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Contractions with rational inequalities in the extended *b*-metric space

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Abstract

In this paper, we prove some fixed point theorems for contractions of rational type in the setting of the extended *b*-metric spaces. We present some examples to illustrate the validity of our results. Our results improve and generalize a number of fixed point results in the literature.

MSC: 46T99; 47H10; 54H25

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1 Introduction and preliminaries

Due to the wide application potential, one of the most discussed theorems in nonlinear analysis is the well-known Banach contraction principle. It has been generalized in several directions, such as, by relaxing the conditions of the abstract spaces, by relaxing the contraction types, and so on. Among them, we shall now mention the interesting papers of Dass and Gupta [3] and Jaggi [4] in which the rational type expressions were considered in the contraction condition (see also, e.g. [1–4, 6–9]). For the sake of completeness, we recollect the main results of these papers.

Theorem 1.1 ([4]) *Let $(\mathcal{M}, d)$ be a complete metric space and $T : \mathcal{M} \to \mathcal{M}$ be a continuous mapping. If there exist $\alpha, \beta \in [0, 1)$, with $\alpha + \beta < 1$ such that*

$$d(Tx, Ty) \leq \alpha \cdot \frac{d(x, Tx)d(y, Ty)}{d(x, y)} + \beta \cdot d(x, y), \tag{1}$$

for all distinct $x, y \in \mathcal{M}$, then T possesses a unique fixed point in $\mathcal{M}$.

Theorem 1.2 ([3]) *Let $(\mathcal{M}, d)$ be a complete metric space and $T : \mathcal{M} \to \mathcal{M}$ be a mapping. If there exist $\alpha, \beta \in [0, 1)$, with $\alpha + \beta < 1$ such that*

$$d(Tx, Ty) \leq \alpha \cdot d(y, Ty)\frac{1 + d(x, Tx)}{1 + d(x, y)} + \beta \cdot d(x, y) \tag{2}$$

for all $x, y \in \mathcal{M}$, then T has a unique fixed point $u \in X$. Moreover, the sequence $\{T^n x\}$ converges to the fixed point u for all $x \in \mathcal{M}$.

Throughout this paper, we shall denote the set of positive numbers and the set of real numbers by $\mathbb{N}$ and $\mathbb{R}$, respectively.

In this note, we shall reconsider the results of Dass and Gupta [3] and Jaggi [4] in a newly introduced abstract space, known as extended b-metric space. The notion of the extended b-metric space was introduced by Kamran *et al.* [5] as an extension of b-metric space.

Definition 1.1 ([5]) Let $\mathcal{M}$ be a nonempty set and $\theta : \mathcal{M} \times \mathcal{M} \to [1, \infty)$. A function $d_\theta : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ is called an extended b-metric if for all $x, y, z \in \mathcal{M}$ it satisfies

$(d_\theta 1)$ $d_\theta(x, y) = 0$ if and only if $x = y$;

$(d_\theta 2)$ $d_\theta(x, y) = d_\theta(y, x)$;

$(d_\theta 3)$ $d_\theta(x, z) \leq \theta(x, z)[d_\theta(x, y) + d_\theta(y, z)]$.

The pair $(\mathcal{M}, d_\theta)$ is called an extended b-metric space.

It is clear that an extended b-metric space coincides with the corresponding b-metric space, for $\theta(x, y) = s \geq 1$ where $s \in \mathbb{R}$ and it turns to be standard metric if $s = 1$.

As expected, the basic topological notions were defined analogously.

Definition 1.2 ([5]) Let $(\mathcal{M}, d_\theta)$ be an extended b-metric space.

(i) A sequence x_n in $\mathcal{M}$ is said to converge to $x \in \mathcal{M}$, if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $d_\theta(x_n, x) < \epsilon$, for all $n \geq N$. In this case, we write $\lim_{n \to \infty} x_n = x$.

(ii) A sequence x_n in $\mathcal{M}$ is said to be Cauchy if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $d_\theta(x_m, x_n) < \epsilon$, for all $m, n \geq N$.

Definition 1.3 ([5]) An extended b-metric space $(\mathcal{M}, d_\theta)$ is complete if every Cauchy sequence in $\mathcal{M}$ is convergent.

Notice that an extended b-metric need not be continuous.

Lemma 1.1 ([5]) *Let* $(\mathcal{M}, d_\theta)$ *be an extended b-metric space. If d_θ is continuous, then every convergent sequence has a unique limit.*

For the sake of simplicity, throughout the paper, we assume that $(\mathcal{M}, d_\theta)$ represents a complete extended b-metric space. In addition, we assume that d_θ is a continuous functional unless otherwise stated.

In what follows, we recollect the main results of Kamran *et al.* [5] which is an analog of the Banach contraction principle in the context of extended b-metric space.

