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Controllability and constrained controllability for nonlocal Hilfer fractional differential systems with Clarke's subdifferential

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Abstract

Sobolev-type nonlocal fractional differential systems with Clarke's subdifferential are studied. Sufficient conditions for controllability and constrained controllability for Sobolev-type nonlocal fractional differential systems with Clarke's subdifferential are established, where the time fractional derivative is the Hilfer derivative. An example is given to illustrate the obtained results.

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1 Introduction

A Sobolev-type equation appears in several physical problems such as flow of fluids through fissured rocks, thermodynamics and propagation of long waves of small amplitude (see [1–3]). Nonlinear fractional differential equations can be observed in many areas such as population dynamics, heat conduction in materials with memory, seepage flow in porous media, autonomous mobile robots, fluid dynamics, traffic models, electro magnetic, aeronautics, economics (see [4–13]). Controllability means to steer a dynamical system from an arbitrary initial state to the desired final state in a given finite interval of time by using the admissible controls, and controllability results for linear and nonlinear integer order differential systems were studied by several authors (see [14–27]). The constrained controllability is concerned with the existence of an admissible control that steers the state to a given target set from a specified initial state. Few authors studied constrained controllability; for example Son [28] studied constrained approximate controllability for the heat equations and retarded equations, Klamka [29] studied constrained controllability of nonlinear systems, Klamka [30] studied constrained controllability of semilinear systems with delays, Sikora and Klamka [31] studied constrained controllability of fractional linear systems with delays in control. Furthermore, the Clarke subdifferential has been applied in mechanics and engineering, especially in nonsmooth analysis and optimization [32, 33].

However, the controllability and the constrained controllability of nonlocal Hilfer fractional differential equations with the Clarke subdifferential have not yet been considered in the literature, and this fact motivates this work. The purpose of this paper is to study the controllability of Sobolev-type nonlocal Hilfer fractional differential equation system with the Clarke subdifferential in Banach spaces and to study the constrained local controllability of Sobolev-type nonlocal Hilfer fractional differential system with the Clarke subdifferential in Banach spaces.

2 Preliminaries

In order to study the controllability and constrained controllability for Clarke subdifferential Hilfer fractional differential equations with nonlocal condition, we need the following basic definitions and lemmas.

Definition 2.1 (see [34]) The fractional integral operator of order $\mu > 0$ for a function f can be defined as

$$I^{\mu}f(t) = \frac{1}{\Gamma(\mu)}\int_0^t \frac{f(s)}{(t-s)^{1-\mu}}\,ds, \quad t > 0,$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2 (see [35, 36]) The Hilfer fractional derivative of order $0 \le \nu \le 1$ and $0 < \mu < 1$ is defined as

$$D_{0+}^{\nu,\mu}f(t) = I_{0+}^{\nu(1-\mu)}\frac{d}{dt}I_{0+}^{(1-\nu)(1-\mu)}f(t).$$

Next we recall some definitions from multi-valued analysis (see [37])

(i) For a given Banach space X, a multi-valued map $F : X \to 2^X \setminus \{\emptyset\} := P(X)$ is convex (closed) valued, if $F(x)$ is convex (closed) for all $x \in X$.

(ii) F is called upper semi-continuous (u.s.c) on X, if for each $x \in X$, the set $F(x)$ is a non-empty, closed subset of X, and if for each open set V of X containing $F(x)$, there exists an open neighborhood N of x such that $F(N) \subseteq V$.

(iii) F is said to be completely continuous if $F(V)$ is relatively compact, for every bounded subset $V \subseteq X$.

(iv) Let (Ω, Σ) be a measurable space and (X, d) a separable metric space. A multi-valued map $F : J \to P(X)$ is said to be measurable, if for every closed set $C \subseteq X$, we have $F^{-1} = \{t \in J : F(t) \cap C \neq \emptyset\} \in \Sigma$.

Throughout this paper, let X is a Banach spaces with $\|\cdot\|$ and let $C(J, X)$ be the Banach space of all continuous maps from $J = (0, a]$ into X.

Define $Y = \{x : \cdot^{(1-\nu)(1-\mu)}x(\cdot) \in C(J, X)\}$, with norm $\|\cdot\|_Y$ defined by

$$\|\cdot\|_Y = \sup_{t\in J}\left\| t^{(1-\nu)(1-\mu)}x(t)\right\|.$$

Obviously, Y is a Banach space.

Introduce the set $B_r = \{x \in Y : \|x\|_Y \le r\}$, where $r > 0$.

For $x \in X$, we define two families of operators $\{S_{\nu,\mu}(t) : t > 0\}$ and $\{P_\mu(t) : t > 0\}$ by

$$\begin{aligned} &S_{\nu,\mu}(t) = I_{0+}^{\nu(1-\mu)} P_\mu(t), \qquad P_\mu(t) = t^{\mu-1} T_\mu(t), \\ &T_\mu(t) = \int_0^\infty \mu\theta \Psi_\mu(\theta) S\big(t^\mu \theta\big)\, d\theta, \end{aligned} \tag{2.1}$$

where

$$\Psi_\mu(\theta) = \sum_{n=1}^{\infty} \frac{(-\theta)^{n-1}}{(n-1)!\Gamma(1-n\mu)}, \quad 0 < \mu < 1, \theta \in (0,\infty), \tag{2.2}$$

is a function of Wright-type which satisfies

$$\int_0^\infty \theta^\tau \Psi_\mu(\theta)\, d\theta = \frac{\Gamma(1+\tau)}{\Gamma(1+\mu\tau)}, \quad \theta \ge 0.$$

Lemma 2.1 (see [38]) *The operators $S_{\nu,\mu}$ and P_μ have the following properties.*

(i) *$\{P_\mu(t) : t > 0\}$ is continuous in the uniform operator topology.*

(ii) *For any fixed $t > 0$, $S_{\nu,\mu}(t)$ and $P_\mu(t)$ are linear and bounded operators, and*

$$\big\|P_\mu(t)x\big\| \le \frac{Mt^{\mu-1}}{\Gamma(\mu)}\|x\|, \qquad \big\|S_{\nu,\mu}(t)x\big\| \le \frac{Mt^{(\nu-1)(1-\mu)}}{\Gamma(\nu(1-\mu)+\mu)}\|x\|. \tag{2.3}$$

(iii) *$\{P_\mu(t) : t > 0\}$ and $\{S_{\nu,\mu}(t) : t > 0\}$ are strongly continuous.*

(iv) *For every $t > 0$, $\{P_\mu(t)\}$ and $\{S_{\nu,\mu}(t)\}$ are also compact operators if $T(t)$, $t > 0$ is compact.*

The operators $A : D(A) \subset X \to Y$ and $E : D(E) \subset X \to Y$ satisfy the following conditions:

(H1) A and E are closed linear operators.

(H2) $D(E) \subset D(A)$ and E is bijective.

(H3) $E^{-1} : Y \to D(E)$ is continuous.

Here, (H1) and (H2) together with the closed graph theorem imply the boundedness of the linear operator $AE^{-1} : Y \to Y$.

(H4) For each $t \in J$ and for $\lambda \in \rho(-AE^{-1})$, the resolvent of $-AE^{-1}$, the resolvent of $R(\lambda, -AE^{-1})$ is the compact operator.

Lemma 2.2 (see [39]) *Let $T(t)$ be a uniformly continuous semigroup. If the resolvent set $R(\lambda, A)$ of A is compact for every $\lambda \in \rho(A)$, then $T(t)$ is a compact semigroup.*

From the above fact, $-AE^{-1}$ generates a compact semigroup $\{S(t), t > 0\}$ in Y, which means that there exists $M > 1$ such that $\sup_{t\in J} \|S(t)\| \le M$.

Definition 2.3 (see [33, 37]) Let X be a Banach space with the dual space X^* and $Z : X \to R$, be a locally Lipschitz functional on X. The Clarke generalized directional derivative of Z at the point $x \in X$ in the direction $\nu \in X$, denoted by $Z^0(x;\nu)$, is defined by

$$Z^0(x;\nu) = \lim_{\lambda\to 0^+} \sup_{y\to x} \frac{Z(y+\lambda\nu) - Z(y)}{\lambda}.$$

The Clarke generalized gradient of Z at $x \in X$, denoted by $\partial Z(x)$, is a subset of X^* given by

$$\partial Z(x) = x^* \in X^* : Z^0(x;\nu) \geq \langle x^*, \nu \rangle, \quad \forall \nu \in X.$$

(H5) The functional $Z : J \times X \to R$ satisfies the following conditions:
  (i) $Z(\cdot, x) : J \to R$ is measurable for all $x \in X$;
  (ii) $Z(t, \cdot) : X \to R$ is locally Lipschitz continuous for a.e. $t \in J$;
  (iii) there exist a function $\zeta \in L^p(J, R^+)$ $(0 < \frac{1}{p} < \mu < 1)$ and constant $k > 0$ satisfying

$$\left\| \partial Z(t,x) \right\|_X = \sup\left\{ \|z\|_X : z \in \partial Z(t,x) \right\} \leq \zeta(t) + k\|x\|_X, \quad \forall x \in X, \text{a.e. } t \in J.$$

Now we define an operator $N : L^2(J,X) \to 2^{L^2(J,X)}$ as follows:

$$N(x) = \left\{ w \in L^2(J,X) : w(t) \in \partial Z(t,x) \text{ a.e. } t \in J \right\}, \quad \text{for } x \in L^2(J,X).$$

Lemma 2.3 *If* (H5) *holds, then for $x \in L^2(J,X)$ the set $N(x)$ has non-empty, convex and weakly compact values.*

Lemma 2.4 *If* (H5) *holds, then the operator N satisfies: if $x_n \to x$ in $L^2(J,X)$, $w_n \to w$ weakly in $L^2(J,X)$ and $w_n \in N(x_n)$, then we have $w \in N(x)$.*

Theorem 2.1 *Let X be a Banach space and $F : X \to 2^X$ be a compact convex valued, u.s.c. multi-valued maps such that there exists a closed neighborhood V of* 0 *for which $F(V)$ is a relatively compact set. If the set $\Omega = \{x \in X : \lambda x \in F(x)$ for some $\lambda > 1\}$ is bounded, then F has a fixed point.*

3 Controllability results

In this section, we present and prove main results of controllability for a Sobolev-type nonlocal Hilfer fractional differential system with the Clarke subdifferential in Banach spaces in the following form:

$$\begin{cases} D_{0+}^{\nu,\mu}(Ex(t)) + Ax(t) \\ \quad = Bu(t) + f(t,x(t)) + \int_0^t g(t,s,x(s), \int_0^s H(s,\tau,x(\tau))\,d\tau)\,ds \\ \qquad + \partial Z(t,x(t)), \quad t \in J = (0,a], \\ I_{0+}^{(1-\nu)(1-\mu)}x(0) + q(x) = x_0, \end{cases} \tag{3.1}$$

where $D_{0+}^{\nu,\mu}$ is the Hilfer fractional derivative, $0 \leq \nu \leq 1$, $0 < \mu < 1$, A and E are closed, linear and densely defined operators with domain contained in the Banach space X and ranges contained in the Banach space Y. The state $x(\cdot)$ takes values in the Banach space X and the control function $u(\cdot)$ is given in $L^2(J,U)$. The Banach space of admissible control functions with U a Banach space. The symbol B stands for a bounded linear from U into Y. The nonlinear operators $f : J \times X \to Y$, $H : J \times J \times X \to X$, $g : J \times J \times X \times X \to Y$ and $\partial Z(t,\cdot)$ is the Clarke subdifferential of $Z(t,\cdot)$.

