)}} \\
&:= E_{3,1} \times E_{3,2}.
\end{aligned}
$$

It is obvious that

$$
E_{3,1} = E_{1,1} \lesssim \|f_1\|_{M\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot),\lambda_1}(w_1)}.
$$

Since $n\delta_{12} + \alpha_2(0) > 0$, by (7) and Lemma 7 we obtain

$$
\begin{aligned}
E_{3,2} &\lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_2} \left(\sum_{k=-\infty}^{L} 2^{k\alpha_2(0)q_2(0)} \right. \\
&\quad \left. \times \left(\sum_{j=k+2}^{\infty} 2^{(k-j)n\delta_{12}} \|f_{2j}\|_{L^{p_2(\cdot)}(w_2)} \right)^{q_2(0)} \right)^{\frac{1}{q_2(0)}} \\
&\lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_2} \\
&\quad \times \left(\sum_{k=-\infty}^{L} \left(\sum_{j=k+2}^{\infty} 2^{j\alpha_2(0)} \|f_{2j}\|_{L^{p_2(\cdot)}(w_2)} 2^{(k-j)(n\delta_{12}+\alpha_2(0))} \right)^{q_2(0)} \right)^{\frac{1}{q_2(0)}} \\
&\lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_2} \left(\sum_{j=-\infty}^{L+2} 2^{j\alpha_2(0)q_2(0)} \|f_{2j}\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)} \right)^{\frac{1}{q_2(0)}} \\
&\lesssim \|f_2\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)},
\end{aligned}
$$

where we wrote $2^{-|k-j|(n\delta_{12}+\alpha_2(0))} \lesssim 2^{-|k-j|\eta_2}$ for some $\eta_2 \in (0, n\delta_{12} + \alpha_2(0))$. Thus, we have

$$E_3 \lesssim \|f_1\|_{M\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot),\lambda_1}(w_1)} \|f_2\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}.$$

To estimate E_5, using Hölder's inequality and Lemma 8, we have

$$\begin{aligned}
E_5 &\lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda} \left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)} \left\| \sum_{l=k-1}^{k+1} \sum_{j=k-1}^{k+1} T(f_{1l}, f_{2j})\chi_k \right\|_{L^{p(\cdot)}(w)}^{q(0)} \right)^{\frac{1}{q(0)}} \\
&\lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda} \left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)} \left(\|f_1\|_{L^{p_1(\cdot)}(w_1)} \|f_2\|_{L^{p_2(\cdot)}(w_2)} \right)^{q(0)} \right)^{\frac{1}{q(0)}} \\
&\lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_1} \left(\sum_{k=-\infty}^{L} 2^{k\alpha_1(0)q_1(0)} \|f_1\|_{L^{p_1(\cdot)}(w_1)}^{q_1(0)} \right)^{\frac{1}{q_1(0)}} \\
&\quad \times 2^{-L\lambda_2} \left(\sum_{k=-\infty}^{L} 2^{k\alpha_2(0)q_2(0)} \|f_2\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)} \right)^{\frac{1}{q_2(0)}} \\
&\lesssim \|f_1\|_{M\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot),\lambda_1}(w_1)} \|f_2\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}.
\end{aligned}$$

To estimate E_6, since $k-1 \le l \le k+1$ and $j \ge k+2$, we obtain

$$|x-y_1| > 2^{k-2}, \qquad |x-y_2| > 2^{j-2}, \quad x \in D_k, y_1 \in D_l, y_2 \in D_j.$$

Thus, $\forall x \in D_k$, $k-1 \le l \le k+1$ and $j \ge k+2$, we obtain

$$\begin{aligned}
\left|T(f_{1l}, f_{2j})(x)\right| &\lesssim \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f_{1l}(y_1)||f_{2j}(y_2)|}{(|x-y_1| + |x-y_2|)^{2n}} \, dy_1 \, dy_2 \\
&\lesssim 2^{-kn} 2^{-jn} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left|f_{1l}(y_1)\right| \left|f_{2j}(y_2)\right| \, dy_1 \, dy_2.
\end{aligned}$$

