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On decay and blow-up of solutions for a system of viscoelastic equations with weak damping and source terms

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Abstract

In this article, we investigate a system of two viscoelastic equations with Dirichlet boundary conditions. Under some suitable assumptions on the function $g_i(\cdot)$, $f_i(\cdot,\cdot)$ ($i = 1,2$) and the initial data, we obtain general and optimal decay results. Moreover, for certain initial data, we establish a finite time blow-up result. This work generalizes and improves earlier results in the literature. The conditions of the relaxation functions $g_1(t)$ and $g_2(t)$ in our work are weak and seldom appear in previous literature, which is an important breakthrough.

Keywords: Decay rate; Blow up; Viscoelastic equations; Nonlinear damping

1 Introduction

In this paper, we investigate the following initial-boundary problem:

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g_1(t-\tau)\Delta u(\tau)\,d\tau + u_t = f_1(u,v), & x \in \Omega, t > 0, \\ v_{tt} - \Delta v + \int_0^t g_2(t-\tau)\Delta v(\tau)\,d\tau + v_t = f_2(u,v), & x \in \Omega, t > 0, \\ u(x,t) = v(x,t) = 0, \quad x \in \partial\Omega, t \geq 0, \\ u(x,0) = u_0, \quad u_t(x,0) = u_1, \quad v(x,0) = v_0, \quad v_t(x,0) = v_1, \quad x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of R^n ($n \geq 1$) with a smooth boundary $\partial\Omega$, u and v represent the transverse displacements of wave. The functions g_1 and g_2 denote the kernel of the memory term, and the nonlinear functions f_1 and f_2 will be specified later. Problem (1.1) describes the propagation of some material possessing a capacity of storage and dissipation of mechanical energy. Models of this type arise in the theory of viscoelasticity and physics.

Firstly, we present some results related to viscoelastic equations. The single viscoelastic equation of the form

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau + a|u_t|^{m-2}u_t = b|u|^{p-2}u, \\ \quad \text{in } \Omega \times (0,+\infty), \\ u(x,t) = 0, \quad \text{on } \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \text{in } \Omega, \end{cases} \tag{1.2}$$

which has been extensively studied by many authors, and related results concerning existence, decay, and blow-up have been recently established (see [6, 11]). Here, we understand $\triangle u_{tt}$, $|u_t|^{m-2}u_t$, $\int_0^t g(t-\tau)\triangle u(\tau)d\tau$, and $|u|^{p-2}u$ to be the dispersion term, weak damping term, viscoelasticity dissipative term, and source term, respectively. This type of problem usually appears as a model in nonlinear viscoelasticity.

As $a = b = 0$ in the presence of the strong damping term, Cavalcanti et al. [1] dealt with the equation

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - \triangle u_{tt} + \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau - \gamma\Delta u_t = 0, & \text{in } \Omega \times (0,+\infty), \\ u(x,t) = 0, \quad \text{on } \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad \text{in } \Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in R^n ($n \geq 1$) and $\rho > 0$. They established a global existence result when the constant $\gamma \geq 0$ and an exponential decay result for the case $\gamma > 0$. Later, this result was improved by Messaoudi and Tatar [8] to a situation where a source term is presented. By using perturbation techniques, they established a global existence and an exponential decay result.

When $a = b = 1$ in the absence of the dispersion term, the following problem

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + |u_t|^{m-2}u_t = |u|^{p-2}u, & \text{in } \Omega \times [0,T], \\ u(x,t) = 0, \quad x \in \partial\Omega, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in \Omega, \end{cases} \tag{1.4}$$

in a bounded domain and $m > 2$, $p > 2$ was studied by Song [10]. The author proved the nonexistence of global solutions of problem (1.4) with bounded positive initial energy.

The case of $\rho = 0$ in the absence of the viscoelasticity term problem (1.2) has been discussed by many authors. For example, Chen and Liu [2] studied the following equation:

$$u_{tt} - \Delta u - \omega\Delta u_t + a|u_t|^{m-2}u_t = b|u|^{p-2}u \tag{1.5}$$

subject to the same boundary and initial conditions as problem (1.4). Under some suitable conditions on the initial data, they proved the global existence of solutions in both cases which are polynomial and exponential decay in the energy space respectively.

In the case of $a = b = 1$, $m = 2$, in the presence of the strong damping term and without the dispersion term, the following viscoelastic equation

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau - \Delta u_t + u_t = |u|^{p-2}u, & \text{in } \Omega \times (0,+\infty), \\ u(x,t) = 0, \quad \text{on } \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad \text{in } \Omega, \end{cases} \tag{1.6}$$

was studied by Li and He [4]. By using some properties of the convex functions, they obtained a general decay rate result. Moreover, they established a finite time blow-up result for solutions with negative initial energy and positive initial energy.

For a coupled system, the following system of viscoelastic equations

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g_1(t-\tau)\Delta u(\tau)\,d\tau + f_1(u,v) = 0, & x \in \Omega, t > 0, \\ v_{tt} - \Delta v + \int_0^t g_2(t-\tau)\Delta v(\tau)\,d\tau + f_2(u,v) = 0, & x \in \Omega, t > 0, \\ u(x,t) = v(x,t) = 0, \quad x \in \partial\Omega, t \geq 0, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ v(x,0) = v_0(x), \qquad v_t(x,0) = v_1(x), \quad x \in \Omega, \end{cases} \tag{1.7}$$

was considered by Mustafa [9]. Under some suitable conditions, they proved the well-posedness and obtained a generalized stability result.

In the work [5], Liu considered the following problem:

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - \gamma_1 \Delta u_{tt} + \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau + f(u,v) = 0, & x \in \Omega, t > 0, \\ |v_t|^\rho v_{tt} - \Delta v - \gamma_2 \Delta v_{tt} + \int_0^t h(t-\tau)\Delta v(\tau)\,d\tau + k(u,v) = 0, & x \in \Omega, t > 0, \\ u(x,t) = v(x,t) = 0, \quad x \in \partial\Omega, t \geq 0, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ v(x,0) = v_0(x), \qquad v_t(x,0) = v_1(x), \quad x \in \Omega, \end{cases} \tag{1.8}$$

where $\gamma_1, \gamma_2 > 0$ and $0 < \rho \leq \frac{2}{n-2}$ if $n \geq 3$ or $\rho > 0$ if $n = 1, 2$. Under the following assumptions on the relaxation functions:

$$g_1'(t) \leq -\zeta_1(t) g_1^p(t), \qquad g_2'(t) \leq -\zeta_2(t) g_2^q(t), \quad t \geq 0,$$

where ζ_1 and ζ_2 are positive constants. By exploiting the perturbed energy method, a uniform decay of the energy result was established.

Recently, Houari et al. [3] investigated the following system:

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g_1(t-\tau)\Delta u(\tau)\,d\tau + |u_t|^{m-1}u_t = f_1(u,v), & \text{in } \Omega \times (0,+\infty), \\ v_{tt} - \Delta v + \int_0^t g_2(t-\tau)\Delta v(\tau)\,d\tau + |v_t|^{r-1}v_t = f_2(u,v), & \text{in } \Omega \times (0,+\infty), \\ u(x,t) = v(x,t) = 0, \quad \text{on } \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad \text{in } \Omega, \\ v(x,0) = v_0(x), \qquad v_t(x,0) = v_1(x), \quad \text{in } \Omega, \end{cases} \tag{1.9}$$

where Ω is an open bounded domain of R^n with a smooth boundary $\partial\Omega$, $m, r \geq 1$. Under some suitable conditions, they obtained a general decay, which depends on the relaxation functions.

As for a single wave equation, in the absence of the source term, the damping term assures global existence. On the other hand, without the damping term, the source term may cause finite time blow-up of solution. Hence, it is valuable to study the viscoelastic equation with damping and source terms.

Our aim in this work is to establish the global existence, decay, and blow-up result of solutions to problem (1.1). By adopting and modifying the methods used in [9], we establish the general and optimal decay and blow-up results, while we should overcome the additional difficulty caused by the changes of the conditions of the relaxation functions

$g_1(t)$ and $g_2(t)$. In [9], the relaxation functions $g_i(t)$ $(i = 1, 2)$ satisfy $g_i'(t) \le -\zeta_i g_i(t)$ for all $t \ge 0$, where $\zeta_i(t)$ are positive nonincreasing functions. In this paper, the conditions have been replaced by $g_i'(t) \le -\zeta_i(t) g_i^{\gamma}(t)$, $\gamma \in [1, 3/2)$. As far as we know, the conditions of the relaxation functions $g_i(t)$ in our work seldom appear in previous literature, which is an important breakthrough. Our main novel contribution is an extension and improvement of the previous result from [9].

This paper is organized as follows. In Sect. 2, we give material needed for our work. In Sect. 3, we prove the global existence. In Sect. 4, we present some technical lemmas needed in the proof of our result. Section 5 is devoted to the general decay result. In Sect. 6, we carry out the proof of finite time blow-up result.

2 Preliminaries

In this section, we present some material needed for our work. First, we make the following assumptions.