To establish the result, we need the following additional hypotheses:

(H6) $f: J \times X \to Y$ is a continuous function and there exist constants $N_1 > 0$ and $N_2 > 0$ such that, for all $t \in J$, $v_1, v_2 \in X$ we have

$$\|f(t,v_1) - f(t,v_2)\| \le N_1 \|v_1 - v_2\|, \qquad N_2 = \|f(t,0)\|.$$

(H7) $g: J \times J \times X \times X \to Y$ is a continuous function and there exist constants $L_1 > 0$ and $L_2 > 0$ such that, for all $t, s \in J$, $v_1, v_2, w_1, w_2 \in X$ we have

$$\|g(t,s,v_1,w_1) - g(t,s,v_2,w_2)\| \le L_1\big[\|v_1 - v_2\| + \|w_1 - w_2\|\big],$$
$$L_2 = \|g(t,s,0,0)\|.$$

(H8) $H: J \times J \times X \to X$ is continuous and there exist constants $L_3 > 0$, $L_4 > 0$, such that for all $t, s \in J$, $v_1, v_2 \in X$ we have

$$\|H(t,s,v_1) - H(t,s,v_2)\| \le L_3 \|v_1 - v_2\|, \qquad L_4 = \|H(t,s,0)\|.$$

(H9) The linear operator W from U into E defined by

$$Wu = \int_0^a E^{-1} P_\mu(a-s) B u(s)\, ds,$$

has an inverse operator W^{-1} which takes values in $L^2(J,U) \setminus \ker W$, where the kernel space of W is defined by $\ker W = \{x \in L^2(J,U) : Wx = 0\}$ and B is a bounded operator.

Definition 3.1 We say $x \in C(J,X)$ is a mild solution of the system (3.1) if it satisfies the integral equation

$$\begin{aligned} x(t) = {} & E^{-1} S_{\nu,\mu}(t) E\big[x_0 - q(x)\big] + \int_0^t E^{-1} P_\mu(t-s) f\big(s, x(s)\big)\, ds + \int_0^t E^{-1} P_\mu(t-s) B u(s)\, ds \\ & + \int_0^t E^{-1} P_\mu(t-s) \left\{ \int_0^s g\big(s, \tau, x(\tau), R(\tau)\big)\, d\tau \right\} ds \\ & + \int_0^t E^{-1} P_\mu(t-s) z(s)\, ds, \quad t \in J, \end{aligned} \tag{3.2}$$

where

$$R(\tau) = \int_0^\tau H\big(\tau, \eta, x(\eta)\big)\, d\eta.$$

The proof of mild solution of Eq. (3.1) is similar to the proof of mild solution of Eq. (1.1) in [38].

Definition 3.2 The system (3.1) is said to be controllable on J, if for every $x_0, x_1 \in X$, there exists a control $u \in L^2(J,U)$ such that the mild solution $x(t)$ of the system (3.1) satisfies $x(a) = x_1$, where x_1 and a are the preassigned terminal state and time, respectively.

Theorem 3.1 *If the hypotheses* (H1)–(H9) *are satisfied, then the system* (3.1) *is controllable on* J *provided that there exists a constant* $r > 0$ *such that*

$$\begin{aligned} M\|E^{-1}\| \left(1 + \frac{Ma^{\mu}\|E^{-1}\|\|B\|\|W^{-1}\|}{\Gamma(\mu+1)}\right) \biggl[\frac{\|E\|(\|x_0\| + \|q\|)}{\Gamma(\nu(1-\mu)+\mu)} \\ + \frac{Ma^{\nu(\mu-1)+1}}{\Gamma(\mu+1)} \left(N_1 r + N_2 + \frac{a}{\mu+1}\left(L_1 \left(r + \frac{a}{\mu+2}(L_3 r + L_4)\right) + L_2 \right) + \|\zeta\| + kr \right) \biggr] \\ + \frac{Ma^{\nu(\mu-1)+1}}{\Gamma(\mu+1)} \|E^{-1}\|\|B\|\|W^{-1}\|\|x_1\| \le r. \end{aligned}$$

Proof For any $x \in C(J,X) \subset L^2(J,X)$ from Lemma 2.3 we consider the map $V_r : C(J,X) \to 2^{C(J,X)}$ as follows:

$$\begin{aligned} V_r(x) = \biggl\{ h \in C(J,X) : h(t) = E^{-1}S_{\nu,\mu}(t)E\bigl[x_0 - q(x)\bigr] + \int_0^t E^{-1}P_\mu(t-s) f\bigl(s, x(s)\bigr)\, ds \\ + \int_0^t E^{-1}P_\mu(t-s)Bu(s)\, ds + \int_0^t E^{-1}P_\mu(t-s) \int_0^s g\bigl(s,\tau,x(\tau),R(\tau)\bigr)\, d\tau\, ds \\ + \int_0^t E^{-1}P_\mu(t-s)z(s)\, ds, z \in N(x) \biggr\}, \quad \text{for } x \in C(J,X). \end{aligned}$$

We will show V_r has a fixed point using Theorem 2.1. Note $V_r(x)$ is convex from convexity of $N(x)$. We divide the proof into five steps.

Step 1: V_r maps bounded sets into bounded sets in $C(J,X)$.

For any $x \in B_r$ and $\Phi \in V_r(x)$, we choose a $z \in N(x)$ with

$$\begin{aligned} \Phi(t) = E^{-1}S_{\nu,\mu}(t)E\bigl[x_0 - q(x)\bigr] + \int_0^t E^{-1}P_\mu(t-s) f\bigl(s, x(s)\bigr)\, ds + \int_0^t E^{-1}P_\mu(t-s)Bu(s)\, ds \\ + \int_0^t E^{-1}P_\mu(t-s) \int_0^s g\bigl(s,\tau,x(\tau),R(\tau)\bigr)\, d\tau\, ds + \int_0^t E^{-1}P_\mu(t-s)z(s)\, ds. \end{aligned}$$

Using the assumption (H9) for any arbitrary function $x(\cdot)$, define the control

$$\begin{aligned} u(t) = W^{-1}\biggl\{ x_1 - E^{-1}S_{\nu,\mu}(a)E\bigl[x_0 - q(x)\bigr] - \int_0^a E^{-1}P_\mu(a-s) f\bigl(s, x(s)\bigr)\, ds \\ - \int_0^a E^{-1}P_\mu(a-s) \int_0^s g\bigl(s,\tau,x(\tau),R(\tau)\bigr)\, d\tau\, ds - \int_0^a E^{-1}P_\mu(a-s)z(s)\, ds \biggr\}(t), \end{aligned}$$

then the operator Φ takes the form

$$\begin{aligned} \Phi(t) = E^{-1}S_{\nu,\mu}(t)E\bigl[x_0 - q(x)\bigr] + \int_0^t E^{-1}P_\mu(t-s) f\bigl(s, x(s)\bigr)\, ds + \int_0^t E^{-1}P_\mu(t-s)BW^{-1} \\ \times \biggl\{ x_1 - E^{-1}S_{\nu,\mu}(a)E\bigl(x_0 - q(x)\bigr) - \int_0^a E^{-1}P_\mu(a-\eta) f\bigl(\eta, x(\eta)\bigr)\, d\eta \\ - \int_0^a E^{-1}P_\mu(a-\eta) \biggl\{ \int_0^\eta g\bigl(\eta,\tau,x(\tau),R(\tau)\bigr)\, d\tau \biggr\}\, d\eta \end{aligned}$$

$$
\begin{aligned}
&- \int_0^a E^{-1} P_\mu(a-\eta) z(\eta)\, d\eta \Big\}(s)\, ds \\
&+ \int_0^t E^{-1} P_\mu(t-s) \int_0^s g\big(s,\tau,x(\tau),R(\tau)\big)\, d\tau\, ds + \int_0^t E^{-1} P_\mu(t-s) z(s)\, ds. \qquad (3.3)
\end{aligned}
$$

From (H7), (H8) and the Beta function, we have

$$
\begin{aligned}
&\int_0^t (t-s)^{\mu-1} \int_0^s \left\| g\left(s,\tau,x(\tau), \int_0^\tau H\big(\tau,\eta,x(\eta)\big)\, d\eta\right) d\tau \right\| ds \\
&\quad \leq \int_0^t (t-s)^{\mu-1} \int_0^s \left(L_1\left(\|x\| + \int_0^\tau \big\| H\big(\tau,\eta,x(\eta)\big)\big\|\, d\eta \right) + L_2 \right) d\tau\, ds \\
&\quad \leq \int_0^t (t-s)^{\mu-1} \int_0^s \left(L_1\left(r + \int_0^\tau (L_3 r + L_4)\, d\eta \right) + L_2 \right) d\tau\, ds \\
&\quad \leq \int_0^t (t-s)^{\mu-1} \int_0^s \big(L_1\big(r + \tau(L_3 r + L_4)\big) + L_2 \big)\, d\tau\, ds \\
&\quad \leq \int_0^t (t-s)^{\mu-1} \left[\left(L_1\left(sr + \frac{s^2}{2}(L_3 r + L_4) \right) + sL_2 \right) \right] ds \\
&\quad \leq L_1\left(r t^{\mu+1} \frac{\Gamma(\mu)\Gamma(2)}{\Gamma(\mu+2)} + \frac{1}{2} t^{\mu+2} \frac{\Gamma(\mu)\Gamma(3)}{\Gamma(\mu+3)} (L_3 r + L_4) \right) + L_2 t^{\mu+1} \frac{\Gamma(\mu)\Gamma(2)}{\Gamma(\mu+2)} \\
&\quad \leq \frac{a^{\mu+1}}{\mu(\mu+1)} \left[L_1\left(r + \frac{a}{\mu+2}(L_3 r + L_4) \right) + L_2 \right].
\end{aligned}
$$