Theorem 1.3 ([5]) *Let $T : \mathcal{M} \to \mathcal{M}$ be mapping. If there exists $k \in [0, 1)$ such that*

$$d_\theta(Tx, Ty) \leq k \cdot d_\theta(x, y) \tag{3}$$

for all $x, y \in \mathcal{M}$, where for each $x_0 \in \mathcal{M}$, $\lim_{n,m \to \infty} \theta(x_n, x_m) < \frac{1}{k}$, where $x_n = T^n x_0$, $n \in \mathbb{N}$ Then T has precisely one fixed point u. Moreover, for each $y \in \mathcal{M}$, $T^n y \to u$.

The main purpose of this paper is to extend the results of Kamran *et al.* [5] for the well-known fixed point results, including the interesting theorems of Dass and Gupta [3] and Jaggi [4] in the frame of an extended b-metric space.

2 Main results

Theorem 2.1 *Let $T : \mathcal{M} \to \mathcal{M}$ be a continuous mapping such that, for all distinct $x, y \in \mathcal{M}$,*

$$d_\theta(Tx, Ty) \le k \cdot \mathcal{R}(x, y), \tag{4}$$

where $k \in [0, 1)$ and

$$\mathcal{R}(x, y) = \max\left\{ \frac{d_\theta(x, Tx)[1 + d_\theta(y, Ty)]}{1 + d_\theta(x, y)}, \frac{d_\theta(y, Ty)[1 + d_\theta(x, Tx)]}{1 + d_\theta(x, y)}, \right.$$
$$\left. \frac{d_\theta(y, Ty) d_\theta(x, Tx)}{d_\theta(x, y)}, d_\theta(x, y) \right\}. \tag{5}$$

Suppose also that, for each $x_0 \in \mathcal{M}$, $\lim_{n,m \to \infty} \theta(x_n, x_m) < \frac{1}{k}$, where $x_n = T^n x_0$, $n \in \mathbb{N}$. Then T has a fixed point u. Moreover, for each $x \in \mathcal{M}$, we have $T^n x \to u$.

Proof By presumptions, for given $x_0 \in \mathcal{M}$ we construct the sequence $\{x_n\}$ in $\mathcal{M}$ as $x_n = T^n x_0 = T x_{n-1}$, for $n \in \mathbb{N}$. If $x_{n_0} = x_{n_0+1} = T x_{n_0}$ for some $n_0 \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, then $x^* = x_{n_0}$ forms a fixed point for T which completes the proof. Consequently, throughout the proof, we assume that

$$x_n \ne x_{n+1} \quad \text{for all } n \in \mathbb{N}_0. \tag{6}$$

By taking $x = x_{n-1}$ and $y = x_n$ in the inequality (4), we derive that

$$d_\theta(x_n, x_{n+1}) = d_\theta(T x_{n-1}, T x_n) \le k \cdot \mathcal{R}(x_{n-1}, x_n) \tag{7}$$

with

$$\mathcal{R}(x_{n-1}, x_n) = \max \left\{ \begin{array}{c} \frac{d_\theta(x_{n-1}, Tx_{n-1}) d_\theta(x_n, Tx_n)}{d_\theta(x_{n-1}, x_n)}, \frac{d_\theta(x_{n-1}, Tx_{n-1})[1 + d_\theta(x_n, Tx_n)]}{1 + d_\theta(x_{n-1}, x_n)}, \\ \frac{d_\theta(x_n, Tx_n)[1 + d_\theta(x_{n-1}, Tx_{n-1})]}{1 + d_\theta(x_{n-1}, x_n)}, d_\theta(x_{n-1}, x_n) \end{array} \right\}$$

$$= \max \left\{ \begin{array}{c} \frac{d_\theta(x_{n-1}, x_n) d_\theta(x_n, x_{n+1})}{d_\theta(x_{n-1}, x_n)}, \frac{d_\theta(x_{n-1}, x_n)[1 + d_\theta(x_n, x_{n+1})]}{1 + d_\theta(x_{n-1}, x_n)}, \\ \frac{d_\theta(x_n, x_{n+1})[1 + d_\theta(x_{n-1}, x_n)]}{1 + d_\theta(x_{n-1}, x_n)}, d_\theta(x_{n-1}, x_n) \end{array} \right\}$$

$$= \max \left\{ \frac{d_\theta(x_{n-1}, x_n)[1 + d_\theta(x_n, x_{n+1})]}{1 + d_\theta(x_{n-1}, x_n)}, d_\theta(x_n, x_{n+1}), d_\theta(x_{n-1}, x_n) \right\}.$$

Thus,

$$d_\theta(x_n, x_{n+1}) \le k \max \left\{ d_\theta(x_n, x_{n+1}), d_\theta(x_{n-1}, x_n), \frac{d_\theta(x_{n-1}, x_n)[1 + d_\theta(x_n, x_{n+1})]}{1 + d_\theta(x_{n-1}, x_n)} \right\}. \tag{8}$$

For refining the inequality above, we shall consider the following cases:

Case (i). If $\mathcal{R}(x_{n-1}, x_n) = d_\theta(x_n, x_{n+1})$, then $d_\theta(x_n, x_{n+1}) \le k \cdot d_\theta(x_n, x_{n+1})$, which is a contradiction.