(G1) $g_i : \mathbb{R}_+ \to \mathbb{R}_+$ (for $i = 1, 2$) are C^1 nonincreasing functions satisfying

$$1 - \int_0^{\infty} g_1(\tau)\, d\tau = l > 0, \qquad 1 - \int_0^{\infty} g_2(\tau)\, d\tau = k > 0, \quad g_i(0) > 0. \tag{2.1}$$

(G2) There exist two nonincreasing differentiable functions $\zeta_1, \zeta_2 : \mathbb{R}_+ \to \mathbb{R}_+$, with $\zeta_1(0), \zeta_2(0) > 0$ and satisfying

$$g_i'(t) \le -\zeta_i(t) g_i^{\gamma}(t), \quad t \ge 0, 1 \le \gamma < \frac{3}{2}, \text{for } i = 1, 2. \tag{2.2}$$

(G3) For the functions f_1 and f_2, we note that

$$f_1(u, v) = |u + v|^{2(p+1)}(u + v) + |u|^p u |v|^{p+2}, \tag{2.3}$$

$$f_2(u, v) = |u + v|^{2(p+1)}(u + v) + |u|^{p+2} v |v|^p, \tag{2.4}$$

where p satisfies

$$p > -1 \quad \text{if } n = 1, 2 \quad \text{and} \quad -1 < p \le \frac{3 - n}{n - 2} \quad \text{if } n \ge 3. \tag{2.5}$$

It is easy to verify that

$$u f_1 + v f_2 = 2(p + 2) F(u, v), \tag{2.6}$$

where

$$F(u, v) = \frac{1}{2(p + 2)}\big[|u + v|^{2(p+2)} + 2|uv|^{p+2}\big]. \tag{2.7}$$

Remark 2.1 There are many functions $g_i(t)$ and $\zeta_i(t)$ (for $i = 1, 2$) satisfying (G1) and (G2). An example of such functions is

$$\begin{aligned} g_1(t) &= \frac{1}{(1 + t)^4}, \qquad \zeta_1(t) = \frac{4}{1 + t}, \\ g_2(t) &= \frac{1}{16(1 + t)^4}, \qquad \zeta_2(t) = \frac{8}{1 + t}, \qquad \gamma = \frac{5}{4}. \end{aligned} \tag{2.8}$$

Definition 2.2 A pair of functions (u,v) defined on $[0,T]$ is called a weak solution of system (1.1) if $u,v\in C_w([0,T],H_0^1(\Omega))$, $u_t,v_t\in C_w([0,T],L^2(\Omega))$, $(u(x,0),v(x,0))=(u_0,v_0)\in H_0^1(\Omega)\times H_0^1(\Omega)$, $(u_t(x,0),v_t(x,0))=(u_1,v_1)\in H_0^1(\Omega)\times H_0^1(\Omega)$ and (u,v) satisfies

$$\begin{aligned} &\int_\Omega u_{tt}\phi\,dx-\int_0^t g_1(t-\tau)\int_\Omega \nabla\phi(\tau)\nabla u(\tau)\,dx\,d\tau \\ &\quad+\int_\Omega \nabla\phi\nabla u\,dx+\int_\Omega \phi u_t\,dx=\int_\Omega f_1(u,v)\phi\,dx, \end{aligned}\tag{2.9}$$

$$\begin{aligned} &\int_\Omega v_{tt}\psi\,dx-\int_0^t g_2(t-\tau)\int_\Omega \nabla\psi(\tau)\nabla v(\tau)\,dx\,d\tau \\ &\quad+\int_\Omega \nabla\psi\nabla v\,dx+\int_\Omega \psi v_t\,dx=\int_\Omega f_2(u,v)\psi\,dx \end{aligned}\tag{2.10}$$

for a.e. $t\in[0,T]$ and all test functions $\phi,\psi\in H_0^1(\Omega)$. Here, $C_w([0,T],X)$ denotes the space of weakly continuous functions from $[0,T]$ into Banach space X.

Proposition 2.3 *Let* $(u_0,v_0)\in H_0^1(\Omega)\times H_0^1(\Omega)$ *and* $(u_1,v_1)\in L^2(\Omega)\times L^2(\Omega)$. *Assume that* (G1)–(G3) *hold. Then there exists a unique local weak solution* (u,v) *of system* (1.1) *defined in* $[0,T_m]$ *for some* $T_m>0$ *small enough.*

3 Global existence

In this section, we prove the global existence of the solution to system (1.1).

First, we define

$$\begin{aligned} I(t)={}&\left(1-\int_0^t g_1(\tau)\,d\tau\right)\|\nabla u\|_2^2+\left(1-\int_0^t g_2(\tau)\,d\tau\right)\|\nabla v\|_2^2 \\ &+(g_1\circ\nabla u)(t)+(g_2\circ\nabla v)(t)-2(p+2)\int_\Omega F(u,v)\,dx, \end{aligned}\tag{3.1}$$

$$\begin{aligned} J(t)={}&\frac{1}{2}\left(1-\int_0^t g_1(\tau)\,d\tau\right)\|\nabla u\|_2^2+\frac{1}{2}\left(1-\int_0^t g_2(\tau)\,d\tau\right)\|\nabla v\|_2^2 \\ &+\frac{1}{2}(g_1\circ\nabla u)(t)+\frac{1}{2}(g_2\circ\nabla v)(t)-\int_\Omega F(u,v)\,dx, \end{aligned}\tag{3.2}$$

and the energy function associated with system (1.1)

$$\begin{aligned} E(t)={}&\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\|v_t\|_2^2+\frac{1}{2}\left(1-\int_0^t g_1(\tau)\,d\tau\right)\|\nabla u\|_2^2+\frac{1}{2}(g_1\circ\nabla u)(t) \\ &+\frac{1}{2}\left(1-\int_0^t g_2(\tau)\,d\tau\right)\|\nabla v\|_2^2+\frac{1}{2}(g_2\circ\nabla v)(t)-\int_\Omega F(u,v)\,dx, \end{aligned}\tag{3.3}$$

where

$$(g_1\circ\nabla u)(t)=\int_0^t g_1(t-\tau)\big\|\nabla u(t)-\nabla u(\tau)\big\|_2^2\,d\tau,\tag{3.4}$$

$$(g_2\circ\nabla v)(t)=\int_0^t g_2(t-\tau)\big\|\nabla v(t)-\nabla v(\tau)\big\|_2^2\,d\tau.\tag{3.5}$$

Lemma 3.1 *Let* (G1)–(G3) *hold and* (u,v) *be the solution of* (1.1), *then we have*

$$\begin{aligned}\frac{dE(t)}{dt} &= \frac{1}{2}\bigl[\bigl(g_1'\circ\nabla u\bigr)(t)+\bigl(g_2'\circ\nabla v\bigr)(t)\bigr]-\|u_t\|_2^2\\&\quad-\|v_t\|_2^2-\frac{1}{2}\bigl[g_1(t)\|\nabla u\|_2^2-g_2(t)\|\nabla v\|_2^2\bigr]\le 0.\end{aligned}\tag{3.6}$$

Proof Multiplying the first equation in system (1.1) by u_t and integrating over Ω gives

$$\begin{aligned}&\int_\Omega u_t u_{tt}\,dx-\int_0^t g_1(t-\tau)\int_\Omega \nabla u_t(t)\nabla u(\tau)\,dx\,d\tau\\&\quad+\int_\Omega \nabla u_t\nabla u\,dx+\|u_t\|_2^2=-\int_\Omega u_t f_1(u,v)\,dx.\end{aligned}\tag{3.7}$$

Then

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{1}{2}\int_\Omega |u_t|^2\,dx+\frac{1}{2}\int_\Omega |\nabla u|^2\,dx\right\}+\|u_t\|_2^2\\&\quad-\int_0^t g_1(t-\tau)\int_\Omega \nabla u_t(t)\nabla u(\tau)\,dx\,d\tau=-\int_\Omega u_t f_1(u,v)\,dx.\end{aligned}\tag{3.8}$$

For the last term on the left-hand side of (3.8), we get

$$\begin{aligned}&\int_0^t g_1(t-\tau)\int_\Omega \nabla u_t(t)\nabla u(\tau)\,dx\,d\tau\\&\quad=\int_0^t g_1(t-\tau)\int_\Omega \nabla u_t(t)\bigl[\nabla u(\tau)-\nabla u(t)\bigr]\,dx\,d\tau\\&\qquad+\int_0^t g_1(t-\tau)\int_\Omega \nabla u_t(t)\nabla u(t)\,dx\,d\tau\\&\quad=-\frac{1}{2}\int_0^t g_1(t-\tau)\frac{d}{dt}\int_\Omega \bigl|\nabla u(\tau)-\nabla u(t)\bigr|^2\,dx\,d\tau\\&\qquad+\int_0^t g_1(\tau)\left(\frac{d}{dt}\frac{1}{2}\int_\Omega \bigl|\nabla u(t)\bigr|^2\,dx\right)d\tau\\&\quad=-\frac{1}{2}\frac{d}{dt}\left[\int_0^t g_1(t-\tau)\int_\Omega \bigl|\nabla u(\tau)-\nabla u(t)\bigr|^2\,dx\,d\tau\right]\\&\qquad+\frac{1}{2}\frac{d}{dt}\left[\int_0^t g_1(\tau)\int_\Omega \bigl|\nabla u(t)\bigr|^2\,dx\,d\tau\right]-\frac{1}{2}g_1(\tau)\|\nabla u\|_2^2\\&\qquad+\frac{1}{2}\int_0^t g_1'(t-\tau)\int_\Omega \bigl|\nabla u(\tau)-\nabla u(t)\bigr|^2\,dx\,d\tau.\end{aligned}\tag{3.9}$$