From (H5)–(H9), Lemma 2.1 and Hölder's inequality, we have

$$
\begin{aligned}
\|\Phi\|_Y &= \sup_{t\in J} t^{(1-\nu)(1-\mu)} \big\|\Phi(t)\big\| \\
&\leq \sup_{t\in J} t^{(1-\nu)(1-\mu)} \Big\{ \big\|E^{-1}\big\| \big\|S_{\nu,\mu}(t)\big\| \|E\| \big\|x_0 - q(x)\big\| \\
&\quad + \int_0^t \big\|E^{-1}\big\| \big\|P_\mu(t-s)\big\| \big\|f\big(s,x(s)\big)\big\|\, ds + \int_0^t \big\|E^{-1}\big\| \big\|P_\mu(t-s)\big\| \|B\| \big\|W^{-1}\big\| \\
&\quad \times \Big\| x_1 - E^{-1} S_{\nu,\mu}(a) E\big(x_0 - q(x)\big) - \int_0^a E^{-1} P_\mu(a-\eta) f\big(\eta,x(\eta)\big)\, d\eta \\
&\quad - \int_0^a E^{-1} P_\mu(a-\eta) \Big\{ \int_0^\eta g\big(\eta,\tau,x(\tau),R(\tau)\big)\, d\tau \Big\}\, d\eta \\
&\quad - \int_0^a E^{-1} P_\mu(a-\eta) z(\eta)\, d\eta \Big\| (s)\, ds \\
&\quad + \int_0^t \big\|E^{-1}\big\| \big\|P_\mu(t-s)\big\| \int_0^s \big\| g\big(s,\tau,x(\tau),R(\tau)\big)\, d\tau \big\|\, ds \\
&\quad + \int_0^t \big\|E^{-1}\big\| \big\|P_\mu(t-s)\big\| \big\|z(s)\big\|\, ds \Big\} \\
&\leq \frac{M}{\Gamma(\nu(1-\mu)+\mu)} \big\|E^{-1}\big\| \|E\| \big(\|x_0\| + \big\|q(x)\big\| \big) \\
&\quad + \frac{M a^{\nu(\mu-1)+1} \|E^{-1}\|}{\Gamma(\mu+1)}
\end{aligned}
$$

$$\times \left[N_1 r + N_2 + \frac{a}{\mu+1}\left(L_1\left(r + \frac{a}{\mu+2}(L_3 r + L_4)\right) + L_2\right) + \|\zeta\| + kr\right]$$
$$+ \frac{Ma^{\nu(\mu-1)+1}}{\Gamma(\mu+1)}\left\|E^{-1}\right\|\|B\|\left\|W^{-1}\right\|\|x_1\|$$
$$+ \frac{M^2 a^{\mu}\|E^{-1}\|^2\|B\|\|W^{-1}\|\|E\|}{\Gamma(\mu+1)\Gamma(\nu(1-\mu)+\mu)}\left(\|x_0\| + \left\|q(x)\right\|\right)$$
$$+ \frac{M^2 a^{\nu(\mu-1)+1}\|E^{-1}\|^2\|B\|\|W^{-1}\|a^{\mu}}{\Gamma(\mu+1)^2}$$
$$\times \left[N_1 r + N_2 + \frac{a}{\mu+1}\left(L_1\left(r + \frac{a}{\mu+2}(L_3 r + L_4)\right) + L_2\right) + \|\zeta\| + kr\right]$$
$$= \frac{M}{\Gamma(\nu(1-\mu)+\mu)}\left\|E^{-1}\right\|\|E\|\left(\|x_0\| + \left\|q(x)\right\|\right)\left(1 + \frac{Ma^{\mu}\|E^{-1}\|\|B\|\|W^{-1}\|}{\Gamma(\mu+1)}\right)$$
$$+ \frac{Ma^{\nu(\mu-1)+1}\|E^{-1}\|}{\Gamma(\mu+1)}$$
$$\times \left[N_1 r + N_2 + \frac{a}{\mu+1}\left(L_1\left(r + \frac{a}{\mu+2}(L_3 r + L_4)\right) + L_2\right) + \|\zeta\| + kr\right]$$
$$\times \left(1 + \frac{Ma^{\mu}\|E^{-1}\|\|B\|\|W^{-1}\|}{\Gamma(\mu+1)}\right) + \frac{Ma^{\nu(\mu-1)+1}}{\Gamma(\mu+1)}\left\|E^{-1}\right\|\|B\|\left\|W^{-1}\right\|\|x_1\|$$
$$= M\left\|E^{-1}\right\|\left(1 + \frac{Ma^{\mu}\|E^{-1}\|\|B\|\|W^{-1}\|}{\Gamma(\mu+1)}\right)$$
$$\times \left[\frac{\|E\|(\|x_0\| + \|q\|)}{\Gamma(\nu(1-\mu)+\mu)} + \frac{Ma^{\nu(\mu-1)+1}}{\Gamma(\mu+1)}\right.$$
$$\left.\times \left(N_1 r + N_2 + \frac{a}{\mu+1}\left(L_1\left(r + \frac{a}{\mu+2}(L_3 r + L_4)\right) + L_2\right) + \|\zeta\| + kr\right)\right]$$
$$+ \frac{Ma^{\nu(\mu-1)+1}}{\Gamma(\mu+1)}\left\|E^{-1}\right\|\|B\|\left\|W^{-1}\right\|\|x_1\| \le r.$$

Thus $V_r(B_r)$ is bounded in $C(J, X)$.

Step 2: $\{V_r(x) : x \in B_r\}$ is equicontinuous (for all $r > 0$).

For any $x \in B_r$ and $\Phi \in V_r(x)$ and $z \in N(x)$ and from Lemma 2.1(ii) and Hölder's inequality, we have

$$\left\|\Phi(t) - \Phi(0)\right\|_Y$$
$$= \sup_{t\in J} t^{(1-\nu)(1-\mu)}\left\|\Phi(t) - \Phi(0)\right\|$$
$$\le M\left\|E^{-1}\right\|\left(1 + \frac{Ma^{\mu}\|E^{-1}\|\|B\|\|W^{-1}\|}{\Gamma(\mu+1)}\right)$$
$$\times \left[\frac{\|E\|(\|x_0\| + \|q\|)}{\Gamma(\nu(1-\mu)+\mu)} + \frac{Ma^{\nu(\mu-1)+1}}{\Gamma(\mu+1)}\right.$$
$$\left.\times \left(N_1 r + N_2 + \frac{a}{\mu+1}\left(L_1\left(r + \frac{a}{\mu+2}(L_3 r + L_4)\right) + L_2\right) + \|\zeta\| + kr\right)\right]$$
$$+ \frac{Ma^{\nu(\mu-1)+1}}{\Gamma(\mu+1)}\left\|E^{-1}\right\|\|B\|\left\|W^{-1}\right\|\|x_1\| + \|x_0\| + \|q\|.$$

Thus, for all $\varepsilon > 0$ and for sufficiently small $\delta_1 > 0$, with $0 < t \le \delta_1$, we have $\|\Phi(t) - \Phi(0)\|_Y < \frac{\varepsilon}{2}$. Hence, for all $\varepsilon > 0$, $\forall \tau_1, \tau_2 \in [0, \delta_1]$ and $\forall \Phi \in V_r(B_r)$, we have $\|\Phi(\tau_2) - \Phi(\tau_1)\|_Y < \varepsilon$. For any $x \in B_r$, and $\frac{\delta_1}{2} \le \tau_1 < \tau_2 \le a$, we obtain

$$\begin{aligned}
&\big\|\Phi(\tau_2) - \Phi(\tau_1)\big\| \\
&\quad\le \|E^{-1}\| \biggl\{ \big\|\big(S_{\nu,\mu}(\tau_2) - S_{\nu,\mu}(\tau_1)\big)E\big(x_0 - q(x)\big)\big\| + \biggl\| \int_{\tau_1}^{\tau_2} P_\mu(\tau_2 - s) f\big(s, x(s)\big)\,ds \biggr\| \\
&\qquad + \biggl\| \int_{\tau_1}^{\tau_2} P_\mu(\tau_2 - s) B W^{-1} \biggl\{ x_1 - E^{-1} S_{\nu,\mu}(a) E\big(x_0 - q(x)\big) \\
&\qquad - \int_0^a E^{-1} P_\mu(a - \eta) f\big(\eta, x(\eta)\big)\,d\eta - \int_0^a E^{-1} P_\mu(a - \eta) \int_0^\eta g\big(\eta, \tau, x(\tau), R(\tau)\big)\,d\tau\,d\eta \\
&\qquad - \int_0^a E^{-1} P_\mu(a - \eta) z(\eta)\,d\eta \biggr\}(s)\,ds \biggr\| \\
&\qquad + \biggl\| \int_{\tau_1}^{\tau_2} P_\mu(\tau_2 - s) \int_0^s g\big(s, \tau, x(\tau), R(\tau)\big)\,d\tau\,ds \biggr\| + \biggl\| \int_{\tau_1}^{\tau_2} P_\mu(\tau_2 - s) z(s)\,ds \biggr\| \\
&\qquad + \biggl\| \int_0^{\tau_1} \big[P_\mu(\tau_2 - s) - P_\mu(\tau_1 - s)\big] f\big(s, x(s)\big)\,ds \biggr\| \\
&\qquad + \biggl\| \int_0^{\tau_1} \big[P_\mu(\tau_2 - s) - P_\mu(\tau_1 - s)\big] B W^{-1} \biggl\{ x_1 - E^{-1} S_{\nu,\mu}(a) E\big(x_0 - q(x)\big) \\
&\qquad - \int_0^a E^{-1} P_\mu(a - \eta) f\big(\eta, x(\eta)\big)\,d\eta - \int_0^a E^{-1} P_\mu(a - \eta) \int_0^\eta g\big(\eta, \tau, x(\tau), R(\tau)\big)\,d\tau\,d\eta \\
&\qquad - \int_0^a E^{-1} P_\mu(a - \eta) z(\eta)\,d\eta \biggr\}(s)\,ds \biggr\| \\
&\qquad + \biggl\| \int_0^{\tau_1} \big[P_\mu(\tau_2 - s) - P_\mu(\tau_1 - s)\big] \int_0^s g\big(s, \tau, x(\tau), R(\tau)\big)\,d\tau\,ds \biggr\| \\
&\qquad + \biggl\| \int_0^{\tau_1} \big[P_\mu(\tau_2 - s) - P_\mu(\tau_1 - s)\big] z(s)\,ds \biggr\| \biggr\}.
\end{aligned} \tag{3.4}$$

From the compactness of $T(t), t > 0$, Lemma 2.1(ii), we see that the right hand side of inequality (3.4) tends to zero as $\tau_2 \to \tau_1$. Thus we see that $\|(\Phi)(\tau_2) - (\Phi)(\tau_1)\|_Y$ tends to zero.