Case (ii). If $\mathcal{R}(x_{n-1}, x_n) = d_\theta(x_{n-1}, x_n)$, then the inequality (4) turns into the inequality below:

$$0 < d_\theta(x_n, x_{n+1}) \le k \cdot d_\theta(x_{n-1}, x_n). \tag{9}$$

Case (iii). Suppose that $\mathcal{R}(x_{n-1}, x_n) = \frac{d_\theta(x_{n-1}, x_n)[1 + d_\theta(x_n, x_{n+1})]}{1 + d_\theta(x_{n-1}, x_n)}$; this yields

$$\max\left\{ d_\theta(x_{n-1}, x_n), d_\theta(x_n, x_{n+1}) \right\} \le \frac{d_\theta(x_{n-1}, x_n)[1 + d_\theta(x_n, x_{n+1})]}{1 + d_\theta(x_{n-1}, x_n)}. \tag{10}$$

We shall illustrate that this case is not possible. For this reason, we consider the following subcases:

Case (iii)$_a$. Suppose that $\max\{d_\theta(x_{n-1}, x_n), d_\theta(x_n, x_{n+1})\} = d_\theta(x_n, x_{n+1})$, that is,

$$d_\theta(x_{n-1}, x_n) < d_\theta(x_n, x_{n+1}). \tag{11}$$

On the other hand, from (10), we have

$$d_\theta(x_n, x_{n+1}) \le \frac{d_\theta(x_{n-1}, x_n)[1 + d_\theta(x_n, x_{n+1})]}{1 + d_\theta(x_{n-1}, x_n)}. \tag{12}$$

By a simple calculation, we derive, from the inequality above, that

$$d_\theta(x_n, x_{n+1}) \le d_\theta(x_{n-1}, x_n),$$

which contradicts the assumption (11).

Case (iii)$_b$. Assume that $\max\{d_\theta(x_{n-1}, x_n), d_\theta(x_n, x_{n+1})\} = d_\theta(x_{n-1}, x_n)$, that is,

$$d_\theta(x_{n-1}, x_n) > d_\theta(x_n, x_{n+1}). \tag{13}$$

Furthermore, from (10), we observe that

$$d_\theta(x_{n-1}, x_n) \le \frac{d_\theta(x_{n-1}, x_n)[1 + d_\theta(x_n, x_{n+1})]}{1 + d_\theta(x_{n-1}, x_n)}. \tag{14}$$

A simple evaluation implies, from the inequality above, that

$$d_\theta(x_{n-1}, x_n) \le d_\theta(x_n, x_{n+1}),$$

which contradicts the assumption (13). Hence, Case (iii) does not occur.

Consequently, we can state that the inequality (9) holds for all these three cases and by applying it recursively we obtain

$$0 < d_\theta(x_n, x_{n+1}) \le k^n d_\theta(x_0, x_1). \tag{15}$$

Since $k \in [0, 1)$, we find that

$$\lim_{n \to \infty} d_\theta(x_n, x_{n+1}) = 0. \tag{16}$$

On the other hand, by $(d_\theta 3)$, together with the triangular inequality, for $p \ge 1$, we derive that

$$d_\theta(x_n, x_{n+p})$$
$$\le \theta(x_n, x_{n+p}) \cdot \left[d_\theta(x_n, x_{n+1}) + d_\theta(x_{n+1}, x_{n+p}) \right]$$

$$\leq \theta(x_n, x_{n+p}) d_\theta(x_n, x_{n+1}) + \theta(x_n, x_{n+p}) d_\theta(x_{n+1}, x_{n+p})$$

$$\leq \theta(x_n, x_{n+p}) k^n d_\theta(x_0, x_1) + \theta(x_n, x_{n+p}) \theta(x_{n+1}, x_{n+p}) \big[d_\theta(x_{n+1}, x_{n+2}) + d_\theta(x_{n+2}, x_{n+p}) \big]$$

$$\leq \theta(x_n, x_{n+p}) \cdot k^n d_\theta(x_0, x_1) + \theta(x_n, x_{n+p}) \theta(x_{n+1}, x_{n+p}) \cdot k^{n+1} d_\theta(x_0, x_1) + \cdots$$

$$+ \theta(x_n, x_{n+p}) \cdot \cdots \cdot \theta(x_{n+p-1}, x_{n+p}) \cdot k^{n+p-1} d_\theta(x_0, x_1)$$

$$= d_\theta(x_0, x_1) \sum_{i=1}^{n+p-1} k^i \prod_{j=1}^{i} \theta(x_{n+j}, x_{n+p}). \tag{17}$$

Notice the inequality above is dominated by $\sum_{i=1}^{n+p-1} k^i \prod_{j=1}^{i} \theta(x_{n+j}, x_{n+p}) \leq \sum_{i=1}^{n+p-1} k^i \times \prod_{j=1}^{i} \theta(x_j, x_{n+p})$.