By combining (3.4), (3.8), and (3.9), we deduce

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{1}{2}\int_\Omega |u_t|^2\,dx+\frac{1}{2}\int_\Omega |\nabla u|^2\,dx\right\}-\frac{1}{2}\frac{d}{dt}\left[\int_0^t g_1(\tau)\bigl\|\nabla u(t)\bigr\|_2^2 d\tau\right]\\&\qquad+\frac{1}{2}\frac{d}{dt}\left[\int_0^t g_1(t-\tau)\int_\Omega \bigl|\nabla u(\tau)-\nabla u(t)\bigr|^2\,dx\,d\tau\right]\\&\quad=-\int_\Omega u_t f_1(u,v)\,dx+\frac{1}{2}\bigl(g_1'\circ\nabla u\bigr)(t)-\frac{1}{2}g_1(t)\bigl\|\nabla u(t)\bigr\|_2^2-\|u_t\|_2^2.\end{aligned}\tag{3.10}$$

Similarly, multiplying the second equation in (1.1) by v_t and integrating over Ω yields

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{1}{2}\int_\Omega |v_t|^2\,dx+\frac{1}{2}\int_\Omega |\nabla v|^2\,dx\right\}-\frac{1}{2}\frac{d}{dt}\left[\int_0^t g_2(\tau)\left\|\nabla v(t)\right\|_2^2 d\tau\right]\\&\quad+\frac{1}{2}\frac{d}{dt}\left[\int_0^t g_2(t-\tau)\int_\Omega \left|\nabla v(\tau)-\nabla v(t)\right|^2 dx\,d\tau\right]\\&=-\int_\Omega v_t f_2(u,v)\,dx+\frac{1}{2}\left(g_2'\circ\nabla v\right)(t)-\frac{1}{2}g_2(t)\left\|\nabla v(t)\right\|_2^2-\|v_t\|_2^2.\end{aligned}\tag{3.11}$$

Finally, by adding (3.10) to (3.11), (3.6) is established. □

The following lemma is important to prove the global existence of solution.

Lemma 3.2 ([3]) *Assume that* (2.5) *holds. Then there exists* $\eta > 0$ *such that, for any* $(u,v) \in H_0^1(\Omega)\times H_0^1(\Omega)$, *we obtain*

$$\|u+v\|_{2(p+2)}^{2(p+2)}+2\|uv\|_{p+2}^{p+2}\le \eta\left(l\|\nabla u\|_2^2+k\|\nabla v\|_2^2\right)^{p+2}.\tag{3.12}$$

Proof Exploiting Minkowski's inequality, we have

$$\|u+v\|_{2(p+2)}^2\le 2\left(\|u\|_{2(p+2)}^2+\|v\|_{2(p+2)}^2\right).\tag{3.13}$$

From Hölder's inequality and Young's inequality, we derive

$$\|uv\|_{p+2}\le \|u\|_{2(p+2)}\|v\|_{2(p+2)}\le c\left(l\|\nabla u\|_2^2+k\|\nabla v\|_2^2\right).\tag{3.14}$$

Then, combining (3.13), (3.14) and the embedding $H_0^1(\Omega)\hookrightarrow L^{2(p+2)}(\Omega)$ leads to (3.12). □

Lemma 3.3 *Let* $(u_0,v_0)\in H_0^1(\Omega)\times H_0^1(\Omega)$ *and* $(u_1,v_1)\in L^2(\Omega)\times L^2(\Omega)$. *Assume that* (G1)–(G3) *hold. If*

$$I(0)=I(u_0,v_0)>0\quad\text{and}\quad \beta=\eta\left(\frac{2(p+2)}{p+1}E(0)\right)^{p+1}<1,\tag{3.15}$$

then $I(t)>0$ *for* $t\in[0,T_m]$.

Proof Since $I(0)>0$, then by the continuity of $I(t)$, there exists $T^*<T_m$ such that $I(t)>0$, $\forall t\in[0,T^*]$. By using (3.1) and (3.2), we have

$$\begin{aligned}J(t)&=\frac{p+1}{2(p+2)}\left\{\left(1-\int_0^t g_1(\tau)\,d\tau\right)\|\nabla u\|_2^2+\left(1-\int_0^t g_2(\tau)\,d\tau\right)\|\nabla v\|_2^2\right.\\&\quad\left.+(g_1\circ\nabla u)(t)+(g_2\circ\nabla v)(t)\right\}+\frac{1}{2(p+2)}I(t)\\&\ge\frac{p+1}{2(p+2)}\left\{\left(1-\int_0^t g_1(\tau)\,d\tau\right)\|\nabla u\|_2^2+\left(1-\int_0^t g_2(\tau)\,d\tau\right)\|\nabla v\|_2^2\right.\\&\quad\left.+(g_1\circ\nabla u)(t)+(g_2\circ\nabla v)(t)\right\}.\end{aligned}\tag{3.16}$$

From (2.1), (3.1), (3.2), and (3.6), we infer that

$$\begin{aligned} l\|\nabla u\|_2^2 + k\|\nabla v\|_2^2 &\le \left(1 - \int_0^t g_1(\tau)\,d\tau\right)\|\nabla u\|_2^2 \\ &\quad + \left(1 - \int_0^t g_2(\tau)\,d\tau\right)\|\nabla v\|_2^2 \\ &\le \frac{2(p+2)}{p+1}J(t) \\ &\le \frac{2(p+2)}{p+1}E(t) \\ &\le \frac{2(p+2)}{p+1}E(0). \end{aligned} \tag{3.17}$$

It follows from (2.7), (3.12), and (3.15) that

$$\begin{aligned} &2(p+2)\int_\Omega F(u,v)\,dx \\ &\quad \le \eta\bigl(l\|\nabla u\|_2^2 + k\|\nabla v\|_2^2\bigr)^{p+2} \\ &\quad \le \eta\bigl(l\|\nabla u\|_2^2 + k\|\nabla v\|_2^2\bigr)^{p+1}\bigl(l\|\nabla u\|_2^2 + k\|\nabla v\|_2^2\bigr) \\ &\quad \le \eta\left[\frac{2(p+2)}{p+1}E(0)\right]^{p+1}\bigl(l\|\nabla u\|_2^2 + k\|\nabla v\|_2^2\bigr), \quad \forall t \in [0, T^*]. \end{aligned} \tag{3.18}$$

Combining (3.15) and (3.18), we deduce that

$$\begin{aligned} &2(p+2)\int_\Omega F(u,v)\,dx \\ &\quad \le \beta\bigl(l\|\nabla u\|_2^2 + k\|\nabla v\|_2^2\bigr) \\ &\quad \le \beta\left(1 - \int_0^t g_1(\tau)\,d\tau\right)\|\nabla u\|_2^2 + \beta\left(1 - \int_0^t g_2(\tau)\,d\tau\right)\|\nabla v\|_2^2 \\ &\quad \le \left(1 - \int_0^t g_1(\tau)\,d\tau\right)\|\nabla u\|_2^2 + \left(1 - \int_0^t g_2(\tau)\,d\tau\right)\|\nabla v\|_2^2, \quad \forall t \in [0, T^*]. \end{aligned} \tag{3.19}$$

Therefore

$$\begin{aligned} I(t) &= \left(1 - \int_0^t g_1(\tau)\,d\tau\right)\|\nabla u\|_2^2 + \left(1 - \int_0^t g_2(\tau)\,d\tau\right)\|\nabla v\|_2^2 + (g_1 \circ \nabla u)(t) \\ &\quad + (g_2 \circ \nabla v)(t) - 2(p+2)\int_\Omega F(u,v)\,dx > 0, \quad \forall t \in [0, T^*]. \end{aligned} \tag{3.20}$$

By repeating this procedure, T^* is extended to T_m. □

Lemma 3.4 *Assume that* (G1)–(G3) *hold. If* $(u_0, v_0) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *and* $(u_1, v_1) \in L^2(\Omega) \times L^2(\Omega)$ *and satisfy* (3.15), *then the solution of system* (1.1) *is bounded and global in time.*

Proof From Lemma 3.3, (3.6), and (3.17), we see that

$$\begin{aligned} E(0) \geq E(t) &= \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|v_t\|_2^2 + J(t) \\ &\geq \frac{p+1}{2(p+2)}\left(l\|\nabla u\|_2^2 + k\|\nabla v\|_2^2\right), \end{aligned} \tag{3.21}$$

therefore

$$l\|\nabla u\|_2^2 + k\|\nabla v\|_2^2 \leq \frac{2(p+2)}{p+1}E(0), \tag{3.22}$$

which implies that the solution of system (1.1) is global and bounded. □

4 Technical lemmas

In this section, we present some lemmas needed for the proof of our result.