For $\forall \varepsilon > 0, \forall \tau_1, \tau_2 \in (0, a], |\tau_1 - \tau_2| < \delta_1, \forall \Phi \in V_r(B_r)$ we see that $\|(\Phi)(\tau_2) - (\Phi)(\tau_1)\|_Y < \varepsilon$ independently of $x \in B_r$. Therefore, we deduce that $\{V_r(x) : x \in B_r\}$ is an equicontinuous family of functions in $C(J, X)$.

Step 3: V_r is completely continuous.

We prove that, for all $t \in J$, $r > 0$, the set $\prod(t) = \{\Phi(t) : \Phi \in V_r(B_r)\}$ is relatively compact in X. Obviously, $\prod(0) = x_0 - q(x)$ is compact, so we only need to consider $t > 0$. Let $0 < t < a$ be fixed. For any $x \in B_r$, $\Phi \in V_r(x)$, we choose $z \in N(x)$ with

$$\begin{aligned}
\Phi(t) &= E^{-1} S_{\nu,\mu}(t) E\big[x_0 - q(x)\big] + \int_0^t E^{-1} P_\mu(t - s) f\big(s, x(s)\big)\,ds + \int_0^t E^{-1} P_\mu(t - s) B W^{-1} \\
&\quad \times \biggl\{ x_1 - E^{-1} S_{\nu,\mu}(a) E\big(x_0 - q(x)\big) - \int_0^a E^{-1} P_\mu(a - \eta) f\big(\eta, x(\eta)\big)\,d\eta \\
&\quad - \int_0^a E^{-1} P_\mu(a - \eta) \biggl\{ \int_0^\eta g\big(\eta, \tau, x(\tau), R(\tau)\big)\,d\tau \biggr\}\,d\eta
\end{aligned}$$

$$- \int_0^a E^{-1}P_\mu(a-\eta)z(\eta)\,d\eta\biggr\}(s)\,ds + \int_0^t E^{-1}P_\mu(t-s)\int_0^s g\bigl(s,\tau,x(\tau),R(\tau)\bigr)\,d\tau\,ds$$
$$+ \int_0^t E^{-1}P_\mu(t-s)z(s)\,ds, \quad t \in J.$$

For each $\epsilon \in (0,t)$, $t \in (0,a]$, $x \in B_r$, and any $\delta > 0$, we define

$$\begin{aligned}
\Phi^{\epsilon,\delta}(t) &= \frac{\mu}{\Gamma(\nu(1-\mu))}\int_0^t\int_\delta^\infty E^{-1}\theta(t-s)^{\nu(1-\mu)-1}s^{\mu-1}\Psi_\mu(\theta)S\bigl(s^\mu\theta\bigr)E\bigl[x_0-q(x)\bigr]\,d\theta\,ds \\
&\quad + \mu\int_0^{t-\epsilon}\int_\delta^\infty E^{-1}\theta(t-s)^{\mu-1}\Psi_\mu(\theta)S\bigl((t-s)^\mu\theta\bigr)f\bigl(s,x(s)\bigr)\,d\theta\,ds \\
&\quad + \mu\int_0^{t-\epsilon}\int_\delta^\infty E^{-1}\theta(t-s)^{\mu-1}\Psi_\mu(\theta)S\bigl((t-s)^\mu\theta\bigr)BW^{-1} \\
&\quad \times\biggl[x_1 - \frac{\mu}{\Gamma(\nu(1-\mu))}\int_0^a\int_0^\infty E^{-1}\theta(a-\eta)^{\nu(1-\mu)-1}\eta^{\mu-1}\Psi_\mu(\theta)S\bigl(\eta^\mu\theta\bigr) \\
&\quad \times E\bigl[x_0-q(x)\bigr]\,d\theta\,d\eta \\
&\quad - \mu\int_0^a\int_0^\infty E^{-1}\theta(a-s)^{\mu-1}\Psi_\mu(\theta)S\bigl((a-s)^\mu\theta\bigr)f\bigl(\eta,x(\eta)\bigr)\,d\theta\,d\eta \\
&\quad - \mu\int_0^a\int_0^\infty E^{-1}\theta(a-s)^{\mu-1}\Psi_\mu(\theta)S\bigl((a-s)^\mu\theta\bigr) \\
&\quad \times\biggl\{\int_0^\eta g\bigl(\eta,\tau,x(\tau),R(\tau)\bigr)\,d\tau\biggr\}\,d\theta\,d\eta \\
&\quad - \mu\int_0^a\int_0^\infty E^{-1}\theta(a-s)^{\mu-1}\Psi_\mu(\theta)S\bigl((a-s)^\mu\theta\bigr)z(\eta)\,d\theta\,d\eta\biggr](s)\,d\theta\,ds \\
&\quad + \mu\int_0^{t-\epsilon}\int_\delta^\infty E^{-1}\theta(t-s)^{\mu-1}\Psi_\mu(\theta)S\bigl((t-s)^\mu\theta\bigr)\int_0^s g\bigl(s,\tau,x(\tau),R(\tau)\bigr)\,d\tau\,d\theta\,ds \\
&\quad + \mu\int_0^{t-\epsilon}\int_\delta^\infty E^{-1}\theta(t-s)^{\mu-1}\Psi_\mu(\theta)S\bigl((t-s)^\mu\theta\bigr)z(s)\,d\theta\,ds \\
&= \frac{\mu S(\epsilon^\mu\delta)}{\Gamma(\nu(1-\mu))}\int_0^t\int_\delta^\infty E^{-1}\theta(t-s)^{\nu(1-\mu)-1}s^{\mu-1}\Psi_\mu(\theta)S\bigl(s^\mu\theta-\epsilon^\mu\delta\bigr) \\
&\quad \times E\bigl[x_0-q(x)\bigr]\,d\theta\,ds \\
&\quad + \mu S\bigl(\epsilon^\mu\delta\bigr)\int_0^{t-\epsilon}\int_\delta^\infty E^{-1}\theta(t-s)^{\mu-1}\Psi_\mu(\theta)S\bigl((t-s)^\mu\theta-\epsilon^\mu\delta\bigr)f\bigl(s,x(s)\bigr)\,d\theta\,ds \\
&\quad + \mu S\bigl(\epsilon^\mu\delta\bigr)\int_0^{t-\epsilon}\int_\delta^\infty E^{-1}\theta(t-s)^{\mu-1}\Psi_\mu(\theta)S\bigl((t-s)^\mu\theta-\epsilon^\mu\delta\bigr)BW^{-1} \\
&\quad \times\biggl[x_1 - \frac{\mu}{\Gamma(\nu(1-\mu))}\int_0^a\int_0^\infty E^{-1}\theta(a-\eta)^{\nu(1-\mu)-1}\eta^{\mu-1}\Psi_\mu(\theta)S\bigl(\eta^\mu\theta\bigr) \\
&\quad \times E\bigl[x_0-q(x)\bigr]\,d\theta\,d\eta \\
&\quad - \mu\int_0^a\int_0^\infty E^{-1}\theta(a-s)^{\mu-1}\Psi_\mu(\theta)S\bigl((a-s)^\mu\theta\bigr)f\bigl(\eta,x(\eta)\bigr)\,d\theta\,d\eta \\
&\quad - \mu\int_0^a\int_0^\infty E^{-1}\theta(a-s)^{\mu-1}\Psi_\mu(\theta)S\bigl((a-s)^\mu\theta\bigr) \\
&\quad \times\biggl\{\int_0^\eta g\bigl(\eta,\tau,x(\tau),R(\tau)\bigr)\,d\tau\biggr\}\,d\theta\,d\eta
\end{aligned}$$

$$
\begin{aligned}
&- \mu \int_0^a \int_0^\infty E^{-1}\theta(a-s)^{\mu-1}\Psi_\mu(\theta)S\bigl((a-s)^\mu\theta\bigr)z(\eta)\,d\theta\,d\eta\biggr](s)\,d\theta\,ds \\
&+ \mu S\bigl(\epsilon^\mu\delta\bigr)\int_0^{t-\epsilon}\int_\delta^\infty E^{-1}\theta(t-s)^{\mu-1}\Psi_\mu(\theta)S\bigl((t-s)^\mu\theta-\epsilon^\mu\delta\bigr) \\
&\times \int_0^s g\bigl(s,\tau,x(\tau),R(\tau)\bigr)\,d\tau\,d\theta\,ds \\
&+ \mu S\bigl(\epsilon^\mu\delta\bigr)\int_0^{t-\epsilon}\int_\delta^\infty E^{-1}\theta(t-s)^{\mu-1}\Psi_\mu(\theta)S\bigl((t-s)^\mu\theta-\epsilon^\mu\delta\bigr)z(s)\,d\theta\,ds.
\end{aligned}
$$

From the compactness of $S(\epsilon^\mu\delta)$, $\epsilon^\mu\delta > 0$ and the bounded of $u(s)$ we see that the set $\prod_{\epsilon,\delta}(t) = \{\Phi^{\epsilon,\delta}(t) : \Phi \in V_r(B_r)\}$ is relatively compact in X for each $\epsilon \in (0,t)$ and $\delta > 0$. Moreover, we have