On the other hand, by employing the ratio test, we conclude that the series $\sum_{i=1}^{\infty} k^i \times \prod_{j=1}^{i} \theta(x_j, x_{n+p})$ converges to some $S \in (0, \infty)$. Indeed, $\lim_{i \to \infty} \frac{a_{i+1}}{a_i} = \lim_{i \to \infty} k\theta(x_i, x_{i+p}) < 1$ and hence we get the desired result. Thus, we have

$$S = \sum_{i=1}^{\infty} k^i \prod_{j=1}^{i} \theta(x_j, x_{n+p}) \quad \text{with the partial sum } S_n = \sum_{i=1}^{n} k^i \prod_{j=1}^{i} \theta(x_j, x_{n+p}).$$

Consequently, we observe, for $n \leq 1, p \leq 1$, that

$$d_\theta(x_n, x_{n+p}) \leq k^n d_\theta(x_0, x_1)[S_{n+p-1} - S_{n-1}]. \tag{18}$$

Letting $n \to \infty$ in (18), we conclude that the constructed sequence $\{x_n\}$ is Cauchy in the extended b-metric space $(\mathcal{M}, d_\theta)$. Regarding the assumption of the completeness, we conclude that there exists $u \in \mathcal{M}$ such that $x_n \to u$ as $n \to \infty$. Due to the continuity of T, we shall show that the limit point u is a fixed point of T. Indeed, we have

$$Tu = T\Big(\lim_{n \to \infty} x_n \Big) = \lim_{n \to \infty} T(x_n) = \lim_{n \to \infty} x_{n+1} = u. \qquad \square$$

Corollary 2.1 *A continuous mapping $T : \mathcal{M} \to \mathcal{M}$ has a fixed point provided that, for all distinct $x, y \in \mathcal{M}$,*

$$d_\theta(Tx, Ty) \leq a_1 \frac{d_\theta(x, Tx)[1 + d_\theta(y, Ty)]}{1 + d_\theta(x, y)} + a_2 \frac{d_\theta(y, Ty)[1 + d_\theta(x, Tx)]}{1 + d_\theta(x, y)}$$

$$+ a_3 \frac{d_\theta(x, Tx) d_\theta(y, Ty)}{d_\theta(x, y)} + a_4 d_\theta(x, y), \tag{19}$$

where $a_i \in [0, 1)$, $i \in \{1, 2, 3, 4\}$ with $\sum_{i=1}^{4} a_i < 1$ and for each $x_0 \in \mathcal{M}$, $\lim_{n,m \to \infty} \theta(x_n, x_m) < \frac{1}{\sum_{i=1}^{4} a_i}$, where $x_n = T^n x_0$, $n \in \mathbb{N}$.

Proof The proof follows from Theorem 2.1 by letting $k = \sum_{i=1}^{4} a_i$. Indeed, we have

$$d_\theta(Tx, Ty) \leq a_1 \frac{d_\theta(x, Tx)[1 + d_\theta(y, Ty)]}{1 + d_\theta(x, y)}$$

$$+ a_2 \frac{d_\theta(y, Ty)[1 + d_\theta(x, Tx)]}{1 + d_\theta(x, y)}$$

$$+ a_3 \frac{d_\theta(x, Tx) d_\theta(y, Ty)}{d_\theta(x, y)} + a_4 d_\theta(x, y)$$

$$\leq (a_1 + a_2 + a_3 + a_4) \cdot \mathcal{R}(x, y). \tag{20}$$

Regarding the analogy, we skip the details.   □

Corollary 2.2 *Let* $T : \mathcal{M} \to \mathcal{M}$ *be a continuous mapping. Suppose that, for all distinct* $x, y \in \mathcal{M}$, *we have the inequality*

$$d_\theta(Tx, Ty) \leq \alpha \cdot \frac{d_\theta(x, Tx) d_\theta(y, Ty)}{d_\theta(x, y)} + \beta d_\theta(x, y), \tag{21}$$

where $\alpha, \beta \in [0, 1)$, $\alpha + \beta < 1$. *Suppose also that, for each* $x_0 \in \mathcal{M}$, $\lim_{n,m \to \infty} \theta(x_n, x_m) < \frac{1}{\alpha + \beta}$, *where* $x_n = T^n x_0$, $n \in \mathbb{N}$. *Then* T *has a unique fixed point* u. *Moreover, for each* $x \in \mathcal{M}$, *we have* $T^n x \to u$.