Therefore, by Hölder's inequality, we obtain

$$\begin{aligned}
&\left\| \sum_{l=k-1}^{k+1} \sum_{j=k+2}^{\infty} T(f_{1l}, f_{2j})\chi_k \right\|_{L^{p(\cdot)}(w)} \\
&\quad \lesssim 2^{-kn} 2^{-jn} \left\| \sum_{l=k-1}^{k+1} \int_{\mathbb{R}^n} \left|f_{1l}(y_1)\right| dy_1 \sum_{j=-\infty}^{k-2} \int_{\mathbb{R}^n} \left|f_{2j}(y_2)\right| dy_2 \chi_k \right\|_{L^{p(\cdot)}(w)} \\
&\quad \lesssim \left\| \sum_{l=k-1}^{k+1} 2^{-kn} \int_{\mathbb{R}^n} \left|f_{1l}(y_1)\right| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)} \\
&\qquad \times \left\| \sum_{j=-\infty}^{k-2} 2^{-jn} \int_{\mathbb{R}^n} \left|f_{2j}(y_2)\right| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}.
\end{aligned} \tag{11}$$

Since $1/q(0) = 1/q_1(0) + 1/q_2(0)$, $\lambda = \lambda_1 + \lambda_2$, by Hölder's inequality, we have

$$\begin{aligned} E_6 &\lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda} \Biggl(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)} \Biggl\| \sum_{l=k-1}^{k+1} 2^{-kn} \int_{\mathbb{R}^n} \bigl|f_{1l}(y_1)\bigr|\,dy_1 \chi_k \Biggr\|_{L^{p_1(\cdot)}(w_1)}^{q(0)} \\ &\quad \times \Biggl\| \sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} \bigl|f_{2j}(y_2)\bigr|\,dy_2 \chi_k \Biggr\|_{L^{p_2(\cdot)}(w_2)}^{q(0)} \Biggr)^{\frac{1}{q(0)}} \\ &\lesssim \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_1} \\ &\quad \times \Biggl(\sum_{k=-\infty}^{L} 2^{k\alpha_1(0)q_1(0)} \Biggl\| \sum_{l=k-1}^{k+1} 2^{-kn} \int_{\mathbb{R}^n} \bigl|f_{1l}(y_1)\bigr|\,dy_1 \chi_k \Biggr\|_{L^{p_1(\cdot)}(w_1)}^{q_1(0)} \Biggr)^{\frac{1}{q_1(0)}} \\ &\quad \times \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda_2} \\ &\quad \times \Biggl(\sum_{k=-\infty}^{L} 2^{k\alpha_2(0)q_2(0)} \Biggl\| \sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} \bigl|f_{2j}(y_2)\bigr|\,dy_2 \chi_k \Biggr\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)} \Biggr)^{\frac{1}{q_2(0)}} \\ &:= E_{6,1} \times E_{6,2}. \end{aligned}$$

By the symmetry of f_1 and f_2, we can know that the estimate $E_{6,1}$ is similar to the estimated $E_{2,2}$ and $E_{6,2} = E_{3,2}$.

To estimate E_9, since $l, j \ge k+2$, for $i = 1, 2$, we get

$$|x - y_i| > 2^{k-2}, \quad x \in D_k, y_1 \in D_l, y_2 \in D_j.$$

Therefore, $\forall x \in D_k$, $l, j \ge k+2$, we have

$$\begin{aligned} \bigl|T(f_{1l}, f_{2j})(x)\bigr| &\lesssim \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f_{1l}(y_1)||f_{2j}(y_2)|}{(|x-y_1| + |x-y_2|)^{2n}}\,dy_1\,dy_2 \\ &\lesssim 2^{-ln} 2^{-jn} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \bigl|f_{1l}(y_1)\bigr|\bigl|f_{2j}(y_2)\bigr|\,dy_1\,dy_2. \end{aligned}$$