$$
\begin{aligned}
&\bigl\|\Phi(t)-\Phi^{\epsilon,\delta}(t)\bigr\|_Y \\
&= \sup_{t\in J} t^{(1-\nu)(1-\mu)}\bigl\|\Phi(t)-\Phi^{\epsilon,\delta}(t)\bigr\| \\
&\leq \sup_{t\in J} t^{(1-\nu)(1-\mu)}\biggl\{\biggl\|\frac{\mu}{\Gamma(\nu(1-\mu))}\int_0^t\int_0^\delta E^{-1}\theta(t-s)^{\nu(1-\mu)-1}s^{\mu-1}\Psi_\mu(\theta)S\bigl(s^\mu\theta\bigr) \\
&\quad\times E\bigl[x_0-q(x)\bigr]\,d\theta\,ds\biggr\| \\
&\quad+ \mu\biggl\|\int_0^t\int_0^\delta E^{-1}\theta(t-s)^{\mu-1}\Psi_\mu(\theta)S\bigl((t-s)^\mu\theta\bigr)f\bigl(s,x(s)\bigr)\,d\theta\,ds\biggr\| \\
&\quad+ \mu\biggl\|\int_0^t\int_0^\delta E^{-1}\theta(t-s)^{\mu-1}\Psi_\mu(\theta)S\bigl((t-s)^\mu\theta\bigr)BW^{-1} \\
&\quad\times\biggl[x_1-\frac{\mu}{\Gamma(\nu(1-\mu))}\int_0^a\int_0^\infty E^{-1}\theta(a-\eta)^{\nu(1-\mu)-1}\eta^{\mu-1}\Psi_\mu(\theta)S\bigl(\eta^\mu\theta\bigr) \\
&\quad\times E\bigl[x_0-q(x)\bigr]\,d\theta\,d\eta \\
&\quad- \mu\int_0^a\int_0^\infty E^{-1}\theta(a-s)^{\mu-1}\Psi_\mu(\theta)S\bigl((a-s)^\mu\theta\bigr)f\bigl(\eta,x(\eta)\bigr)\,d\theta\,d\eta \\
&\quad- \mu\int_0^a\int_0^\infty E^{-1}\theta(a-s)^{\mu-1}\Psi_\mu(\theta)S\bigl((a-s)^\mu\theta\bigr)\biggl\{\int_0^\eta g\bigl(\eta,\tau,x(\tau),R(\tau)\bigr)\,d\tau\biggr\}\,d\theta\,d\eta \\
&\quad- \mu\int_0^a\int_0^\infty E^{-1}\theta(a-s)^{\mu-1}\Psi_\mu(\theta)S\bigl((a-s)^\mu\theta\bigr)z(\eta)\,d\theta\,d\eta\biggr](s)\,d\theta\,ds\biggr\| \\
&\quad+ \mu\biggl\|\int_0^t\int_0^\delta E^{-1}\theta(t-s)^{\mu-1}\Psi_\mu(\theta)S\bigl((t-s)^\mu\theta\bigr)\int_0^s g\bigl(s,\tau,x(\tau),R(\tau)\bigr)\,d\tau\,d\theta\,ds\biggr\| \\
&\quad+ \mu\biggl\|\int_0^t\int_0^\delta E^{-1}\theta(t-s)^{\mu-1}\Psi_\mu(\theta)S\bigl((t-s)^\mu\theta\bigr)z(s)\,d\theta\,ds\biggr\| \\
&\quad+ \mu\biggl\|\int_{t-\epsilon}^t\int_\delta^\infty E^{-1}\theta(t-s)^{\mu-1}\Psi_\mu(\theta)S\bigl((t-s)^\mu\theta\bigr)f\bigl(s,x(s)\bigr)\,d\theta\,ds\biggr\| \\
&\quad+ \mu\biggl\|\int_{t-\epsilon}^t\int_\delta^\infty E^{-1}\theta(t-s)^{\mu-1}\Psi_\mu(\theta)S\bigl((t-s)^\mu\theta\bigr)BW^{-1} \\
&\quad\times\biggl[x_1-\frac{\mu}{\Gamma(\nu(1-\mu))}\int_0^a\int_0^\infty E^{-1}\theta(a-\eta)^{\nu(1-\mu)-1}\eta^{\mu-1}\Psi_\mu(\theta)S\bigl(\eta^\mu\theta\bigr)
\end{aligned}
$$

$$
\begin{aligned}
&\times E\big[x_0 - q(x)\big]\,d\theta\,d\eta \\
&- \mu \int_0^a \int_0^\infty E^{-1}\theta(a-s)^{\mu-1}\Psi_\mu(\theta)S\big((a-s)^\mu\theta\big)f\big(\eta, x(\eta)\big)\,d\theta\,d\eta \\
&- \mu \int_0^a \int_0^\infty E^{-1}\theta(a-s)^{\mu-1}\Psi_\mu(\theta)S\big((a-s)^\mu\theta\big)\left\{\int_0^\eta g\big(\eta,\tau,x(\tau),R(\tau)\big)\,d\tau\right\}d\theta\,d\eta \\
&- \mu \int_0^a \int_0^\infty E^{-1}\theta(a-s)^{\mu-1}\Psi_\mu(\theta)S\big((a-s)^\mu\theta\big)z(\eta)\,d\theta\,d\eta\Bigg](s)\,d\theta\,ds\Bigg\| \\
&+ \mu\left\|\int_{t-\epsilon}^t \int_\delta^\infty E^{-1}\theta(t-s)^{\mu-1}\Psi_\mu(\theta)S\big((t-s)^\mu\theta\big)\int_0^s g\big(s,\tau,x(\tau),R(\tau)\big)\,d\tau\,d\theta\,ds\right\| \\
&+ \mu\left\|\int_{t-\epsilon}^t \int_\delta^\infty E^{-1}\theta(t-s)^{\mu-1}\Psi_\mu(\theta)S\big((t-s)^\mu\theta\big)z(s)\,d\theta\,ds\right\|\Bigg\} \\
\le{}& \frac{\mu M\|E^{-1}\|\|E\|[\|x_0\| + \|q(x)\|]}{\Gamma(\nu(1-\mu))}\sup_{t\in J} t^{(1-\nu)(1-\mu)}\int_0^t (t-s)^{\nu(1-\mu)-1}s^{\mu-1}\,ds\int_0^\delta \theta\Psi_\mu(\theta)\,d\theta \\
&+ \mu M\big\|E^{-1}\big\|\sup_{t\in J} t^{(1-\nu)(1-\mu)}\int_0^t (t-s)^{\mu-1}g_k(s)\,ds\int_0^\delta \theta\Psi_\mu(\theta)\,d\theta \\
&+ \mu M\big\|E^{-1}\big\|\|B\|\big\|W^{-1}\big\|\sup_{t\in J} t^{(1-\nu)(1-\mu)}\int_0^t (t-s)^{\mu-1} \\
&\times\Bigg[\|x_1\| + \frac{\mu M\|E^{-1}\|\|E\|[\|x_0\| + \|q(x)\|]}{\Gamma(\nu(1-\mu))}\int_0^a (a-\eta)^{\nu(1-\mu)-1}\eta^{\mu-1}\,d\eta \\
&+ \mu M\big\|E^{-1}\big\|\int_0^a (a-s)^{\mu-1}g_k(\eta)\,d\eta + \mu M\big\|E^{-1}\big\|\int_0^a (a-s)^{\mu-1}h_k(\eta)\,d\eta \\
&+ \mu M\big\|E^{-1}\big\|\int_0^a (a-s)^{\mu-1}z(\eta)\,d\eta\Bigg](s)\,ds\int_0^\delta \theta\Psi_\mu(\theta)\,d\theta \\
&+ \mu M\big\|E^{-1}\big\|\sup_{t\in J} t^{(1-\nu)(1-\mu)}\int_0^t (t-s)^{\mu-1}h_k(s)\,ds\int_0^\delta \theta\Psi_\mu(\theta)\,d\theta \\
&+ \mu M\big\|E^{-1}\big\|\sup_{t\in J} t^{(1-\nu)(1-\mu)}\int_0^t (t-s)^{\mu-1}z(s)\,ds\int_0^\delta \theta\Psi_\mu(\theta)\,d\theta \\
&+ \mu M\big\|E^{-1}\big\|\sup_{t\in J} t^{(1-\nu)(1-\mu)}\int_{t-\epsilon}^t (t-s)^{\mu-1}g_k(s)\,ds\int_\delta^\infty \theta\Psi_\mu(\theta)\,d\theta \\
&+ \mu M\big\|E^{-1}\big\|\|B\|\big\|W^{-1}\big\|\sup_{t\in J} t^{(1-\nu)(1-\mu)}\int_{t-\epsilon}^t (t-s)^{\mu-1} \\
&\times\Bigg[\|x_1\| + \frac{\mu M\|E^{-1}\|\|E\|[\|x_0\| + \|q(x)\|]}{\Gamma(\nu(1-\mu))}\int_0^a (a-\eta)^{\nu(1-\mu)-1}\eta^{\mu-1}\,d\eta \\
&+ \mu M\big\|E^{-1}\big\|\int_0^a (a-s)^{\mu-1}g_k(\eta)\,d\eta + \mu M\big\|E^{-1}\big\|\int_0^a (a-s)^{\mu-1}h_k(\eta)\,d\eta \\
&+ \mu M\big\|E^{-1}\big\|\int_0^a (a-s)^{\mu-1}z(\eta)\,d\eta\Bigg](s)\,ds\int_\delta^\infty \theta\Psi_\mu(\theta)\,d\theta \\
&+ \mu M\big\|E^{-1}\big\|\sup_{t\in J} t^{(1-\nu)(1-\mu)}\int_{t-\epsilon}^t (t-s)^{\mu-1}h_k(s)\,ds\int_\delta^\infty \theta\Psi_\mu(\theta)\,d\theta \\
&+ \mu M\big\|E^{-1}\big\|\sup_{t\in J} t^{(1-\nu)(1-\mu)}\int_{t-\epsilon}^t (t-s)^{\mu-1}z(s)\,ds\int_\delta^\infty \theta\Psi_\mu(\theta)\,d\theta.
\end{aligned}
$$

Now we see that $\|\Phi(t)-\Phi^{\epsilon,\delta}(t)\|_Y \to 0$ as $\epsilon \to 0$, $\delta \to 0$. Therefore, the set $\prod(t)$, $t>0$ is totally bounded, i.e., relatively compact in X. From the above (and step 2) and the Ascoli–Arzela theorem, we see that V_r is completely continuous.

Step 4: V_r has a closed graph.

Let $x_n \to x_*$ as $n\to\infty$ in $C(J,X)$, $\Phi_n \in V_r(x_n)$ and $\Phi_n \to \Phi_*$ as $n\to\infty$ in $C(J,X)$. We prove that $\Phi_* \in V_r(x_*)$. Now $\Phi_n \in V_r(x_n)$, so there exist $z_n \in N(x_n)$, $f_n = f(t,x_n(t))$, $R_n(\tau) = \int_0^\tau H(\tau,\eta,x_n(\eta))\,d\eta$ and $g_n = g(t,s,x_n(s),R_n(\tau))$ in $L^2(J,X)$ with

$$\begin{aligned}\Phi_n(t) &= E^{-1}S_{\nu,\mu}(t)E\big[x_0-q(x)\big] + \int_0^t E^{-1}P_\mu(t-s)f_n\big(s,x(s)\big)\,ds + \int_0^t E^{-1}P_\mu(t-s)Bu(s)\,ds \\ &\quad + \int_0^t E^{-1}P_\mu(t-s)\int_0^s g_n\big(s,\tau,x(\tau),R_n(\tau)\big)\,d\tau\,ds + \int_0^t E^{-1}P_\mu(t-s)z_n(s)\,ds. \qquad (3.5)\end{aligned}$$

From (H5)–(H8), $\{z_n,f_n,g_n\}_{n\ge1} \subseteq L^2(J,X)$ are bounded. Hence we assume that

$$z_n \to z_*, \qquad f_n \to f_*, \qquad g_n \to g_*, \quad \text{weakly in } L^2(J,X). \qquad (3.6)$$

From (3.5), (3.6) and compactness of $P_\mu(t)$, we have

$$\begin{aligned}\Phi_n(t) &\to E^{-1}S_{\nu,\mu}(t)E\big[x_0-q(x)\big] + \int_0^t E^{-1}P_\mu(t-s)f_*\big(s,x(s)\big)\,ds \\ &\quad + \int_0^t E^{-1}P_\mu(t-s)Bu(s)\,ds + \int_0^t E^{-1}P_\mu(t-s)\int_0^s g_*\big(s,\tau,x(\tau),R_*(\tau)\big)\,d\tau\,ds \\ &\quad + \int_0^t E^{-1}P_\mu(t-s)z_*(s)\,ds.\end{aligned}$$

Note that $\Phi_n \to \Phi_*$ in $C(J,X)$ and $z_n \in N(x_n)$. Hence, from Lemma 2.4 we obtain $z_* \in N(x_*)$ and $\Phi_* \in V_r(x_*)$, which implies V_r has a closed graph and V_r is u.s.c.