Lemma 4.1 ([3]) *There exist two constants $a_0 > 0$ and $a_1 > 0$ such that*

$$\frac{a_0}{2(p+2)}\left(|u|^{2(p+2)} + |v|^{2(p+2)}\right) \leq F(u,v) \leq \frac{a_1}{2(p+2)}\left(|u|^{2(p+2)} + |v|^{2(p+2)}\right). \tag{4.1}$$

Proof The right-hand side of (4.1) is trivial. If $u = v = 0$, for the left-hand side of (4.1), the result is also trivial. If, without loss of generality, $v \neq 0$, then either $|u| \leq |v|$ or $|u| > |v|$.

If $|u| \leq |v|$, we have

$$F(u,v) = \frac{1}{2(p+2)}|v|^{2(p+2)}\left[\left|1 + \frac{u}{v}\right|^{2(p+2)} + 2\left|\frac{u}{v}\right|^{p+2}\right]. \tag{4.2}$$

Now we consider the continuous function

$$j(\omega) = |1+\omega|^{2(p+2)} + 2|\omega|^{p+2} \quad \text{over } [-1,1], \tag{4.3}$$

then we obtain that $\min j(\omega) \geq 0$. If $\min j(\omega) = 0$, then there exists $\omega_0 \in [-1,1]$ such that

$$j(\omega_0) = |1+\omega_0|^{2(p+2)} + 2|\omega_0|^{p+2} = 0. \tag{4.4}$$

This infers that $|1 + \omega_0| = |\omega_0| = 0$, which is impossible. Hence $\min j(\omega) = 2a_0 > 0$. Thus

$$F(u,v) \geq \frac{a_0}{p+2}|v|^{2(p+2)} \geq \frac{a_0}{p+2}|u|^{2(p+2)}. \tag{4.5}$$

It follows from (4.5) that

$$2F(u,v) \geq \frac{a_0}{p+2}\left\{|v|^{2(p+2)} + |u|^{2(p+2)}\right\}, \tag{4.6}$$

and then

$$\frac{a_0}{2(p+2)}\left\{|v|^{2(p+2)} + |u|^{2(p+2)}\right\} \leq F(u,v). \tag{4.7}$$

If $|u| > |v|$, we deduce that

$$\begin{aligned} F(u,v) &= \frac{1}{2(p+2)}|u|^{2(p+2)}\left[\left|1+\frac{v}{u}\right|^{2(p+2)} + 2\left|\frac{v}{u}\right|^{p+2}\right] \\ &\geq \frac{a_0}{p+2}|u|^{2(p+2)} \geq \frac{a_0}{p+2}|v|^{2(p+2)}. \end{aligned} \tag{4.8}$$

Hence, this gives the desired result. □

Lemma 4.2 ([3]) *There exist two positive constants* λ_1 *and* λ_2 *such that*

$$\int_\Omega \left|f_i(u,v)\right|^2 dx \leq \lambda_i\left(l\|\nabla u\|_2^2 + k\|\nabla v\|_2^2\right)^{2p+3}, \quad i = 1,2. \tag{4.9}$$

Proof From (2.3), we easily get

$$\begin{aligned} \left|f_1(u,v)\right| &\leq C\left(|u+v|^{2p+3} + |u|^{p+1}|v|^{p+2}\right) \\ &\leq C\left(|u|^{2p+3} + |v|^{2p+3} + |u|^{p+1}|v|^{p+2}\right). \end{aligned} \tag{4.10}$$

By (4.10) and Young's inequality, with $q = \frac{2p+3}{p+1}$ and $q' = \frac{2p+3}{p+2}$, we have

$$|u|^{p+1}|v|^{p+2} \leq c_1|u|^{2p+3} + c_2|v|^{2p+3}, \tag{4.11}$$

therefore

$$\left|f_1(u,v)\right| \leq C\left(|u|^{2p+3} + |v|^{2p+3}\right). \tag{4.12}$$

Then, by using Poincare's inequality and (2.5), we observe that

$$\begin{aligned} \int_\Omega \left|f_1(u,v)\right|^2 dx &\leq C\left(\|\nabla u\|_2^{2(2p+3)} + \|\nabla v\|_2^{2(2p+3)}\right) \\ &\leq \lambda_1\left(l\|\nabla u\|_2^2 + k\|\nabla v\|_2^2\right)^{2p+3}. \end{aligned} \tag{4.13}$$

In the same way, we have

$$\begin{aligned} \int_\Omega \left|f_2(u,v)\right|^2 dx &\leq C\left(\|\nabla u\|_2^{2(2p+3)} + \|\nabla v\|_2^{2(2p+3)}\right) \\ &\leq \lambda_2\left(l\|\nabla u\|_2^2 + k\|\nabla v\|_2^2\right)^{2p+3}. \end{aligned} \tag{4.14}$$

□

Lemma 4.3 (Jensen inequality) *Suppose that* G *is a concave function on* $[a,b]$, $f : \Omega \to [a,b]$ *and* h *are integrable functions on* Ω, *with* $h(x) \geq 0$, *and* $\int_\Omega h(x)\,dx = r > 0$, *then*

$$\frac{1}{r}\int_\Omega G\left[f(x)\right]h(x)\,dx \leq G\left[\frac{1}{r}\int_\Omega f(x)h(x)\,dx\right]. \tag{4.15}$$

For the special case $G(y) = y^{\frac{1}{\gamma}}$, $y \geq 0$, $\gamma > 1$, we obtain

$$\frac{1}{r}\int_{\Omega}\big[f(x)\big]^{\frac{1}{\gamma}}h(x)\,dx \leq \left[\frac{1}{r}\int_{\Omega}f(x)h(x)\,dx\right]^{\frac{1}{\gamma}}. \tag{4.16}$$

Lemma 4.4 *Assume that g_i satisfies* (G1) *and* (G2) *for $i = 1, 2$, then*

$$\int_0^{+\infty}\zeta_i(t)g_i^{1-\theta}(t)\,dt < +\infty, \quad \forall 0 \leq \theta < 2-\gamma. \tag{4.17}$$

Proof From (G1) and (G2), we see that

$$\zeta_i(t)g_i^{1-\theta}(t) = \zeta_i(t)g_i^{\gamma}(t)g_i^{1-\theta-\gamma}(t) \leq -g_i'(t)g_i^{1-\theta-\gamma}(t). \tag{4.18}$$

Integrating (4.18) over $(0, +\infty)$ and using the fact that $0 \leq \theta < 2-\gamma$, we get

$$\int_0^{+\infty}\zeta_i(t)g_i^{1-\theta}(t)\,dt \leq -\int_0^{+\infty}g_i'(t)g_i^{1-\theta-\gamma}(t)\,dt = \frac{-g_i^{2-\theta-\gamma}(t)}{2-\theta-\gamma}\bigg|_0^{+\infty} < +\infty. \tag{4.19}$$

□

Lemma 4.5 ([7]) *If* (G1)–(G3) *hold, $u \in L^{\infty}(0, T, H_0^1(\Omega))$, for $0 < \theta < 1$, we obtain*

$$(g_1 \circ \nabla u)(t) \leq C\left\{\left(\int_0^{+\infty}g_1^{1-\theta}(t)\,dt\right)E(0)\right\}^{\frac{\gamma-1}{\gamma-1+\theta}}\left\{\left(g_1^{\gamma} \circ \nabla u\right)\right\}^{\frac{\theta}{\gamma-1+\theta}}(t). \tag{4.20}$$

By taking $\theta = \frac{1}{2}$, we have

$$(g_1 \circ \nabla u)(t) \leq C\left\{\int_0^t g_1^{\frac{1}{2}}(\tau)\,d\tau\right\}^{\frac{2\gamma-2}{2\gamma-1}}\left\{\left(g_1^{\gamma} \circ \nabla u\right)\right\}^{\frac{1}{2\gamma-1}}(t). \tag{4.21}$$

Proof For $q > 1$, we derive

$$\begin{aligned}(g_1 \circ \nabla u)(t) = {}& \int_0^t g_1^{\frac{1-\theta}{q}}(t-\tau)\big\|\nabla u(t) - \nabla u(\tau)\big\|_2^{\frac{2}{q}} \\ & \times g_1^{\frac{q-1+\theta}{q}}(t-\tau)\big\|\nabla u(t) - \nabla u(\tau)\big\|_2^{\frac{2q-2}{q}}\,d\tau.\end{aligned} \tag{4.22}$$