Thus, by Hölder's inequality, we have

$$\begin{aligned} &\Biggl\| \sum_{l=k+2}^{\infty} \sum_{j=k+2}^{\infty} T(f_{1l}, f_{2j}) \chi_k \Biggr\|_{L^{p(\cdot)}(w)} \\ &\quad \lesssim 2^{-ln} 2^{-jn} \Biggl\| \sum_{l=k+2}^{\infty} \int_{\mathbb{R}^n} \bigl|f_{1l}(y_1)\bigr|\,dy_1 \sum_{j=k+2}^{\infty} \int_{\mathbb{R}^n} \bigl|f_{2j}(y_2)\bigr|\,dy_2 \chi_k \Biggr\|_{L^{p(\cdot)}(w)} \\ &\quad \lesssim \Biggl\| \sum_{l=k+2}^{\infty} 2^{-ln} \int_{\mathbb{R}^n} \bigl|f_{1l}(y_1)\bigr|\,dy_1 \chi_k \Biggr\|_{L^{p_1(\cdot)}(w_1)} \\ &\quad \quad \times \Biggl\| \sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} \bigl|f_{2j}(y_2)\bigr|\,dy_2 \chi_k \Biggr\|_{L^{p_2(\cdot)}(w_2)}. \end{aligned} \tag{12}$$

Since $1/q(0) = 1/q_1(0) + 1/q_2(0)$, $\lambda = \lambda_1 + \lambda_2$, by Hölder's inequality, we have

$$\begin{aligned}
E_9 &\lesssim \sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q(0)}\left\|\sum_{l=k+2}^{\infty} 2^{-ln}\int_{\mathbb{R}^n}\left|f_{1l}(y_1)\right|dy_1\chi_k\right\|_{L^{p_1(\cdot)}(w_1)}^{q(0)}\right.\\
&\quad\left.\times\left\|\sum_{j=k+2}^{\infty} 2^{-jn}\int_{\mathbb{R}^n}\left|f_{2j}(y_2)\right|dy_2\chi_k\right\|_{L^{p_2(\cdot)}(w_2)}^{q(0)}\right)^{\frac{1}{q(0)}}\\
&\lesssim \sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda_1}\\
&\quad\times\left(\sum_{k=-\infty}^{L} 2^{k\alpha_1(0)q_1(0)}\left\|\sum_{l=k+2}^{\infty} 2^{-ln}\int_{\mathbb{R}^n}\left|f_{1l}(y_1)\right|dy_1\chi_k\right\|_{L^{p_1(\cdot)}(w_1)}^{q_1(0)}\right)^{\frac{1}{q_1(0)}}\\
&\quad\times\sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda_2}\\
&\quad\times\left(\sum_{k=-\infty}^{L} 2^{k\alpha_2(0)q_2(0)}\left\|\sum_{j=k+2}^{\infty} 2^{-jn}\int_{\mathbb{R}^n}\left|f_{2j}(y_2)\right|dy_2\chi_k\right\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)}\right)^{\frac{1}{q_2(0)}}\\
&:= E_{9,1}\times E_{9,2}.
\end{aligned}$$

Obviously, the estimate $E_{9,i}$ is similar to the estimated $E_{3,2}$ for $i = 1, 2$.

Taking all estimates for E_i together, $i = 1, 2, \dots, 9$, we obtain

$$E \lesssim \|f_1\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}(w_1)}\|f_2\|_{M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_2(\cdot)}(w_2)}.$$

To go on, we need some further preparation.

If $l < 0$, by Proposition 1, we have

$$\begin{aligned}
\|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)} &= 2^{-l\alpha_i(0)}\left(2^{l\alpha_i(0)q_i(0)}\|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)}^{q_i(0)}\right)^{\frac{1}{q_i(0)}}\\
&\lesssim 2^{-l\alpha_i(0)}\left(\sum_{t=-\infty}^{l} 2^{t\alpha_i(0)q_i(0)}\|f_{it}\chi_t\|_{L^{p_i(\cdot)}(w_i)}^{q_i(0)}\right)^{\frac{1}{q_i(0)}}\\
&\lesssim 2^{l(\lambda_i-\alpha_i(0))}\left(2^{-l\lambda_i}\left(\sum_{t=-\infty}^{l}\left\|2^{t\alpha_i(0)}f_{it}\chi_t\right\|_{L^{p_i(\cdot)}(w_i)}^{q_i(0)}\right)\right)^{\frac{1}{q_i(0)}}\\
&\lesssim 2^{l(\lambda_i-\alpha_i(0))}\|f_i\|_{M\dot{K}_{p_i(\cdot),\lambda_i}^{\alpha_i(\cdot),q_i(\cdot)}(w_i)}.
\end{aligned}\tag{13}$$