Step 5: *A priori* estimate.

From steps 1–4, we see that V_r is u.s.c. and is compact convex valued and $V_r(B_r)$ is a relatively compact set (here $r>0$). We now prove that the set $\Omega = \{x \in C(J,X) : \lambda x \in V_r(x), \lambda > 0\}$ is bounded. For all $x \in \omega$, there exist $z \in N(x)$ and f, g in $L^2(J,X)$ with

$$\begin{aligned}x(t) &= \lambda^{-1}E^{-1}S_{\nu,\mu}(t)E\big[x_0-q(x)\big] + \lambda^{-1}\int_0^t E^{-1}P_\mu(t-s)f\big(s,x(s)\big)\,ds \\ &\quad + \lambda^{-1}\int_0^t E^{-1}P_\mu(t-s)Bu(s)\,ds \\ &\quad + \lambda^{-1}\int_0^t E^{-1}P_\mu(t-s)\bigg\{\int_0^s g\big(s,\tau,x(\tau),R(\tau)\big)\,d\tau\bigg\}\,ds \\ &\quad + \lambda^{-1}\int_0^t E^{-1}P_\mu(t-s)z(s)\,ds. \qquad (3.7)\end{aligned}$$

Then from assumptions (H5)–(H8), we derive

$$\begin{aligned}\big\|x(t)\big\|_Y &= \sup_{t\in J} t^{(1-\nu)(1-\mu)}\big\|x(t)\big\| \\ &= \sup t^{(1-\nu)(1-\mu)}\bigg\{\bigg\|\lambda^{-1}E^{-1}S_{\nu,\mu}(t)E\big[x_0-q(x)\big] + \lambda^{-1}\int_0^t E^{-1}P_\mu(t-s)f\big(s,x(s)\big)\,ds\end{aligned}$$

$$
\begin{aligned}
&+ \lambda^{-1} \int_0^t E^{-1} P_\mu(t-s) B u(s)\, ds \\
&+ \lambda^{-1} \int_0^t E^{-1} P_\mu(t-s) \left\{ \int_0^s g\big(s,\tau,x(\tau),R(\tau)\big)\, d\tau \right\} ds \\
&+ \lambda^{-1} \int_0^t E^{-1} P_\mu(t-s) z(s)\, ds \bigg\| \bigg\} \\
\le{}& \sup t^{(1-\nu)(1-\mu)} \bigg\{ \lambda^{-1} \big\|E^{-1}\big\| \big\|S_{\nu,\mu}(t)\big\| \|E\| \big(\|x_0\| + \big\|q(x)\big\| \big) \\
&+ \lambda^{-1} \int_0^t \big\|E^{-1}\big\| \big\|P_\mu(t-s)\big\| \big\|f\big(s,x(s)\big)\big\|\, ds \\
&+ \lambda^{-1} \int_0^t \big\|E^{-1}\big\| \big\|P_\mu(t-s)\big\| \|B\| \big\|u(s)\big\|\, ds \\
&+ \lambda^{-1} \int_0^t \big\|E^{-1}\big\| \big\|P_\mu(t-s)\big\| \int_0^s \big\|g\big(s,\tau,x(\tau),R(\tau)\big)\, d\tau\big\|\, ds \\
&+ \lambda^{-1} \int_0^t \big\|E^{-1}\big\| \big\|P_\mu(t-s)\big\| \big\|z(s)\big\|\, ds \bigg\} \\
\le{}& \frac{M\lambda^{-1}}{\Gamma(\nu(1-\mu)+\mu)} \big\|E^{-1}\big\| \|E\| \big(\|x_0\| + \big\|q(x)\big\| \big) \\
&+ \frac{M a^{\nu(\mu-1)+1} \lambda^{-1} \|E^{-1}\|}{\Gamma(\mu+1)} \\
&\times \left[N_1 r + N_2 + \frac{a}{\mu+1} \left(L_1 \left(r + \frac{a}{\mu+2} (L_3 r + L_4) \right) + L_2 \right) \right. \\
&\left. + \|\zeta\| + kr + \|B\| \|u\| \right].
\end{aligned}
$$

It follows from (3.7) and $\lambda^{-1} < 1$ that $\|x(t)\|_Y \le r$. Hence, $\|x\|_C = \sup_{t\in J} \|x(t)\|_Y \le r$, which implies the set Ω is bounded.

From Theorem 2.1, V_r has a fixed point, i.e., the system (3.1) is controllable and the proof is complete. □

4 Constrained controllability

In this section, we present the constrained local controllability of Sobolev-type nonlocal Hilfer fractional differential system with the Clarke subdifferential in Banach spaces in the following form:

$$
\begin{cases}
D_{0+}^{\nu,\mu}(Ex(t)) + Ax(t) \\
\quad = Bu(t) + f_1(t,x(t),u(t)) + \int_0^t g_1(t,s,x(s),\int_0^s H(s,\tau,x(\tau))\, d\tau, u(s))\, ds \\
\qquad + \partial Z(t,x(t)), \quad t \in J = (0,a], \\
I_{0+}^{(1-\nu)(1-\mu)} x(0) + q(x) = x_0,
\end{cases}
\tag{4.1}
$$

where the nonlinear operators $f_1 : J \times X \times U \to Y$, $H : J \times J \times X \to X$, $g_1 : J \times J \times X \times X \times U \to Y$ and $\partial Z(t,\cdot)$ is the Clarke subdifferential of $Z(t,\cdot)$.

In this section, we need the following hypotheses:

(H10) Let $\|Bu(t)\| \le M_B \|u(t)\|_U$ for all $u(t) \in U$ on J where $M_B > 0$.

(H11) $f_1 : J \times X \times U \to Y$ is a uniformly continuous function in t and there exist constants $L_5 > 0$ such that for all $t \in J$, $v_1, v_2 \in X$, $u_1, u_2 \in U$ we have

$$\|f_1(t, v_1, u_1) - f_1(t, v_2, u_2)\| \le L_5(\|v_1 - v_2\| + \|u_1 - u_2\|_U).$$

(H12) $g_1 : J \times J \times X \times X \times U \to Y$ is a uniformly continuous function in t and there exist constants $L_6 > 0$ such that for all $t, s \in J$, $v_1, v_2 \in X$, $u_1, u_2 \in U$ we have

$$\|g_1(t, s, v_1, u_1) - g_1(t, s, v_2, u_2)\| \le L_6(\|v_1 - v_2\| + \|u_1 - u_2\|_U).$$

The mild solution of the system (4.1) takes the form

$$\begin{aligned} x(t) = {} & E^{-1}S_{\nu,\mu}(t)E[x_0 - q(x)] + \int_0^t E^{-1}P_\mu(t-s)Bu(s)\,ds \\ & + \int_0^t E^{-1}P_\mu(t-s)f_1(s, x(s), u(s))\,ds \\ & + \int_0^t E^{-1}P_\mu(t-s)\left\{\int_0^s g_1(s, \tau, x(\tau), R(\tau), u(\tau))\,d\tau\right\}ds \\ & + \int_0^t E^{-1}P_\mu(t-s)z(s)\,ds, \quad t \in J, \end{aligned} \tag{4.2}$$

where

$$R(\tau) = \int_0^\tau H(\tau, \eta, x(\eta))\,d\eta.$$

The constrained set of controls is considered to be a closed convex cone with empty interior and vertex at origin. Let $U_0 \subset U$ be the constrained set of controls and let the set of admissible controls be

$$U_{\mathrm{ad}} = L^2(J; U_0) \subset V = L^2(J; U).$$

Definition 4.1 The attainable set at time $a > 0$, denoted by $K_T(U_0)$, is defined as

$$K_T(U_0) = \{x \in X : x = x(a, u), u(a) \in U_0 \text{ a.e. in } J\},$$

where $x(t, u)$ is a solution of (4.1).

Let us consider the Sobolev-type linear Hilfer fractional differential system

$$\begin{cases} D_{0+}^{\nu,\mu}(Ey(t)) + Ay(t) = Bv(t), & t \in J = (0, a], \\ I_{0+}^{(1-\nu)(1-\mu)}y(0) = 0. \end{cases} \tag{4.3}$$

The mild solution of (4.3) is

$$y(t, v) = \int_0^t E^{-1}P_\mu(t-s)Bv(s)\,ds, \quad t \in J. \tag{4.4}$$

Let us define the following operators:
$\mathcal{B} : U \to C(J, X)$ by

$$\mathcal{B}u(\cdot) = \int_0^{\cdot} E^{-1} P_\mu(\cdot - s) B u(s)\, ds,$$

$\mathcal{F} : X \times U \to C(J, X)$ by

$$\mathcal{F}(x, u)(\cdot) = \int_0^{\cdot} E^{-1} P_\mu(\cdot - s) f_1\bigl(s, x(s), u(s)\bigr)\, ds,$$

and
$\mathcal{G} : X \times X \times U \to C(J, X)$ by

$$\mathcal{G}(x, u)(\cdot) = \int_0^{\cdot} E^{-1} P_\mu(\cdot - s) \left\{ \int_0^s g_1\bigl(s, \tau, x(\tau), R(\tau), u(\tau)\bigr)\, d\tau \right\} ds.$$

Let us put the following hypotheses:
(H13) The nonlinear function f_1, g_1 satisfies:

$$\begin{aligned} &f_1\bigl(t, x(t), u(t)\bigr)|_{u=0} = 0, \qquad D_x f_1\bigl(t, x(t), u(t)\bigr)|_{u=0} = 0, \\ &D_u f_1\bigl(t, x(t), u(t)\bigr)|_{u=0} = 0, \qquad g_1\bigl(t, x(t), u(t)\bigr)|_{u=0} = 0, \\ &D_x g_1\bigl(t, x(t), u(t)\bigr)|_{u=0} = 0 \quad \text{and} \quad D_u g_1\bigl(t, x(t), u(t)\bigr)|_{u=0} = 0, \end{aligned}$$

where D_x and D_u denote the Frechet derivative on space U.
(H14) $\mathcal{B}$, $\mathcal{F}$ and $\mathcal{G}$ are continuously differentiable in U.
(H15) The linear control system (4.3) is U_0-exactly globally controllable on J.