Proof The proof follows from Corollary 2.1 by letting $k = \alpha + \beta$, where $a_1 = a_2 = 0$ and $a_3 = \alpha$, $a_4 = \beta$.   □

Example 2.1 Let $(\mathcal{M}, d_\theta)$ be a complete extended b-metric space, where $\mathcal{M} = [0, \infty)$ and $d_\theta : \mathcal{M} \times \mathcal{M} \to [0, \infty)$, $d_\theta(x, y) = (x - y)^2$ and $\theta : \mathcal{M} \times \mathcal{M} \to [1, \infty)$ is defined as $\theta(x, y) = x + y + 2$. Let $T : \mathcal{M} \to \mathcal{M}$ be defined by $Tx = \frac{x}{3}$. Obviously,

$$d_\theta(Tx, Ty) = \frac{(x - y)^2}{9}, \qquad d_\theta(x, Tx) = \frac{4x^2}{9}, \qquad d_\theta(y, Ty) = \frac{4y^2}{9}$$

and, choosing $\alpha = \frac{2}{9}$ and $\beta = \frac{1}{9}$, we have

$$\lim_{n \to \infty} \theta(x_n, x_{n+p}) = \lim_{n \to \infty} \theta\left(T^n x, T^{n+p} x\right)$$

$$= \lim_{n \to \infty} \theta\left(\frac{x}{3^n}, \frac{x}{3^{n+p}}\right)$$

$$= \lim_{n \to \infty} \left(\frac{x}{3^n} + \frac{x}{3^{n+p}} + 2\right) < 3 = \frac{1}{\alpha + \beta}.$$

By routine calculation, we obtain

$$d_\theta(Tx, Ty) = \frac{(x - y)^2}{9} \leq \frac{1}{3}(x - y)^2 \leq \frac{2}{9} \cdot \frac{\frac{4x^2}{9} \cdot \frac{4y^2}{9}}{(x - y)^2} + \frac{1}{9}(x - y)^2$$

$$= \frac{2}{9} \cdot \frac{d_\theta(x, Tx) d_\theta(y, Ty)}{d_\theta(x, y)} + \frac{1}{9} d_\theta(x, y).$$

Therefore, all conditions of Corollary 2.2 are satisfied. Thus, T has a fixed point.

Corollary 2.3 *Let* $T : \mathcal{M} \to \mathcal{M}$ *be a mapping such that, for all* $x, y \in \mathcal{M}$ *we have*

$$d_\theta(Tx, Ty) \leq \alpha \cdot \frac{d_\theta(y, Ty)[1 + d_\theta(x, Tx)]}{1 + d_\theta(x, y)} + \beta \cdot d_\theta(x, y), \tag{22}$$

where $\alpha, \beta \in [0, 1)$, $\alpha + \beta < 1$. *Suppose also that, for each $x_0 \in \mathcal{M}$, $\lim_{n,m \to \infty} \theta(x_n, x_m) < \frac{1}{\alpha + \beta}$, where $x_n = T^n x_0$, $n \in \mathbb{N}$. Then T has a unique fixed point u. Moreover, for each $x \in \mathcal{M}$, we have $T^n x \to u$.*

Proof Letting $k = \alpha + \beta$, where $a_1 = a_3 = 0$ and $a_2 = \alpha$, $a_4 = \beta$ in Corollary 2.1, and following the steps of the proof of Theorem 2.1, we know that there exists $u \in \mathcal{M}$ such that $T^n x \to u$. We must prove that this point is the unique fixed point of T. Indeed,

$$
\begin{aligned}
d_\theta(Tu, u) &\leq \theta(Tu, u) \cdot \left[d_\theta(Tu, Tx_n) + d_\theta(Tx_n, u) \right] \\
&\leq \theta(Tu, u) \left[\alpha \cdot \frac{d_\theta(x_n, x_{n+1})(1 + d_\theta(u, Tu))}{1 + d_\theta(u, x_n)} + \beta \cdot d_\theta(u, x_n) + d_\theta(x_{n+1}, u) \right]. \quad (23)
\end{aligned}
$$

Letting $n \to \infty$ in the above inequality we get $d_\theta(Tu, u) = 0$. Hence $Tu = u$.