If $l \geq 0$, we have

$$\begin{aligned}
\|f_{il}\chi_l\|_{L^{p_i(\cdot)}(w_i)} &= 2^{-l\alpha_{i\infty}}\left(2^{l\alpha_{i\infty}q_{i\infty}}\|f_i\chi_{il}\|_{L^{p_i(\cdot)}(w_i)}^{q_{i\infty}}\right)^{1/q_{i\infty}}\\
&\lesssim 2^{-l\alpha_{i\infty}}\left(\sum_{t=-\infty}^{l} 2^{t\alpha_{i\infty}q_{i\infty}}\|f_{it}\chi_t\|_{L^{p_i(\cdot)}(w_i)}^{q_{i\infty}}\right)^{1/q_{i\infty}}\\
&\lesssim 2^{l(\lambda_i-\alpha_{i\infty})}\left(2^{-l\lambda_i}\left(\sum_{t=-\infty}^{l}\left\|2^{t\alpha_{i\infty}}f_{it}\chi_t\right\|_{L^{p_i(\cdot)}(w_i)}^{q_{i\infty}}\right)\right)^{1/q_{i\infty}}
\end{aligned}$$

$$\lesssim 2^{l(\lambda_i-\alpha_{i\infty})}\|f_i\|_{M\dot{K}^{\alpha_i(\cdot),q_i(\cdot)}_{p_i(\cdot),\lambda_i}(w_i)}. \tag{14}$$

Finally, we estimate G. By the symmetry of f_1 and f_2, it is only necessary to estimate G_1, G_2, G_3, G_5, G_6, and G_9.

To estimate G_1, since $l, j \le k-2$, $1/q_\infty = 1/q_{1\infty} + 1/q_{2\infty}$, $\lambda = \lambda_1 + \lambda_2$, by (8) and Hölder's inequality, we have

$$\begin{aligned}
G_1 &\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty}\left\|\sum_{l=-\infty}^{k-2} 2^{-kn}\int_{\mathbb{R}^n}\left|f_{1l}(y_1)\right|dy_1\chi_k\right\|^{q_\infty}_{L^{p_1(\cdot)}(w_1)}\right. \\
&\quad\left.\times\left\|\sum_{j=-\infty}^{k-2} 2^{-kn}\int_{\mathbb{R}^n}\left|f_{2j}(y_2)\right|dy_2\chi_k\right\|^{q_\infty}_{L^{p_2(\cdot)}(w_2)}\right)^{\frac{1}{q_\infty}} \\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda_1} \\
&\quad\times\left(\sum_{k=0}^{L} 2^{k\alpha_{1\infty}q_{1\infty}}\left\|\sum_{l=-\infty}^{k-2} 2^{-kn}\int_{\mathbb{R}^n}\left|f_{1l}(y_1)\right|dy_1\chi_k\right\|^{q_{1\infty}}_{L^{p_1(\cdot)}(w_1)}\right)^{\frac{1}{q_{1\infty}}} \\
&\quad\times\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda_2} \\
&\quad\times\left(\sum_{k=0}^{L} 2^{k\alpha_{2\infty}q_{2\infty}}\left\|\sum_{j=-\infty}^{k-2} 2^{-kn}\int_{\mathbb{R}^n}\left|f_{2j}(y_2)\right|dy_2\chi_k\right\|^{q_{2\infty}}_{L^{p_2(\cdot)}(w_2)}\right)^{\frac{1}{q_{2\infty}}} \\
&:= G_{1,1}\times G_{1,2},
\end{aligned}$$

where

$$G_{1,i} := \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda_i}\left(\sum_{k=0}^{L} 2^{k\alpha_{i\infty}q_{i\infty}}\left\|\sum_{l=-\infty}^{k-2} 2^{-kn}\int_{\mathbb{R}^n}\left|f_{il}(y_i)\right|dy_i\chi_k\right\|^{q_{i\infty}}_{L^{p_i(\cdot)}(w_i)}\right)^{\frac{1}{q_{i\infty}}}.$$