Definition 4.2 The system (4.1) is said to be U_0-exactly locally controllable on J if the attainable set $K_T(U_0)$ contains a neighborhood of $x(0) \in X$ in the space X.

Definition 4.3 The system (4.1) is said to be U_0-exactly globally controllable on J if $K_T(U_0) = X$.

The main result observes the application of the generalized open mapping theorem, so we recall it in the following lemma.

Lemma 4.1 ([40]) *Let X, Y be Banach spaces and $F : B_r(x_0) \subset X \to Y$ such that*

$$\bigl\| Fx - F\bar{x} - T(x - \bar{x}) \bigr\| \le k\|x - \bar{x}\| \quad \text{on } B_r(x_0) \times B_r(x_0),$$

for some $k > 0$ and $T \in \mathcal{L}(X, Y)$ with $\operatorname{rank}(T) = Y$. Then $B_\rho(Fx_0) \subset FB_r(x_0)$ for some $\rho > 0$ provided that k is sufficiently small.

Theorem 4.1 *Under the assumptions* (H1)–(H4), (H8) *and* (H10)–(H15) *the nonlinear control system* (4.1) *is U_0-exactly locally controllable on J.*

Proof Let us define an operator $\mathcal{H}: \mathcal{U}_{\mathrm{ad}} \to X$ by $\mathcal{H}(u) = x(a,u)$, which maps control to the final state of the trajectory. Then the integral equation (4.2) implies

$$\mathcal{H}(u) = E^{-1}S_{\nu,\mu}(a)E\big[x_0 - q(x)\big] + \mathcal{B}u(a) + \mathcal{F}(x,u)(a) + \mathcal{G}(x,u)(a) + \int_0^a E^{-1}P_\mu(a-s)z(s)\,ds.$$

By hypothesis (H14), $\mathcal{H}$ is differentiable in $\mathcal{U}_{\mathrm{ad}}$. Thus

$$D_u\mathcal{H}(u) = D_u\big((\mathcal{B}u)(a)\big) + D_u\big(\mathcal{F}(x,u)(a)\big) + D_u\big(\mathcal{G}(x,u)(a)\big). \tag{4.5}$$

We have

$$D_u\big((\mathcal{B}u)(a)\big) = \int_0^a E^{-1}P_\mu(a-s)B\,ds,$$
$$D_u\big(\mathcal{F}(x,u)(a)\big) = \int_0^a E^{-1}P_\mu(a-s)D_u f_1\big(s,x(s),u(s)\big)\,ds,$$

and

$$D_u\big(\mathcal{G}(x,u)(a)\big) = \int_0^a E^{-1}P_\mu(a-s)\bigg\{\int_0^s D_u g_1\big(s,\tau,x(\tau),R(\tau),u(\tau)\big)\,d\tau\bigg\}\,ds.$$

Then, by using hypothesis (H13) in (4.5), we get

$$D_u\mathcal{H}(u)|_{u=0}v = \int_0^a E^{-1}P_\mu(a-s)Bv(s)\,ds = y(a,v).$$

By hypothesis (H15), the linear control system (4.3) is $\mathcal{U}_0$-exactly globally controllable, therefore the map $D_u\mathcal{H}(u)|_{u=0}$, mapping $v \mapsto y(a,v)$, is a surjective map with $D_u\mathcal{H}(0)(\mathcal{U}_{\mathrm{ad}}) = X$. Now, let u_1, $u_2 \in \mathcal{U}_{\mathrm{ad}}$ corresponding to $x_1(t) = x(t,u_1)$ and $x_2(t) = x(t,u_2)$, respectively. Then, for all $t \in J$,

$$\begin{aligned}
&\big\|x_1(t) - x_2(t)\big\| \\
&\quad\le \bigg\|\int_0^t E^{-1}P_\mu(t-s)B\big(u_1(s)-u_2(s)\big)\,ds\bigg\| \\
&\qquad+ \bigg\|\int_0^t E^{-1}P_\mu(t-s)\big[f_1\big(s,x(s),u(s)\big) - f_2\big(s,x(s),u(s)\big)\big]\,ds\bigg\| \\
&\qquad+ \bigg\|\int_0^t E^{-1}P_\mu(t-s) \\
&\qquad\times \bigg\{\int_0^s \big[g_1\big(s,\tau,x(\tau),R(\tau),u(\tau)\big) - g_2\big(s,\tau,x(\tau),R(\tau),u(\tau)\big)\big]\,d\tau\,ds\bigg\}\bigg\| \\
&\quad\le \frac{M\|E^{-1}\|}{\Gamma(\mu)}\int_0^t M_B(t-s)^{\mu-1}\big\|\big(u_1(s)-u_2(s)\big)\big\|\,ds \\
&\qquad+ \frac{M\|E^{-1}\|}{\Gamma(\mu)}\int_0^t L_5(t-s)^{\mu-1}\big[\big\|\big(x_1(s)-x_2(s)\big)\big\| + \big\|\big(u_1(s)-u_2(s)\big)\big\|\big]\,ds \\
&\qquad+ \frac{M\|E^{-1}\|}{\Gamma(\mu)}\int_0^t L_6(t-s)^{\mu-1}\int_0^s \big[\big\|\big(x_1(\tau)-x_2(\tau)\big)\big\| + \big\|\big(u_1(\tau)-u_2(\tau)\big)\big\|\big]\,d\tau\,ds
\end{aligned}$$

$$\begin{aligned}
&\le \frac{M\|E^{-1}\|}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}\left[(M_B+L_5)\big\|\big(u_1(s)-u_2(s)\big)\big\| + L_6\int_0^s \big\|\big(u_1(\tau)-u_2(\tau)\big)\big\|\,d\tau\right]ds \\
&\quad + \frac{M\|E^{-1}\|}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}\left[L_5\big\|\big(x_1(s)-x_2(s)\big)\big\| + L_6\int_0^s \big\|\big(x_1(\tau)-x_2(\tau)\big)\big\|\,d\tau\right]ds.
\end{aligned}$$

By Gronwall's inequality,

$$\begin{aligned}
\big\|x_1(t)-x_2(t)\big\| \le{}& \frac{M\|E^{-1}\|}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1} \\
&\times \left[(M_B+L_5)\big\|\big(u_1(s)-u_2(s)\big)\big\| + L_6\int_0^s \big\|\big(u_1(\tau)-u_2(\tau)\big)\big\|\,d\tau\right]ds \\
&\times e^{(\frac{M\|E^{-1}\|}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}(L_5+L_5 s)\,ds)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\big\|\mathcal{H}(u_1)-\mathcal{H}(u_2)\big\| &\le \big\|x_1(a)-x_2(a)\big\| \\
&\le \frac{Ma^\mu\|E^{-1}\|}{\Gamma(\mu+1)}\left(M_B+L_5+\frac{a}{\mu+1}L_6\right)e^{(\frac{Ma^\mu\|E^{-1}\|}{\Gamma(\mu+1)}(L_5+\frac{a}{\mu+1}L_6))}\|u_1-u_2\|_V
\end{aligned}$$

and

$$\begin{aligned}
\big\|D_u\mathcal{H}(0)(u_1-u_2)\big\| &= \left\|\int_0^a E^{-1}P_\mu(a-s)B\big(u_1(s)-u_2(s)\big)\,ds\right\| \\
&\le \frac{Ma^\mu\|E^{-1}\|M_B}{\Gamma(\mu+1)}\|u_1-u_2\|_V.
\end{aligned}$$

Now

$$\begin{aligned}
&\big\|\mathcal{H}(u_1)-\mathcal{H}(u_2)-D_u\mathcal{H}(0)(u_1-u_2)\big\| \\
&\quad\le \big\|\mathcal{H}(u_1)-\mathcal{H}(u_2)\big\| + \big\|D_u\mathcal{H}(0)(u_1-u_2)\big\| \\
&\quad\le \frac{Ma^\mu\|E^{-1}\|}{\Gamma(\mu+1)}\left(M_B+L_5+\frac{a}{\mu+1}L_6\right)e^{(\frac{Ma^\mu\|E^{-1}\|}{\Gamma(\mu+1)}(L_5+\frac{a}{\mu+1}L_6))}\|u_1-u_2\|_V \\
&\quad\quad + \frac{Ma^\mu\|E^{-1}\|M_B}{\Gamma(\mu+1)}\|u_1-u_2\|_V \\
&\quad\le \frac{Ma^\mu\|E^{-1}\|}{\Gamma(\mu+1)}\left[\left(M_B+L_5+\frac{a}{\mu+1}L_6\right)e^{(\frac{Ma^\mu\|E^{-1}\|(L_5+\frac{a}{\mu+1}L_6)}{\Gamma(\mu+1)})} + M_B\right]\|u_1-u_2\|_V.
\end{aligned}$$

Thus, by Lemma 4.1, the operator $\mathcal{H}$ transforms a neighborhood of zero in U_{ad} onto a neighborhood of $\mathcal{H}(0)$ in the Banach space X. This proves the theorem. □

Remark 1 Controllability for a nonlinear fractional system was studied by many authors. However, to the best of our knowledge, there are no results on the controllability of nonlocal Hilfer fractional differential equations with the Clarke subdifferential.

Remark 2 Constrained controllability of for nonlinear fractional system was studied by few authors. However, to the best of our knowledge, there are no results on the constrained local controllability of nonlocal Hilfer fractional differential equations with the Clarke subdifferential.

Remark 3 The study may be improved by finding the sufficient conditions for controllability and constrained local controllability of a Sobolev-type nonlocal Hilfer fractional stochastic differential equation system with the Clarke subdifferential.