Applying Hölder's inequality, we deduce

$$\begin{aligned}(g_1 \circ \nabla u)(t) \leq {}& \left(\int_0^t g_1^{1-\theta}(t-\tau)\big\|\nabla u(t) - \nabla u(\tau)\big\|_2^2\,d\tau\right)^{\frac{1}{q}} \\ & \times \left(\int_0^t g_1^{\frac{q-1+\theta}{q-1}}(t-\tau)\big\|\nabla u(t) - \nabla u(\tau)\big\|_2^2\,d\tau\right)^{\frac{q-1}{q}}.\end{aligned} \tag{4.23}$$

By taking $q = \frac{\gamma-1+\theta}{\gamma-1}$, we arrive at

$$(g_1 \circ \nabla u)(t) \leq C\left\{\left(\int_0^t g_1^{1-\theta}(\tau)\,d\tau\right)\|\nabla u\|^2_{L^{\infty}(0,T,H_0^1(\Omega))}\right\}^{\frac{\gamma-1}{\gamma-1+\theta}}\left\{\left(g_1^{\gamma} \circ \nabla u\right)\right\}^{\frac{\theta}{\gamma-1+\theta}}(t)$$

$$\leq C\left\{\left(\int_0^t g_1^{1-\theta}(\tau)\,d\tau\right)\sup E(t)\right\}^{\frac{\gamma-1}{\gamma-1+\theta}}\left\{\left(g_1^{\gamma}\circ\nabla u\right)\right\}^{\frac{\theta}{\gamma-1+\theta}}(t)$$
$$\leq C\left\{\left(\int_0^t g_1^{1-\theta}(\tau)\,d\tau\right)\sup E(0)\right\}^{\frac{\gamma-1}{\gamma-1+\theta}}\left\{\left(g_1^{\gamma}\circ\nabla u\right)\right\}^{\frac{\theta}{\gamma-1+\theta}}(t)$$
$$\leq C\left\{\left(\int_0^t g_1^{1-\theta}(\tau)\,d\tau\right)\right\}^{\frac{\gamma-1}{\gamma-1+\theta}}\left\{\left(g_1^{\gamma}\circ\nabla u\right)\right\}^{\frac{\theta}{\gamma-1+\theta}}(t). \tag{4.24}$$

Then, by taking $\theta=\frac{1}{2}$ in (4.24), (4.21) is established. □

Similarly,

$$(g_2\circ\nabla v)(t)\leq C\left\{\left(\int_0^t g_2^{1-\theta}(\tau)\,d\tau\right)E(0)\right\}^{\frac{\gamma-1}{\gamma-1+\theta}}\left\{\left(g_2^{\gamma}\circ\nabla v\right)\right\}^{\frac{\theta}{\gamma-1+\theta}}(t) \tag{4.25}$$

and

$$(g_2\circ\nabla v)(t)\leq C\left\{\int_0^t g_2^{\frac{1}{2}}(\tau)\,d\tau\right\}^{\frac{2\gamma-2}{2\gamma-1}}\left\{\left(g_2^{\gamma}\circ\nabla v\right)\right\}^{\frac{1}{2\gamma-1}}(t). \tag{4.26}$$

Lemma 4.6 ([7]) *Suppose that g_1 satisfies* (G1) *and* (G2), *then we have*

$$\zeta_1(t)(g_1\circ\nabla u)(t)\leq C\left[-E'(t)\right]^{\frac{1}{2\gamma-1}}. \tag{4.27}$$

Proof Multiplying both sides of (4.21) by $\zeta_1(t)$ and by using (G2), (3.6), and (4.17) gives

$$\zeta_1(t)(g_1\circ\nabla u)(t)\leq C\zeta_1^{\frac{2\gamma-2}{2\gamma-1}}(t)\left[\int_0^t g_1^{\frac{1}{2}}(\tau)\,d\tau\right]^{\frac{2\gamma-2}{2\gamma-1}}\zeta_1^{\frac{1}{2\gamma-1}}(t)\left(g_1^{\gamma}\circ\nabla u\right)^{\frac{1}{2\gamma-1}}(t)$$
$$\leq C\left[\int_0^t \zeta_1(\tau)g_1^{\frac{1}{2}}(\tau)d\tau\right]^{\frac{2\gamma-2}{2\gamma-1}}\left(\zeta_1 g_1^{\gamma}\circ\nabla u\right)^{\frac{1}{2\gamma-1}}(t)$$
$$\leq C\left[\int_0^t \zeta_1(\tau)g_1^{\frac{1}{2}}(\tau)d\tau\right]^{\frac{2\gamma-2}{2\gamma-1}}\left(-g_1'\circ\nabla u\right)^{\frac{1}{2\gamma-1}}(t)$$
$$\leq C\left[-E'(t)\right]^{\frac{1}{2\gamma-1}}. \tag{4.28}$$

□

Likewise,

$$\zeta_2(t)(g_2\circ\nabla v)(t)\leq C\left[-E'(t)\right]^{\frac{1}{2\gamma-1}}. \tag{4.29}$$

5 The decay result

In this section, we establish three related lemmas before proving our result.

Lemma 5.1 *If* (G1)–(G3) *and* (3.15) *hold, the functional $\phi(t)$ defined by*

$$\phi(t):=\int_{\Omega}u_t u\,dx+\int_{\Omega}v_t v\,dx$$

satisfies, along solutions of (1.1),

$$\begin{aligned}\phi'(t) \le& -\frac{l}{4}\|\nabla u\|_2^2 + \frac{1-l}{2l}(g_1 \circ \nabla u)(t) + \left(1 + \frac{1}{4\beta_1}\right)\|u_t\|_2^2 \\ &- \frac{k}{4}\|\nabla v\|_2^2 + \frac{1-k}{2k}(g_2 \circ \nabla v)(t) + \left(1 + \frac{1}{4\beta_2}\right)\|v_t\|_2^2 \\ &+ 2(p+2)\int_\Omega F(u,v)\,dx. \end{aligned} \tag{5.1}$$

Proof A differentiation of $\phi(t)$ with respect to t, it follows from system (1.1) that

$$\begin{aligned}\phi'(t) =& \int_\Omega u_{tt}u\,dx + \|u_t\|_2^2 + \int_\Omega v_{tt}v\,dx + \|v_t\|_2^2 \\ =& \int_\Omega \nabla u(t)\int_0^t g_1(t-\tau)\nabla u(\tau)d\tau\,dx + \|u_t\|_2^2 - \|\nabla u\|_2^2 - \int_\Omega uu_t\,dx \\ &+ \int_\Omega \nabla v(t)\int_0^t g_2(t-\tau)\nabla v(\tau)\,d\tau\,dx + \|v_t\|_2^2 - \|\nabla v\|_2^2 - \int_\Omega vv_t\,dx \\ &+ \int_\Omega uf_1(u,v)\,dx + \int_\Omega vf_2(u,v)\,dx. \end{aligned} \tag{5.2}$$

For the first term on the right-hand side of (5.2), by using Young's inequality and the fact that $\int_0^t g_1(s)\,ds \le \int_0^\infty g_1(s)\,ds = 1-l$, for $\eta = \frac{l}{1-l} > 0$, we get

$$\begin{aligned}&\int_\Omega \nabla u(t)\int_0^t g_1(t-\tau)\nabla u(\tau)d\tau\,dx \\ &\le \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_\Omega\left(\int_0^t g_1(t-\tau)\left(\left|\nabla u(\tau)-\nabla u(t)\right| + \left|\nabla u(t)\right|\right)d\tau\right)^2 dx \\ &\le \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}(1+\eta)\int_\Omega\left(\int_0^t g_1(t-\tau)\left|\nabla u(t)\right|d\tau\right)^2 dx \\ &\quad + \frac{1}{2}\left(1+\frac{1}{\eta}\right)\int_\Omega\left(\int_0^t g_1(t-\tau)\left|\nabla u(\tau)-\nabla u(t)\right|d\tau\right)^2 dx \\ &\le \frac{1}{2}\|\nabla u\|_2^2 + \frac{1+\eta}{2}(1-l)^2\|\nabla u\|_2^2 + \frac{1}{2l}\int_\Omega\left(\int_0^t g_1(t-\tau)\left|\nabla u(\tau)-\nabla u(t)\right|d\tau\right)^2 dx \\ &\le \frac{2-l}{2}\|\nabla u\|_2^2 + \frac{1-l}{2l}(g_1\circ\nabla u)(t). \end{aligned} \tag{5.3}$$

Similar calculations also yield, for $\eta_1 = \frac{k}{1-k} > 0$,

$$\int_\Omega \nabla v(t)\int_0^t g_2(t-\tau)\nabla v(\tau)d\tau\,dx \le \frac{2-k}{2}\|\nabla v\|_2^2 + \frac{1-k}{2k}(g_2\circ\nabla v)(t). \tag{5.4}$$

Applying Young's inequality and Poincare's inequality, for some $\beta_1 > 0$, we obtain

$$\begin{aligned}\int_\Omega uu_t\,dx &\le \beta_1\|u\|_2^2 + \frac{1}{4\beta_1}\|u_t\|_2^2 \\ &\le \beta_1 C_*^2\|\nabla u\|_2^2 + \frac{1}{4\beta_1}\|u_t\|_2^2. \end{aligned} \tag{5.5}$$