Since $\lambda_i + n\delta_{2i} > \alpha_{i\infty} \ge \alpha_i(0)$, by (5), (13), (14) and Lemma 7 we obtain

$$\begin{aligned}
G_{1,i} &\lesssim \sup_{L>0,L\in\mathbb{Z}} \|f_i\|_{M\dot{K}^{\alpha_i(\cdot),q_i(\cdot)}_{p_i(\cdot),\lambda_i}(w_i)} 2^{-L\lambda_i}\left\{\sum_{k=0}^{L} 2^{k\alpha_{i\infty}q_{i\infty}}\right. \\
&\quad\left.\times\left(\sum_{l=-\infty}^{-1} 2^{(l-k)n\delta_{2i}}2^{l(\lambda_i-\alpha_i(0))} + \sum_{l=0}^{k} 2^{(l-k)n\delta_{12}}2^{i(\lambda_1-\alpha_{1\infty})}\right)^{q_{i\infty}}\right\}^{\frac{1}{q_{i\infty}}} \\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} \|f_i\|_{M\dot{K}^{\alpha_i(\cdot),q_i(\cdot)}_{p_i(\cdot),\lambda_i}(w_i)} 2^{-L\lambda_i}\left\{\sum_{k=0}^{L} 2^{k\alpha_{i\infty}q_{i\infty}}\right. \\
&\quad\times\left(\sum_{l=-\infty}^{-1} 2^{(l-k)(n\delta_{2i}+\lambda_i-\alpha_{i\infty})}2^{l(\alpha_{i\infty}-\alpha_i(0))}2^{k(\lambda_i-\alpha_{i\infty})}\right. \\
&\quad\left.\left.+\sum_{l=0}^{k} 2^{(l-k)(n\delta_{2i}+\lambda_i-\alpha_{i\infty})}2^{k(\lambda_i-\alpha_{i\infty})}\right)^{q_{i\infty}}\right\}^{\frac{1}{q_{i\infty}}} \\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} \|f_i\|_{M\dot{K}^{\alpha_i(\cdot),q_i(\cdot)}_{p_i(\cdot),\lambda_i}(w_i)}
\end{aligned}$$

$$\times 2^{-L\lambda_i}\left\{\sum_{k=0}^{L}2^{k\lambda_i q_{i\infty}}\left(\sum_{l=-\infty}^{k}2^{(l-k)(n\delta_{2i}-\alpha_{i\infty}+\lambda_i)}\right)^{q_{i\infty}}\right\}^{\frac{1}{q_{i\infty}}}$$

$$\lesssim \sup_{L>0,L\in\mathbb{Z}}\|f_i\|_{M\dot{K}^{\alpha_i(\cdot),q_i(\cdot)}_{p_i(\cdot),\lambda_i}(w_i)}2^{-L\lambda_i}\left(\sum_{k=0}^{L}2^{k\lambda_i q_{i\infty}}\right)^{\frac{1}{q_{i\infty}}}$$

$$\lesssim \|f_i\|_{M\dot{K}^{\alpha_i(\cdot),q_i(\cdot)}_{p_i(\cdot),\lambda_i}(w_i)}.$$

Thus, we get

$$G_1 \lesssim \|f_1\|_{M\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot),\lambda_1}(w_1)}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}.$$