5 Applications

Example 5.1 Consider the following Sobolev-type nonlocal Hilfer fractional differential system with the Clarke subdifferential in Banach spaces:

$$\begin{cases} D_{0+}^{\frac{1}{2},\frac{2}{3}}(x(t,y)-x_{yy}(t,y))-x_{yy}(t,y) \\ \quad = Bu(t,y)+\frac{1}{30}\cos(x(t,y))+\int_0^t(\frac{1}{s^2+9}+\frac{1}{9}\int_0^s\frac{1}{(2+\tau)^2}\,d\tau)\,ds \\ \qquad +\partial Z(t,x(t,y)), \quad 0\le y\le\pi, t\in J=(0,1], \\ x(t,0)=x(t,\pi)=0, \quad t\in J, \\ I_{0+}^{\frac{1}{2},\frac{1}{3}}x(0,y)+\sum_{i=1}^m c_i x(t_i,y)=x_0(y), \end{cases} \tag{5.1}$$

where $D_{0+}^{\frac{1}{2},\frac{2}{3}}$ is the Hilfer fractional derivative, $\nu=\frac{1}{2}$, $\mu=\frac{2}{3}$. Let $X=Y=L^2(0,\pi)$ and define the operators $A:D(A)\subset X\to Y$ and $E:D(E)\subset X\to Y$ by $Ax=-x_{yy}$, $Ex=x-x_{yy}$ where $D(A)$, $D(E)$ is given by $\{x\in X: x,x_y$ are absolutely continuous and $x_{yy}\in X$, $x(0)=x(\pi)=0\}$. The functions $x(t)(y)=x(t,y)$, $Bu(t)(y)=Bu(t,y)$, $f(t,x(t))(y)=\frac{1}{30}\cos(x(t,y))$, $g(t,s,x(s),\int_0^s H(s,\tau,x(\tau))\,d\tau)(y)=\frac{1}{s^2+9}+\frac{1}{9}\int_0^s\frac{1}{(2+\tau)^2}\,d\tau$, $H(s,\tau,x(\tau))=\frac{1}{9}\frac{1}{(2+\tau)^2}$, $\partial Z(t,x(t))(y)=\partial Z(t,x(t,y))$ and $q(x)(y)=\sum_{i=1}^m c_i x(t_i,y)$.

It is easy to verify that the function f satisfies hypothesis (H6) with $N_1=N_2=\frac{1}{30}$.

Then A and E can be written as

$$Ax=\sum_{n=1}^{\infty}n^2(x,x_n)x_n, \quad x\in D(A),$$

$$Ex=\sum_{n=1}^{\infty}(1+n^2)(x,x_n)x_n, \quad x\in D(E),$$

where $x_n(y)=\sqrt{\frac{2}{\pi}}\sin ny$, $n=1,2,3,\ldots$, is the orthogonal set of eigenvectors of A and (x,x_n) is the L^2 inner product. Moreover, for $x\in X$, we get

$$E^{-1}x=\sum_{n=1}^{\infty}\frac{1}{1+n^2}(x,x_n)x_n,$$

$$-AE^{-1}x=\sum_{n=1}^{\infty}\frac{-n^2}{1+n^2}(x,x_n)x_n.$$

It is well known that A generates a compact semigroup $\{T(t),t>0\}$ in X and

$$T(t)x=\sum_{n=1}^{\infty}e^{\frac{-n^2}{1+n^2}t}(x,x_n)x_n, \quad x\in X,$$

with

$$\|T(t)\| \le e^{-t} \le 1.$$

Moreover, the two operators $P_{\frac{2}{3}}(t)$ and $S_{\frac{1}{2},\frac{2}{3}}(t)$ satisfy

$$\|P_{\frac{2}{3}}(t)\| \le \frac{Mt^{\frac{-1}{3}}}{\Gamma(\frac{2}{3})}, \qquad \|S_{\frac{1}{2},\frac{2}{3}}(t)\| \le \frac{Mt^{\frac{-1}{6}}}{\Gamma(\frac{5}{6})}.$$

We note that $L_2 = \frac{1}{9}$, $L_4 = \frac{1}{36}$ and choose other constants such that all hypotheses (H1)–(H9) are satisfied and

$$\begin{aligned} &M\|E^{-1}\|\left(1 + \frac{M\|E^{-1}\|\|B\|\|W^{-1}\|}{\Gamma(\frac{5}{3})}\right)\left[\frac{\|E\|(\|x_0\| + \|q\|)}{\Gamma(\frac{5}{6})} + \frac{M}{\Gamma(\frac{5}{3})}\left(\frac{1}{30}r + \frac{1}{30}\right.\right. \\ &\left.\left.+ \frac{3}{5}\left(L_1\left(r + \frac{3}{8}\left(L_3 r + \frac{1}{36}\right)\right) + \frac{1}{9}\right) + \|\zeta\| + kr\right)\right] + \frac{M}{\Gamma(\frac{5}{3})}\|E^{-1}\|\|B\|\|W^{-1}\|\|x_1\| \le r. \end{aligned}$$

Hence, all the hypotheses of Theorem 3.1 are satisfied and the system (5.1) is controllable on $J = (0, 1]$.

Example 5.2 Consider the following Sobolev-type nonlocal Hilfer fractional differential system with the Clarke subdifferential in Banach spaces:

$$\begin{cases} D_{0+}^{\frac{1}{3},\frac{3}{4}}(x(t,\varsigma) - x_{\varsigma\varsigma}(t,\varsigma)) - x_{\varsigma\varsigma}(t,\varsigma) \\ \quad = Bu(t,\varsigma) + G_1(t, x(t,\varsigma), u(t,\varsigma)) \\ \qquad + \int_0^t G_2(t,s,x(s,\varsigma), \int_0^s G_3(s,\tau,x(\tau,\varsigma))\,d\tau, u(s,\varsigma))\,ds \\ \qquad + \partial Z(t, x(t,\varsigma)), \quad 0 \le \varsigma \le \pi, t \in J = (0,1], \\ x(t,0) = x(t,\pi) = 0, \quad t \in J, \\ I_{0+}^{\frac{2}{3},\frac{1}{4}} x(0,\varsigma) + \sum_{i=1}^m c_i x(t_i,\varsigma) = x_0(\varsigma), \end{cases} \tag{5.2}$$

where $D_{0+}^{\frac{1}{3},\frac{3}{4}}$ is the Hilfer fractional derivative, $\nu = \frac{1}{3}$, $\mu = \frac{3}{4}$. Let $X = Y = L^2(0,\pi)$ and define the operators $A : D(A) \subset X \to Y$ and $E : D(E) \subset X \to Y$ by $Ax = -x_{\varsigma\varsigma}$, $Ex = x - x_{\varsigma\varsigma}$ where $D(A), D(E)$ is given by $\{x \in X : x, x_\varsigma$ are absolutely continuous and $x_{\varsigma\varsigma} \in X, x(0) = x(\pi) = 0\}$. The functions $x(t)(\varsigma) = x(t,\varsigma)$, $Bu(t)(\varsigma) = Bu(t,\varsigma)\partial Z(t,x(t))(\varsigma) = \partial Z(t,x(t,\varsigma))$, $q(x)(\varsigma) = \sum_{i=1}^m c_i x(t_i,\varsigma)$, $f_1(t,x(t),u(t))(\varsigma) = G_1(t,x(t,\varsigma),u(t,\varsigma))$, $g_1(t,s,x(s),\int_0^s H(s,\tau,x(\tau))\,d\tau, u(s))(\varsigma) = G_2(t,s,x(s,\varsigma),\int_0^s G_3(s,\tau,x(\tau,\varsigma))\,d\tau, u(t,\varsigma))$.

Then A and E can be written as

$$Ax = \sum_{n=1}^{\infty} n^2(x,x_n)x_n, \quad x \in D(A),$$

$$Ex = \sum_{n=1}^{\infty} \left(1 + n^2\right)(x,x_n)x_n, \quad x \in D(E),$$

where $x_n(\varsigma) = \sqrt{\frac{2}{\pi}} \sin n\varsigma$, $n = 1, 2, 3, \ldots$, is the orthogonal set of eigenvectors of A and (x, x_n) is the L^2 inner product. Moreover, for $x \in X$, we get

$$E^{-1}x = \sum_{n=1}^{\infty} \frac{1}{1+n^2}(x, x_n)x_n,$$
$$-AE^{-1}x = \sum_{n=1}^{\infty} \frac{-n^2}{1+n^2}(x, x_n)x_n.$$

It is well known that A generates a compact semigroup $\{T(t), t > 0\}$ in X and

$$T(t)x = \sum_{n=1}^{\infty} e^{\frac{-n^2}{1+n^2}t}(x, x_n)x_n, \quad x \in X,$$

with

$$\|T(t)\| \le e^{-t} \le 1.$$

Moreover, the two operators $P_{\frac{3}{4}}(t)$ and $S_{\frac{1}{3},\frac{3}{4}}(t)$ satisfy

$$\|P_{\frac{3}{4}}(t)\| \le \frac{Mt^{\frac{-1}{4}}}{\Gamma(\frac{3}{4})}, \qquad \|S_{\frac{1}{3},\frac{3}{4}}(t)\| \le \frac{Mt^{\frac{-1}{6}}}{\Gamma(\frac{5}{6})}.$$

Take $L^2[0,\pi]$ as the control space and $U_0 = \{u(t) \in U : u(t,\varsigma) \ge 0\}$. The space of admissible controls is $U_{\mathrm{ad}} = L^2(J; U_0) \subset V = L^2(J; U)$ and the attainable set is

$$K_T(U_0) = \{x \in X : x = x(t,u), u(t,\varsigma) \in U_0\}.$$

The associated linear control system of the nonlocal Hilfer fractional differential system with the Clarke subdifferential (5.2) takes the form

$$\begin{cases} D_{0+}^{\frac{1}{3},\frac{3}{4}}(y(t,\varsigma) - y_{\varsigma\varsigma}(t,\varsigma)) - y_{\varsigma\varsigma}(t,\varsigma) \\ \quad = Bv(t,\varsigma), \quad 0 \le \varsigma \le \pi, t \in J = (0,1], \\ y(t,0) = y(t,\pi) = 0, \quad t \in J, \\ I_{0+}^{(\frac{2}{3})(\frac{1}{4})} y(0,\varsigma) = 0, \end{cases} \tag{5.3}$$

with mild solution in the form

$$y(t,v) = \int_0^t E^{-1}P_\mu(t-s)Bv(s,\varsigma)\,ds, \quad t \in J. \tag{5.4}$$

We can prove that all the hypotheses (H10)–(H15) are satisfied. Hence, Theorem 4.1 is satisfied and the nonlinear control system (5.2) is U_0-exactly locally controllable on $J = (0,1]$.

6 Conclusion

In this paper, by using fractional calculus and the Sadovskii fixed point theorem, we studied the sufficient conditions for controllability of Sobolev-type nonlocal Hilfer fractional differential systems with Clarke's subdifferential. In addition, we established the constrained local controllability for Sobolev-type nonlocal Hilfer fractional differential systems with Clarke's subdifferential. Also, we provided two examples to illustrate our results. In the future we aim to study the existence of mild solution for a class of noninstantaneous and nonlocal impulsive Hilfer fractional stochastic integrodifferential equations with fractional Brownian motion and Poisson jumps.

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