In order to show the uniqueness, suppose that there exists $v \in \mathcal{M}$ such that $Tu = u \neq v = Tv$. Then

$$
\begin{aligned}
d_\theta(u, v) &= d_\theta(Tu, Tv) \\
&\leq \alpha \cdot \frac{d_\theta(v, Tv)(1 + d_\theta(u, Tu))}{1 + d_\theta(u, v)} + \beta \cdot d_\theta(u, v) \\
&= \beta \cdot d_\theta(u, v) < d_\theta(u, v),
\end{aligned}
$$

which is a contradiction. Thus, we have completed the proof. □

Example 2.2 Let $\mathcal{M} = \{1, 2, 3, 4, \ldots\}$ and define $d_\theta : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ as

$$
d_\theta(x, y) = (x - y)^4,
$$

where $\theta : \mathcal{M} \times \mathcal{M} \to [1, \infty)$ is a function defined by

$$
\theta(x, y) = \begin{cases} |x - y|^3, & \text{if } x \neq y, \\ 1, & \text{if } x = y. \end{cases}
$$

Then $(\mathcal{M}, d_\theta)$ forms an extended b-metric space (see Example 3.1 in [10]).

Let $T : \mathcal{M} \to \mathcal{M}$ be defined by

$$
Tx = \begin{cases} 3, & \text{if } x = 1, \\ 4, & \text{otherwise.} \end{cases}
$$

Let also $\alpha = \frac{1}{16}$ and $\beta = \frac{1}{8}$. Of course, since $d_\theta(x, 1) = (x - 1)^4 \geq 2^4$ and $d_\theta(Tx, T1) = 1$ for any $x \in \{3, 4, 5, \ldots\}$, the inequality (22) becomes

$$
d_\theta(Tx, T1) = 1 < \frac{1}{8} 2^4 \leq \frac{1}{8}(x - 1)^4 \leq \alpha \cdot \frac{d_\theta(1, T1)[1 + d_\theta(x, Tx)]}{1 + d_\theta(x, 1)} + \beta \cdot d_\theta(x, 1).
$$

For the case $x = 2$ and $y = 1$, we have $d_\theta(2, 1) = 1$ and $\frac{d_\theta(1,T1)[1+d_\theta(2,T2)]}{1+d_\theta(2,1)} = \frac{2^4(1+2^4)}{2}$. Obviously, we have

$$d_\theta(T2, T1) = 1 \le \frac{1}{16} \cdot \frac{2^4(1+2^4)}{2} + \frac{1}{8} = \alpha \cdot \frac{d_\theta(1, T1)[1 + d_\theta(2, T2)]}{1 + d_\theta(2, 1)} + \beta \cdot d_\theta(2, 1).$$

For all other cases, $d_\theta(Tx, Ty) = 0$ and the existence of a fixed point is ensured by Corollary 2.3.

Corollary 2.4 *Theorem* 1.3 *is concluded from Corollary* 2.2 *and hence Theorem* 2.1.

Proof It is sufficient to take $\alpha = 0$. $\qquad\square$

2.1 Removing the necessity of the continuity of the functional d_θ

In the following theorem, we relax the condition by removing the continuity of the functional d_θ in the following setting.

Theorem 2.2 *Let* $T : \mathcal{M} \to \mathcal{M}$ *be a mapping that satisfies the inequality*

$$d_\theta(Tx, Ty) \le k \cdot \max\left\{ \frac{d_\theta(x, Tx)d_\theta(x, Ty) + d_\theta(y, Ty)d_\theta(y, Tx)}{d_\theta(x, Ty) + d_\theta(y, Tx)}, d_\theta(x, y) \right\} \tag{24}$$

for all $x, y \in \mathcal{M}$, *where* $k \in [0, 1)$, *be such that, for each* $x_0 \in \mathcal{M}$, $\lim_{n,m\to\infty} \theta(x_n, x_m) < \frac{1}{k}$, *where* $x_n = T^n x_0$, $n \in \mathbb{N}$. *Then* T *has a unique fixed point* u. *Moreover, for each* $y \in X$, $T^n y \to u$.

Proof By taking $x = x_{n-1}$ and $y = x_n$ in the inequality (24), we get

$$\begin{aligned}
0 < d_\theta(x_n, x_{n+1}) &= d_\theta(Tx_{n-1}, Tx_n) \\
&\le k \cdot \max\left\{ \frac{d_\theta(x_{n-1}, x_n)d_\theta(x_{n-1}, x_{n+1}) + d_\theta(x_n, x_{n+1})d_\theta(x_n, x_n)}{d_\theta(x_{n-1}, x_{n+1}) + d_\theta(x_n, x_n)}, d_\theta(x_{n-1}, x_n) \right\} \\
&= k \cdot \max\left\{ \frac{d_\theta(x_{n-1}, x_n)d_\theta(x_{n-1}, x_{n+1})}{d_\theta(x_{n-1}, x_{n+1})}, d_\theta(x_{n-1}, x_n) \right\} \\
&= k \cdot d_\theta(x_{n-1}, x_n). \tag{25}
\end{aligned}$$