To estimate G_2, since $l \le k-2$, $k-1 \le j \le k+1$, $1/q_\infty = 1/q_{1\infty} + 1/q_{2\infty}$, $\lambda = \lambda_1 + \lambda_2$, by (9) and Hölder's inequality, we have

$$G_2 \lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=0}^{L}2^{k\alpha_\infty q_\infty}\left\|\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\left|f_{1l}(y_1)\right|dy_1\chi_k\right\|^{q_\infty}_{L^{p_1(\cdot)}(w_1)}\right.$$

$$\left.\times\left\|\sum_{j=k-1}^{k+1}2^{-kn}\int_{\mathbb{R}^n}\left|f_{2j}(y_2)\right|dy_2\chi_k\right\|^{q_\infty}_{L^{p_2(\cdot)}(w_2)}\right)^{\frac{1}{q_\infty}}$$

$$\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda_1}$$

$$\times\left(\sum_{k=0}^{L}2^{k\alpha_{1\infty}q_{1\infty}}\left\|\sum_{l=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\left|f_{1l}(y_1)\right|dy_1\chi_k\right\|^{q_{1\infty}}_{L^{p_1(\cdot)}(w_1)}\right)^{\frac{1}{q_{1\infty}}}$$

$$\times \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda_2}$$

$$\times\left(\sum_{k=0}^{L}2^{k\alpha_{2\infty}q_{2\infty}}\left\|\sum_{j=k-1}^{k+1}2^{-kn}\int_{\mathbb{R}^n}\left|f_{2j}(y_2)\right|dy_2\chi_k\right\|^{q_{2\infty}}_{L^{p_2(\cdot)}(w_2)}\right)^{\frac{1}{q_{2\infty}}}$$

$$:= G_{2,1}\times G_{2,2}.$$

It is obvious that

$$G_{2,1} = G_{1,1} \lesssim \|f_1\|_{M\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot),\lambda_1}(w_1)}.$$

Now we estimate $G_{2,2}$. Combining (5), (6) and (7), we have

$$G_{2,2} \lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda_2}\left(\sum_{k=0}^{L}2^{k\alpha_{2\infty}q_{2\infty}}\left\|\sum_{j=k-1}^{k+1}2^{(j-k)n}|f_{2j}|\right\|^{q_{2\infty}}_{L^{p_2(\cdot)}(w_2)}\right)^{\frac{1}{q_{2\infty}}}$$

$$\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda_2}\left(\sum_{k=0}^{L}2^{k\alpha_{2\infty}q_{2\infty}}\left\||f_{2j}|\right\|^{q_{2\infty}}_{L^{p_2(\cdot)}(w_2)}\right)^{\frac{1}{q_{2\infty}}}$$

$$\lesssim \|f_2\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)},$$

where we used $2^{-n\delta_{22}} < 1$ and $2^{(j-k)n(1-\delta_{12})} < 2^{(j-k)n}$ for (5) and (7), respectively. Thus, we obtain

$$G_2 \lesssim \|f_1\|_{M\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot),\lambda_1}(w_1)} \|f_2\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}.$$

To estimate G_3, since $l \le k-2$, $j \ge k+2$, $1/q_\infty = 1/q_{1\infty} + 1/q_{2\infty}$, $\lambda = \lambda_1 + \lambda_2$, by (10) and Hölder's inequality, we have

$$\begin{aligned}
G_3 &\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda} \Bigg(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty} \Bigg\| \sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} \big|f_{1l}(y_1)\big|\, dy_1 \chi_k \Bigg\|^{q_\infty}_{L^{p_1(\cdot)}(w_1)} \\
&\quad \times \Bigg\| \sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} \big|f_{2j}(y_2)\big|\, dy_2 \chi_k \Bigg\|^{q_\infty}_{L^{p_2(\cdot)}(w_2)} \Bigg)^{\frac{1}{q_\infty}} \\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda_1} \\
&\quad \times \Bigg(\sum_{k=0}^{L} 2^{k\alpha_{1\infty} q_{1\infty}} \Bigg\| \sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} \big|f_{1l}(y_1)\big|\, dy_1 \chi_k \Bigg\|^{q_{1\infty}}_{L^{p_1(\cdot)}(w_1)} \Bigg)^{\frac{1}{q_{1\infty}}} \\
&\quad \times \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda_2} \\
&\quad \times \Bigg(\sum_{k=0}^{L} 2^{k\alpha_{2\infty} q_{2\infty}} \Bigg\| \sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} \big|f_{2j}(y_2)\big|\, dy_2 \chi_k \Bigg\|^{q_{2\infty}}_{L^{p_2(\cdot)}(w_2)} \Bigg)^{\frac{1}{q_{2\infty}}} \\
&:= G_{3,1} \times G_{3,2}.
\end{aligned}$$