Likewise, for some $\beta_2 > 0$, we have

$$\begin{aligned}\int_\Omega vv_t\,dx &\le \beta_2\|v\|_2^2 + \frac{1}{4\beta_2}\|v_t\|_2^2 \\ &\le \beta_2 C_*^2\|\nabla v\|_2^2 + \frac{1}{4\beta_2}\|v_t\|_2^2.\end{aligned} \tag{5.6}$$

Inserting (5.3)–(5.6) into (5.2) yields

$$\begin{aligned}\phi'(t) \le{}& -\left(\frac{l}{2} - \beta_1 C_*^2\right)\|\nabla u\|_2^2 + \frac{1-l}{2l}(g_1 \circ \nabla u)(t) \\ &+ \left(1 + \frac{1}{4\beta_1}\right)\|u_t\|_2^2 - \left(\frac{k}{2} - \beta_2 C_*^2\right)\|\nabla v\|_2^2 \\ &+ \frac{1-k}{2k}(g_2 \circ \nabla v)(t) + \left(1 + \frac{1}{4\beta_2}\right)\|v_t\|_2^2 + 2(p+2)\int_\Omega F(u,v)\,dx.\end{aligned} \tag{5.7}$$

Now, we pick $\beta_1, \beta_2 > 0$ small enough such that

$$\frac{l}{2} - \beta_1 C_*^2 \ge \frac{l}{4}, \qquad \frac{k}{2} - \beta_2 C_*^2 \ge \frac{k}{4}. \tag{5.8}$$

Finally, a combination of (5.7) and (5.8) gives (5.1). □

Lemma 5.2 *Suppose that* (G1)–(G3) *and* (3.15) *hold. The functional* $\psi_1(t)$ *defined by*

$$\psi_1(t) := -\int_\Omega u_t \int_0^t g_1(t-\tau)\big(u(t) - u(\tau)\big)\,d\tau\,dx$$

satisfies, along solutions of (1.1),

$$\begin{aligned}\psi_1'(t) \le{}& \big[\delta + 2(1-l)^2\delta + \alpha_2\delta l\big]\|\nabla u\|_2^2 - \frac{g_1(0)C_*^2}{4\delta}\big(g_1' \circ \nabla u\big)(t) \\ &+ \left(2\delta + \frac{1}{2\delta} + \frac{C_*^2}{2\delta}\right)(1-l)(g_1 \circ \nabla u)(t) \\ &+ \alpha_2\delta k\|\nabla v\|_2^2 + \left(2\delta - \int_0^t g_1(\tau)\,d\tau\right)\|u_t\|_2^2,\end{aligned} \tag{5.9}$$

where $\alpha_2 = \lambda_1(\frac{2(p+2)}{p+1}E(0))^{2p+2}$, λ_1 *is the constant in Lemma* 4.2.

Proof Taking the derivative of $\psi_1(t)$ with respect to t and using system (1.1) gives

$$\begin{aligned}\psi_1'(t) ={}& -\int_\Omega u_{tt}\int_0^t g_1(t-\tau)\big(u(t) - u(\tau)\big)\,d\tau\,dx \\ &- \int_\Omega u_t \int_0^t g_1'(t-\tau)\big(u(t) - u(\tau)\big)d\tau\,dx - \int_0^t g_1(\tau)\,d\tau\|u_t\|_2^2 \\ ={}& \int_\Omega \nabla u(t)\int_0^t g_1(t-\tau)\big(\nabla u(t) - \nabla u(\tau)\big)\,d\tau\,dx \\ &- \int_\Omega\left(\int_0^t g_1(t-\tau)\nabla u(\tau)d\tau\right)\left(\int_0^t g_1(t-\tau)\big(\nabla u(t) - \nabla u(\tau)\big)d\tau\right)dx\end{aligned}$$

$$
\begin{aligned}
&+\int_\Omega u_t\int_0^t g_1(t-\tau)\big(u(t)-u(\tau)\big)d\tau\,dx\\
&-\int_\Omega f_1(u,v)\int_0^t g_1(t-\tau)\big(u(t)-u(\tau)\big)\,d\tau\,dx\\
&-\int_\Omega u_t\int_0^t g_1'(t-\tau)\big(\nabla u(t)-\nabla u(\tau)\big)\,d\tau\,dx-\int_0^t g_1(\tau)\,d\tau\,\|u_t\|_2^2. \qquad (5.10)
\end{aligned}
$$

For the first term on the right-hand side of (5.10), by exploiting (G1), Young's inequality, and Cauchy–Schwarz inequality, for any $\delta > 0$, we get

$$
\begin{aligned}
&\int_\Omega \nabla u(t)\int_0^t g_1(t-\tau)\big(\nabla u(t)-\nabla u(\tau)\big)\,d\tau\,dx\\
&\le \delta\|\nabla u\|_2^2+\frac{1}{4\delta}\int_\Omega\left(\int_0^t g_1(t-\tau)\big|\nabla u(\tau)-\nabla u(t)\big|d\tau\right)^2dx\\
&\le \delta\|\nabla u\|_2^2+\frac{1}{4\delta}\int_0^t g_1(\tau)\,d\tau\int_0^t g_1(t-\tau)\int_\Omega\big|\nabla u(\tau)-\nabla u(t)\big|^2\,dx\,d\tau\\
&\le \delta\|\nabla u\|_2^2+\frac{1-l}{4\delta}(g_1\circ\nabla u)(t). \qquad (5.11)
\end{aligned}
$$

As for the second term in (5.10), recall that $(a+b)^2 \le 2(a^2+b^2)$, for $\eta_2 = 1$, we obtain

$$
\begin{aligned}
&\int_\Omega\left(\int_0^t g_1(t-\tau)\nabla u(\tau)\,d\tau\right)\left(\int_0^t g_1(t-\tau)\big(\nabla u(t)-\nabla u(\tau)\big)d\tau\right)dx\\
&\le \delta\int_\Omega\left|\int_0^t g_1(t-\tau)\nabla u(\tau)\,d\tau\right|^2dx\\
&\quad+\frac{1}{4\delta}\int_\Omega\left|\int_0^t g_1(t-\tau)\big(\nabla u(t)-\nabla u(\tau)\big)\,d\tau\right|^2dx\\
&\le \delta\int_\Omega\left(\int_0^t g_1(t-\tau)\big(\big|\nabla u(\tau)-\nabla u(t)\big|+\big|\nabla u(t)\big|\big)\,d\tau\right)^2dx\\
&\quad+\frac{1}{4\delta}\int_\Omega\left|\int_0^t g_1(t-\tau)\big(\nabla u(t)-\nabla u(\tau)\big)\,d\tau\right|^2dx\\
&\le \delta(1+\eta_2)\int_\Omega\left(\int_0^t g_1(t-\tau)\big|\nabla u(t)\big|\,d\tau\right)^2dx\\
&\quad+\delta\left(1+\frac{1}{\eta_2}\right)\int_\Omega\left(\int_0^t g_1(t-\tau)\big|\nabla u(\tau)-\nabla u(t)\big|\,d\tau\right)^2dx\\
&\quad+\frac{1}{4\delta}\int_\Omega\left|\int_0^t g_1(t-\tau)\big(\nabla u(t)-\nabla u(\tau)\big)\,d\tau\right|^2dx\\
&\le 2\delta\int_\Omega\left|\int_0^t g_1(t-\tau)\big(\nabla u(t)-\nabla u(\tau)\big)\,d\tau\right|^2dx\\
&\quad+2\delta(1-l)^2\|\nabla u\|_2^2+\frac{1}{4\delta}\left(\int_0^t g_1(\tau)\,d\tau\right)(g_1\circ\nabla u)(t)\\
&\le 2\delta(1-l)^2\|\nabla u\|_2^2+\left(2\delta+\frac{1}{4\delta}\right)(1-l)(g_1\circ\nabla u)(t). \qquad (5.12)
\end{aligned}
$$

The third term can be handled by

$$\begin{aligned}
&\int_\Omega u_t \int_0^t g_1(t-\tau)\big(u(t)-u(\tau)\big)\,d\tau\,dx \\
&\quad\le \delta\|u_t\|_2^2 + \frac{1}{4\delta}\int_\Omega\left(\int_0^t g_1(t-\tau)\big|u(\tau)-u(t)\big|\,d\tau\right)^2 dx \\
&\quad\le \delta\|u_t\|_2^2 + \frac{C_*^2}{4\delta}\int_0^t g_1(\tau)\,d\tau (g_1\circ\nabla u)(t) \\
&\quad\le \delta\|u_t\|_2^2 + \frac{C_*^2(1-l)}{4\delta}(g_1\circ\nabla u)(t).
\end{aligned} \tag{5.13}$$

For the forth term, it follows from (4.9) that

$$\begin{aligned}
&\int_\Omega f_1(u,v)\int_0^t g_1(t-\tau)\big(u(t)-u(\tau)\big)d\tau\,dx \\
&\quad\le \lambda_1\delta\big(l\|\nabla u\|_2^2 + k\|\nabla v\|_2^2\big)^{2p+3} + \frac{1}{4\delta}\int_\Omega\left(\int_0^t g_1(t-\tau)\big|u(\tau)-u(t)\big|\,d\tau\right)^2 dx \\
&\quad\le \lambda_1\delta\left(\frac{2(p+2)}{p+1}E(0)\right)^{2p+2}\big(l\|\nabla u\|_2^2 + k\|\nabla v\|_2^2\big) + \frac{C_*^2(1-l)}{4\delta}(g_1\circ\nabla u)(t) \\
&\quad= \alpha_2\delta\big(l\|\nabla u\|_2^2 + k\|\nabla v\|_2^2\big) + \frac{C_*^2(1-l)}{4\delta}(g_1\circ\nabla u)(t),
\end{aligned} \tag{5.14}$$

where $\alpha_2 = \lambda_1(\frac{2(p+2)}{p+1}E(0))^{2p+2}$.