Recursively, we derive that

$$d_\theta(x_n, x_{n+1}) \le k^n d_\theta(x_0, x_1), \quad \forall n \in \mathbb{N}, \tag{26}$$

and regarding that $0 \le k < 1$, we derive that

$$\lim_{n\to\infty} d_\theta(x_n, x_{n+1}) = 0. \tag{27}$$

On the other hand, by following the same lines in the previous theorem, we conclude that $\{x_n\}$ is a Cauchy sequence. Since $\mathcal{M}$ is complete, there exists $u \in \mathcal{M}$ such that the sequence $\{x_n = T^n x_0\}$ converges to u, that is,

$$\lim_{n\to\infty} d_\theta(x_n, u) = 0. \tag{28}$$

As a next step, we shall prove that u is a fixed point of T. By using (24) and the triangle inequality, we have

$$
\begin{aligned}
d_\theta(u, Tu) &\le \theta(u, Tu) \cdot \left[d_\theta(u, Tx_n) + d_\theta(Tx_n, Tu) \right] \\
&\le \theta(u, Tu) \cdot \Bigg[d_\theta(u, x_{n+1}) \\
&\quad + k \cdot \max\left\{ \frac{d_\theta(u, Tu) d_\theta(u, x_{n+1}) + d_\theta(x_n, x_{n+1}) d_\theta(x_n, Tu)}{d_\theta(u, x_{n+1}) + d_\theta(x_n, Tu)}, d_\theta(u, x_n) \right\} \Bigg].
\end{aligned}
\tag{29}
$$

Letting $n \to \infty$, and taking (27) and (28) into account, we get

$$
d_\theta(Tu, u) \le 0.
$$

Accordingly, we have $d_\theta(Tu, u) = 0$, that is, $Tu = u$.

Lastly, we shall indicate that this fixed point is unique. Suppose, on the contrary, that it is not unique. Thus, there exists another fixed point v of T that is distinct from u. Now, by using (24) and $k < 1$,

$$
\begin{aligned}
0 < d_\theta(u, v) &= d_\theta(Tu, Tv) \\
&\le k \cdot \max\left\{ \frac{d_\theta(u, u) d_\theta(u, v) + d_\theta(v, v) d_\theta(v, u)}{d_\theta(u, v) + d_\theta(u, v)}, d_\theta(u, v) \right\} \\
&= k \cdot d_\theta(u, v) < d_\theta(u, v),
\end{aligned}
$$

which shows that $u = v$. Therefore, T has a unique fixed point. $\qquad\square$

Example 2.3 Let $\mathcal{M} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}\}$ and the functions $\theta : \mathcal{M} \times \mathcal{M} \to [1, \infty)$, $d_\theta : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ be defined as $\theta(x, y) = x + y + 1$ and

$$
\begin{aligned}
&d_\theta\left(\frac{1}{2}, \frac{1}{4}\right) = d_\theta\left(\frac{1}{4}, \frac{1}{2}\right) = 1, \qquad d_\theta\left(\frac{1}{2}, \frac{1}{8}\right) = d_\theta\left(\frac{1}{8}, \frac{1}{2}\right) = 2, \\
&d_\theta\left(\frac{1}{8}, \frac{1}{4}\right) = d_\theta\left(\frac{1}{4}, \frac{1}{8}\right) = 3, \\
&d_\theta\left(\frac{1}{2}, \frac{1}{2}\right) = d_\theta\left(\frac{1}{4}, \frac{1}{4}\right) = d_\theta\left(\frac{1}{8}, \frac{1}{8}\right) = 0.
\end{aligned}
$$

Let $k = \frac{7}{12}$ and $T : \mathcal{M} \to \mathcal{M}$ be defined by

$$
T\frac{1}{2} = T\frac{1}{4} = \frac{1}{4} \quad \text{and} \quad T\frac{1}{8} = \frac{1}{2}.
$$

We have $T^n x \to 0$ for any $x \in \mathcal{M}$ and $\lim_{n,m\to\infty} \theta(T^n x, T^m x) = \frac{1}{4} + \frac{1}{4} + 1 < \frac{3}{2} < \frac{12}{7} = \frac{1}{k}$.
But

$$
\begin{aligned}
&\theta\left(\frac{1}{2}, \frac{1}{4}\right) = \frac{1}{2} + \frac{1}{4} + 1 = \frac{7}{4}, \\
&\theta\left(\frac{1}{2}, \frac{1}{8}\right) = \frac{1}{2} + \frac{1}{8} + 1 = \frac{13}{8},
\end{aligned}
$$