It is obvious that

$$G_{3,1} = G_{1,1} \lesssim \|f_1\|_{M\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot),\lambda_1}(w_1)}.$$

Since $n\delta_{12} + \alpha_{2\infty} > 0$, by (7) and Lemma 7 we obtain

$$\begin{aligned}
G_{3,2} &\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda_2} \Bigg(\sum_{k=0}^{L} 2^{k\alpha_{2\infty} q_{2\infty}} \Bigg(\sum_{j=k+2}^{\infty} 2^{(k-j)n\delta_{12}} \|f_{2j}\|_{L^{p_2(\cdot)}(w_2)} \Bigg)^{q_{2\infty}} \Bigg)^{\frac{1}{q_{2\infty}}} \\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda_2} \Bigg(\sum_{k=0}^{L} \Bigg(\sum_{j=k+2}^{\infty} 2^{j\alpha_{2\infty}} \|f_{2j}\|_{L^{p_2(\cdot)}(w_2)} 2^{(k-j)(n\delta_{12}+\alpha_{2\infty})} \Bigg)^{q_{2\infty}} \Bigg)^{\frac{1}{q_{2\infty}}} \\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda_2} \Bigg(\sum_{j=2}^{L+2} 2^{j\alpha_{2\infty} q_{2\infty}} \|f_{2j}\|^{q_{2\infty}}_{L^{p_2(\cdot)}(w_2)} \Bigg)^{\frac{1}{q_{2\infty}}} \\
&\lesssim \|f_2\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)},
\end{aligned}$$

where we wrote $2^{-|k-j|(n\delta_{12}+\alpha_{2\infty})} \lesssim 2^{-|k-j|\vartheta_2}$ for some $\vartheta_2 \in (0, n\delta_{12} + \alpha_{2\infty})$. Thus, we get

$$G_3 \lesssim \|f_1\|_{M\dot{K}^{\alpha_1(\cdot),q_1(\cdot)}_{p_1(\cdot),\lambda_1}(w_1)} \|f_2\|_{M\dot{K}^{\alpha_2(\cdot),q_2(\cdot)}_{p_2(\cdot),\lambda_2}(w_2)}.$$

To estimate G_5, using Hölder's inequality and Lemma 8

$$\begin{aligned}
G_5 &\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty}\left\|\sum_{l=k-1}^{k+1}\sum_{j=k-1}^{k+1} T(f_{1l},f_{2j})\chi_k\right\|_{L^{p(\cdot)}(w)}^{q_\infty}\right)^{\frac{1}{q_\infty}}\\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty}\left(\|f_1\|_{L^{p_1(\cdot)}(w_1)}\|f_2\|_{L^{p_2(\cdot)}(w_2)}\right)^{q_\infty}\right)^{\frac{1}{q_\infty}}\\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda_1}\left(\sum_{k=0}^{L} 2^{k\alpha_{1\infty} q_{1\infty}}\|f_1\|_{L^{p_1(\cdot)}(w_1)}^{q_{1\infty}}\right)^{\frac{1}{q_{1\infty}}}\\
&\quad\times 2^{-L\lambda_2}\left(\sum_{k=0}^{L} 2^{k\alpha_{2\infty} q_{2\infty}}\|f_2\|_{L^{p_2(\cdot)}(w_2)}^{q_{2\infty}}\right)^{\frac{1}{q_{2\infty}}}\\
&\lesssim \|f_1\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}(w_1)}\|f_2\|_{M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_2(\cdot)}(w_2)}.
\end{aligned}$$