The fifth term on the right-hand side of (5.10) can be estimated as

$$\begin{aligned}
&\int_\Omega u_t \int_0^t g_1'(t-\tau)\big(u(t)-u(\tau)\big)d\tau\,dx \\
&\quad\le \delta\|u_t\|_2^2 + \frac{1}{4\delta}\int_\Omega\left(\int_0^t g_1'(t-\tau)\big|u(t)-u(\tau)\big|\,d\tau\right)^2 dx \\
&\quad\le \delta\|u_t\|_2^2 - \frac{g_1(0)C_*^2}{4\delta}\big(g_1'\circ\nabla u\big)(t).
\end{aligned} \tag{5.15}$$

Taking into account estimates (5.11)–(5.13), estimate (5.9) is established. □

Similar computations also yield the following.

Lemma 5.3 *Suppose that* (G1)–(G3) *and* (3.15) *hold. The functional* $\psi_2(t)$ *defined by*

$$\psi_2(t) := -v_t\int_0^t g_2(t-\tau)\big(v(t)-v(\tau)\big)\,d\tau\,dx$$

satisfies, along solutions of system (1.1),

$$\begin{aligned}
\psi_2'(t) \le{}& \big[\delta + 2(1-k)^2\delta + \alpha_3\delta k\big]\|\nabla v\|_2^2 - \frac{g_2(0)C_*^2}{4\delta}\big(g_2'\circ\nabla v\big)(t) \\
&+ \left(2\delta + \frac{1}{2\delta} + \frac{C_*^2}{2\delta}\right)(1-k)(g_2\circ\nabla v)(t)
\end{aligned}$$

$$+\alpha_3\delta l\|\nabla u\|_2^2+\left(2\delta-\int_0^t g_2(\tau)\,d\tau\right)\|v_t\|_2^2, \tag{5.16}$$

where $\alpha_3=\lambda_2(\frac{2(p+2)}{p+1}E(0))^{2p+2}$, λ_2 *is the constant in Lemma* 4.2.

Now, we define the functional

$$W(t)=E(t)+\epsilon_1\phi(t)+\epsilon_2\psi(t), \tag{5.17}$$

where ϵ_1 and ϵ_2 are positive constants, $\phi(t)$ is given in Lemma 5.1 and $\psi(t):=\psi_1(t)+\psi_2(t)$.

Lemma 5.4 ([5]) *Let* (u,v) *be the solution of system* (1.1) *and assume that* (3.15) *holds. Then there exist constant* $\epsilon>0$ *small enough and* $M>0$ *large enough such that the following relation*

$$\beta_1 W(t)\le E(t)\le\beta_2 W(t),\quad \forall t\ge 0, \tag{5.18}$$

holds for two positive constants β_1 *and* β_2.

Theorem 5.5 *Assume* (G1)–(G3) *and* (3.15) *hold. Let* $(u_0,v_0)\in H_0^1(\Omega)\times H_0^1(\Omega)$ *and* $(u_1,v_1)\in L^2(\Omega)\times L^2(\Omega)$. *Then, for each* $t_0>0$, *there exist positive constants* K, k, k_1, k_2 *such that the solution of system* (1.1) *satisfies, for all* $t\ge t_0$,

$$E(t)\le Ke^{-k\int_{t_0}^t\zeta(\tau)\,d\tau},\quad \gamma=1, \tag{5.19}$$

$$E(t)\le k_1\left[\frac{1}{1+\int_{t_0}^t\zeta^{2\gamma-1}(\tau)\,d\tau}\right]^{\frac{1}{2\gamma-2}},\quad 1<\gamma<\frac{3}{2}. \tag{5.20}$$

Furthermore, if

$$\int_0^{+\infty}\left[\frac{1}{1+t\zeta^{2\gamma-1}(t)}\right]^{\frac{1}{2\gamma-2}}dt<+\infty,\quad 1<\gamma<\frac{3}{2}, \tag{5.21}$$

then

$$E(t)\le k_2\left[\frac{1}{1+\int_{t_0}^t\zeta^{\gamma}(\tau)d\tau}\right]^{\frac{1}{\gamma-1}},\quad 1<\gamma<\frac{3}{2}, \tag{5.22}$$

where $\zeta(t)=\min\{\zeta_1(t),\zeta_2(t)\}$.

Proof From (G1), we know g_1 and g_2 are positive, then for any $t\ge t_0>0$, we have

$$\int_0^t g_1(\tau)\,d\tau\ge\int_0^{t_0}g_1(\tau)d\tau=g_0,$$
$$\int_0^t g_2(\tau)\,d\tau\ge\int_0^{t_0}g_2(\tau)d\tau=h_0.$$

Taking derivative of (5.17) with respect to t and using (3.6), (5.1), and (5.16) yields

$$W'(t)=E'(t)+\epsilon_1\phi'(t)+\epsilon_2\psi'(t)$$

$$
\begin{aligned}
&\leq \left(\frac{1}{2} - \frac{g_1(0)C_*^2}{4\delta}\epsilon_2\right)\big(g_1' \circ \nabla u\big)(t) + \left(\frac{1}{2} - \frac{g_2(0)C_*^2}{4\delta}\epsilon_2\right)\big(g_2' \circ \nabla v\big)(t) \\
&\quad - \left[\epsilon_2(g_0 - 2\delta) - \epsilon_1\left(1 + \frac{1}{4\beta_1}\right)\right]\|u_t\|_2^2 \\
&\quad - \left[\epsilon_2(h_0 - 2\delta) - \epsilon_1\left(1 + \frac{1}{4\beta_2}\right)\right]\|v_t\|_2^2 + 2(p+2)\epsilon_1 \int_\Omega F(u,v)\,dx \\
&\quad - \left[\frac{l\epsilon_1}{4} - \epsilon_2\big(1 + 2(1-l)^2 + \alpha_2 l + \alpha_3 l\big)\delta\right]\|\nabla u\|_2^2 \\
&\quad - \left[\frac{k\epsilon_2}{4} - \epsilon_2\big(1 + 2(1-k)^2 + \alpha_2 k + \alpha_3 k\big)\delta\right]\|\nabla v\|_2^2 \\
&\quad + \left[\frac{\epsilon_1}{2l} + \left(2\delta + \frac{1}{2\delta} + \frac{C_*^2}{2\delta}\right)\epsilon_2\right](1-l)(g_1 \circ \nabla u)(t) \\
&\quad + \left[\frac{\epsilon_2}{2k} + \left(2\delta + \frac{1}{2\delta} + \frac{C_*^2}{2\delta}\right)\epsilon_2\right](1-k)(g_2 \circ \nabla v)(t).
\end{aligned}
\tag{5.23}
$$

At this point, we pick $\delta > 0$ small enough such that

$$\delta \leq \min\left\{\frac{g_0}{2}, \frac{h_0}{2}\right\}$$

and

$$
\begin{aligned}
&\frac{4}{l}\delta\big(1 + 2(1-l)^2 + \alpha_2 l + \alpha_3 l\big) < \frac{g_0}{4(1 + \frac{1}{4\beta_1})}, \\
&\frac{4}{k}\delta\big(1 + 2(1-k)^2 + \alpha_2 k + \alpha_3 k\big) < \frac{g_0}{4(1 + \frac{1}{4\beta_2})}.
\end{aligned}
$$

As long as δ is fixed, the choice of any two positive constants ϵ_1 and ϵ_2 satisfying

$$\frac{g_0}{4(1 + \frac{1}{4\beta_1})}\epsilon_2 < \epsilon_1 < \frac{g_0}{2(1 + \frac{1}{4\beta_1})}\epsilon_2$$

and

$$\frac{h_0}{4(1 + \frac{1}{4\beta_2})}\epsilon_2 < \epsilon_1 < \frac{h_0}{2(1 + \frac{1}{4\beta_2})}\epsilon_2$$

will make

$$
\begin{aligned}
&k_1 = \epsilon_2(g_0 - 2\delta) - \epsilon_1\left(1 + \frac{1}{4\beta_1}\right) > 0, \\
&k_2 = \epsilon_2(h_0 - 2\delta) - \epsilon_1\left(1 + \frac{1}{4\beta_2}\right) > 0, \\
&k_3 = \frac{l\epsilon_1}{4} - \epsilon_2\big(1 + 2(1-l)^2 + \alpha_2 l + \alpha_3 l\big)\delta > 0, \\
&k_4 = \frac{k\epsilon_2}{4} - \epsilon_2\big(1 + 2(1-k)^2 + \alpha_2 k + \alpha_3 k\big)\delta > 0.
\end{aligned}
$$

Hence, there exist two positive constants m and C such that

$$W'(t) \leq -mE(t) + C\big[(g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)\big], \quad t \geq t_0. \tag{5.24}$$

Case 1. $\gamma = 1$:

Let $\zeta(t) = \min\{\zeta_1(t), \zeta_2(t)\}$, since $\zeta_1(t)$ and $\zeta_2(t)$ are nonincreasing differentiable functions, then we get that $\zeta(t)$ is nonincreasing.