$$\theta\left(\frac{1}{8},\frac{1}{4}\right) = \frac{1}{8} + \frac{1}{4} + 1 = \frac{11}{8}$$

and

$$1 = d_\theta\left(\frac{1}{2},\frac{1}{4}\right) \le \theta\left(\frac{1}{2},\frac{1}{4}\right)\left(d_\theta\left(\frac{1}{2},\frac{1}{8}\right) + d_\theta\left(\frac{1}{8},\frac{1}{4}\right)\right) = \frac{7}{4}(2+3),$$

$$2 = d_\theta\left(\frac{1}{2},\frac{1}{8}\right) \le \theta\left(\frac{1}{2},\frac{1}{8}\right)\left(d_\theta\left(\frac{1}{2},\frac{1}{4}\right) + d_\theta\left(\frac{1}{8},\frac{1}{4}\right)\right) = \frac{13}{8}(1+3) = \frac{13}{2},$$

$$3 = d_\theta\left(\frac{1}{4},\frac{1}{8}\right) \le \theta\left(\frac{1}{4},\frac{1}{8}\right)\left(d_\theta\left(\frac{1}{4},\frac{1}{2}\right) + d_\theta\left(\frac{1}{2},\frac{1}{8}\right)\right) = \frac{11}{8}(1+2) = \frac{33}{8},$$

which proves that d_θ is an extended b-metric on $\mathcal{M}$.

We will consider the following cases:

(i) For $x = \frac{1}{2}$, $y = \frac{1}{4}$, we have $d_\theta(T\frac{1}{2}, T\frac{1}{4}) = d_\theta(\frac{1}{4},\frac{1}{4}) = 0$, so the condition from Theorem 2.2 is satisfied.

(ii) For $x = \frac{1}{2}$, $y = \frac{1}{8}$

$$1 = d_\theta\left(T\frac{1}{2}, T\frac{1}{8}\right)$$
$$\le k \max\left\{\frac{d_\theta(\frac{1}{2},T\frac{1}{2})d_\theta(\frac{1}{2},T\frac{1}{8}) + d_\theta(\frac{1}{8},T\frac{1}{8})d_\theta(\frac{1}{8},T\frac{1}{2})}{d_\theta(\frac{1}{2},T\frac{1}{8}) + d_\theta(\frac{1}{8},T\frac{1}{2})}, d_\theta\left(\frac{1}{2},\frac{1}{8}\right)\right\}$$
$$= \frac{7}{12}\max\left\{\frac{0+2\cdot3}{1+2},2\right\} = \frac{7}{6}.$$

(iii) For $x = \frac{1}{4}$, $y = \frac{1}{8}$

$$1 = d_\theta\left(T\frac{1}{4}, T\frac{1}{8}\right)$$
$$\le k \max\left\{\frac{d_\theta(\frac{1}{4},T\frac{1}{4})d_\theta(\frac{1}{4},T\frac{1}{8}) + d_\theta(\frac{1}{8},T\frac{1}{8})d_\theta(\frac{1}{8},T\frac{1}{4})}{d_\theta(\frac{1}{4},T\frac{1}{8}) + d_\theta(\frac{1}{8},T\frac{1}{4})}, d_\theta\left(\frac{1}{4},\frac{1}{8}\right)\right\}$$
$$= \frac{7}{12}\max\left\{\frac{0+2\cdot3}{1+3},3\right\} = \frac{7}{4}.$$

Therefore all conditions of Theorem 2.2 are satisfied. Hence T has a unique fixed point, $x = \frac{1}{4}$.

Corollary 2.5 *Let $T : \mathcal{M} \to \mathcal{M}$ be a mapping that satisfies*

$$d_\theta(Tx, Ty) \le \alpha \cdot \frac{d_\theta(x, Tx)d_\theta(x, Ty) + d_\theta(y, Ty)d_\theta(y, Tx)}{d_\theta(x, Ty) + d_\theta(y, Tx)} + \beta \cdot d_\theta(x, y) \tag{30}$$

for all $x, y \in \mathcal{M}$, where $\alpha, \beta \in [0, 1)$, $\alpha + \beta < 1$ are such that, for each $x_0 \in \mathcal{M}$, $\lim_{n,m\to\infty} \theta(x_n, x_m) < \frac{1}{\alpha+\beta}$, where $x_n = T^n x_0$, $n \in \mathbb{N}$. Then T has a unique fixed point u. Moreover, for each $x \in \mathcal{M}$, $T^n x \to u$.

3 Conclusion

It is clear that the corresponding results in the setting of both b-metric space and standard metric space can be included in our results, by letting $\theta(x, y) = s \ge 1$ and $\theta(x, y) = 1$,

respectively, in the related places. In particular, the main results of Dass and Gupta [3], Jaggi [4] and also the well-known Banach contraction mapping principle are derived from our results.

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