To estimate G_6, since $k-1\le l\le k+1$ and $j\ge k+2$, $1/q_\infty = 1/q_{1\infty}+1/q_{2\infty}$, $\lambda=\lambda_1+\lambda_2$, by (11) and Hölder's inequality, we have

$$\begin{aligned}
G_6 &\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty}\left\|\sum_{l=k-1}^{k+1} 2^{-kn}\int_{\mathbb{R}^n}\left|f_{1l}(y_1)\right|dy_1\chi_k\right\|_{L^{p_1(\cdot)}(w_1)}^{q_\infty}\right.\\
&\quad\left.\times\left\|\sum_{j=k+2}^{\infty} 2^{-jn}\int_{\mathbb{R}^n}\left|f_{2j}(y_2)\right|dy_2\chi_k\right\|_{L^{p_2(\cdot)}(w_2)}^{q_\infty}\right)^{\frac{1}{q_\infty}}\\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda_1}\\
&\quad\times\left(\sum_{k=0}^{L} 2^{k\alpha_{1\infty} q_{1\infty}}\left\|\sum_{l=k-1}^{k+1} 2^{-kn}\int_{\mathbb{R}^n}\left|f_{1l}(y_1)\right|dy_1\chi_k\right\|_{L^{p_1(\cdot)}(w_1)}^{q_{1\infty}}\right)^{\frac{1}{q_{1\infty}}}\\
&\quad\times\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda_2}\\
&\quad\times\left(\sum_{k=0}^{L} 2^{k\alpha_{2\infty} q_{2\infty}}\left\|\sum_{j=k+2}^{\infty} 2^{-jn}\int_{\mathbb{R}^n}\left|f_{2j}(y_2)\right|dy_2\chi_k\right\|_{L^{p_2(\cdot)}(w_2)}^{q_{2\infty}}\right)^{\frac{1}{q_{2\infty}}}\\
&:= G_{6,1}\times G_{6,2}.
\end{aligned}$$

By the symmetry of f_1 and f_2, we can know that the estimate $G_{6,1}$ is similar to the estimated $G_{2,2}$ and $G_{6,2} = G_{3,2}$.

To estimate G_9, since $l,j\ge k+2$, $1/q_\infty = 1/q_{1\infty}+1/q_{2\infty}$, $\lambda=\lambda_1+\lambda_2$, by (12) and Hölder's inequality, we have

$$\begin{aligned}
G_9 &\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=0}^{L} 2^{k\alpha_\infty q_\infty}\left\|\sum_{l=k+2}^{\infty} 2^{-ln}\int_{\mathbb{R}^n}\left|f_{1l}(y_1)\right|dy_1\chi_k\right\|_{L^{p_1(\cdot)}(w_1)}^{q_\infty}\right.\\
&\quad\left.\times\left\|\sum_{j=k+2}^{\infty} 2^{-jn}\int_{\mathbb{R}^n}\left|f_{2j}(y_2)\right|dy_2\chi_k\right\|_{L^{p_2(\cdot)}(w_2)}^{q_\infty}\right)^{\frac{1}{q_\infty}}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda_1} \\
&\quad\times \left(\sum_{k=0}^{L} 2^{k\alpha_{1\infty}q_{1\infty}} \left\| \sum_{l=k+2}^{\infty} 2^{-ln} \int_{\mathbb{R}^n} \left|f_{1l}(y_1)\right| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q_{1\infty}} \right)^{\frac{1}{q_{1\infty}}} \\
&\quad\times \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda_2} \\
&\quad\times \left(\sum_{k=0}^{L} 2^{k\alpha_{2\infty}q_{2\infty}} \left\| \sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} \left|f_{2j}(y_2)\right| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q_{2\infty}} \right)^{\frac{1}{q_{2\infty}}} \\
&:= G_{9,1} \times G_{9,2}.
\end{aligned}
$$

Obviously, the estimate $G_{9,i}$ is similar to the estimated $G_{3,2}$ for $i = 1, 2$.

Taking all estimates for G_i together, $i = 1, 2, \dots, 9$, we obtain

$$
G \lesssim \|f_1\|_{M\dot{K}_{p_1(\cdot),\lambda_1}^{\alpha_1(\cdot),q_1(\cdot)}(w_1)} \|f_2\|_{M\dot{K}_{p_2(\cdot),\lambda_2}^{\alpha_2(\cdot),q_2(\cdot)}(w_2)}.
$$

This completes the proof. □

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