When $\gamma = 1$, from (2.2), we easily get

$$g_i'(t) \leq -\zeta_i(t) g_i^{\gamma}(t) = -\zeta_i(t) g_i(t) \quad \text{for } i = 1, 2. \tag{5.25}$$

Multiplying both sides of (5.24) by $\zeta(t)$ and applying (3.6), (4.27), (4.29), and (5.25) yields

$$\begin{aligned} \zeta(t)W'(t) &\leq -m\zeta(t)E(t) + C\zeta(t)(g_1 \circ \nabla u)(t) + C\zeta(t)(g_2 \circ \nabla v)(t) \\ &\leq -m\zeta(t)E(t) + C\zeta_1(t)(g_1 \circ \nabla u)(t) + C\zeta_2(t)(g_2 \circ \nabla v)(t) \\ &\leq -m\zeta(t)E(t) + C(\zeta_1 g_1 \circ \nabla u)(t) + C(\zeta_2 g_2 \circ \nabla v)(t) \\ &\leq -m\zeta(t)E(t) - C\big(g_1' \circ \nabla u\big)(t) - C\big(g_2' \circ \nabla v\big)(t) \\ &\leq -m\zeta(t)E(t) - CE'(t). \end{aligned} \tag{5.26}$$

Setting $F(t) = \zeta(t)W(t) + CE(t)$, then clearly $F \sim E$. Recalling that $W \sim E \geq 0$ by (3.21) and (5.18), $\zeta'(t) \leq 0$ by (G2), we get that $\zeta'(t)W(t) \leq 0$, then together with (5.26), we have, for some $k > 0$,

$$\begin{aligned} F'(t) &= \zeta'(t)W(t) + \zeta(t)W'(t) + CE'(t) \\ &\leq \zeta(t)W'(t) + CE'(t) \\ &\leq -m\zeta(t)E(t) - CE'(t) + CE'(t) \\ &\leq -k\zeta(t)F(t). \end{aligned} \tag{5.27}$$

Integrating (5.27) over $[t_0, t]$ gives

$$F(t) \leq F(t_0) e^{-k\int_{t_0}^{t} \zeta(\tau)\,d\tau}, \quad \forall t \geq t_0. \tag{5.28}$$

Therefore, by using the fact that $F(t) \sim E(t)$, we derive

$$E(t) \leq K e^{-k\int_{t_0}^{t} \zeta(\tau)\,d\tau}, \quad \forall t \geq t_0. \tag{5.29}$$

Case 2. $1 < \gamma < \frac{3}{2}$:

Multiplying both sides of (5.24) by $\zeta(t)$, using (4.27) and (4.29) leads to

$$\begin{aligned} \zeta(t)W'(t) &\leq -m\zeta(t)E(t) + C\zeta(t)(g_1 \circ \nabla u)(t) + C\zeta(t)(g_2 \circ \nabla v)(t) \\ &\leq -m\zeta(t)E(t) + C\zeta_1(t)(g_1 \circ \nabla u)(t) + C\zeta_2(t)(g_2 \circ \nabla v)(t) \\ &\leq -m\zeta(t)E(t) + C\big[-E'(t)\big]^{\frac{1}{2\gamma-1}}. \end{aligned} \tag{5.30}$$

Multiplying (5.30) by $\zeta^{\alpha}(t)E^{\alpha}(t)$, with $\alpha = 2\gamma - 2$, we get

$$\zeta^{\alpha+1}(t)E^{\alpha}(t)W'(t) \le -m\zeta^{\alpha+1}(t)E^{\alpha+1}(t) + C(\zeta E)^{\alpha}(t)\big[-E'(t)\big]^{\frac{1}{\alpha+1}}. \tag{5.31}$$

Exploiting Young's inequality with $q = 1 + \alpha$ and $q' = \frac{1+\alpha}{\alpha}$ gives

$$\begin{aligned}\zeta^{\alpha+1}(t)E^{\alpha}(t)W'(t) &\le -m\zeta^{\alpha+1}(t)E^{\alpha+1}(t) + C\big[\epsilon\zeta^{\alpha+1}(t)E^{\alpha+1}(t) - C_{\epsilon}E'(t)\big] \\ &\le -(m - \epsilon C)\zeta^{\alpha+1}(t)E^{\alpha+1}(t) - CE'(t).\end{aligned} \tag{5.32}$$

We pick $\epsilon < \frac{m}{C}$ and recall that $\zeta'(t) \le 0$, $\zeta(t) > 0$ by (G2), $E'(t) \le 0$ by (3.6), and $W \sim E \ge 0$ by (3.21) and (5.18), then together with (5.32) we have, for some $c_1 > 0$,

$$\begin{aligned}\big(\zeta^{\alpha+1}E^{\alpha}W\big)'(t) &= (\alpha+1)\zeta^{\alpha}(t)\zeta'(t)E^{\alpha}(t)W(t) \\ &\quad + \alpha\zeta^{\alpha+1}(t)E^{\alpha-1}(t)E'(t)W(t) \\ &\quad + \zeta^{\alpha+1}(t)E^{\alpha}(t)W'(t) \\ &\le \zeta^{\alpha+1}(t)E^{\alpha}(t)W'(t) \\ &\le -c_1\zeta^{\alpha+1}(t)E^{\alpha+1}(t) - CE'(t).\end{aligned} \tag{5.33}$$

Next, we take $F(t) = \zeta^{\alpha+1}WE^{\alpha} + CE$, which is clearly equivalent to $E(t)$, then there exists $a_0 > 0$ such that

$$F'(t) \le -c_1\zeta^{\alpha+1}(t)E^{\alpha+1}(t) \le -a_0\zeta^{\alpha+1}(t)F^{\alpha+1}(t). \tag{5.34}$$

Integrating (5.34) over (t_0, t) and recalling that $F(t) \sim E(t)$ and $\alpha = 2\gamma - 2$, we obtain

$$E(t) \le k_1\left[\frac{1}{1 + \int_{t_0}^{t}\zeta^{2\gamma-1}(\tau)\,d\tau}\right]^{\frac{1}{2\gamma-2}}, \quad \forall t \ge t_0. \tag{5.35}$$

From (5.21) and (5.35), we infer that

$$\int_{t_0}^{+\infty} E(t)\,dt < +\infty. \tag{5.36}$$

Setting $\lambda_1(t) = \int_0^t \|\nabla u(t) - \nabla u(t-\tau)\|_2^2\,d\tau$, by using (3.3), we deduce

$$\begin{aligned}\lambda_1(t) &= \int_0^t \big\|\nabla u(t) - \nabla u(t-\tau)\big\|_2^2 d\tau \le C\int_0^t \big(\big\|\nabla u(t)\big\|_2^2 + \big\|\nabla u(t-\tau)\big\|_2^2\big)\,d\tau \\ &\le C\int_0^t \big[E(t) + E(t-\tau)\big]\,d\tau \\ &\le 2C\int_0^t E(t-\tau)\,d\tau \\ &= 2C\int_0^t E(\tau)\,d\tau < 2C\int_0^{+\infty} E(\tau)\,d\tau < +\infty.\end{aligned} \tag{5.37}$$

Similarly, let $\lambda_2(t) = \int_0^t \|\nabla v(t) - \nabla v(t-\tau)\|_2^2\,d\tau$, we have $\lambda_2(t) < +\infty$.

From (5.24) and recalling that $\zeta(t) = \min\{\zeta_1(t), \zeta_2(t)\}$, $\zeta_1(t)$ and $\zeta_2(t)$ are nonincreasing differentiable functions, we arrive at

$$\begin{aligned}\zeta(t)W'(t) &\leq -m\zeta(t)E(t) + C\zeta(t)\bigl[(g_1\circ\nabla u)(t) + (g_2\circ\nabla v)(t)\bigr]\\ &\leq -m\zeta(t)E(t) + C\zeta_1(t)(g_1\circ\nabla u)(t) + C\zeta_2(t)(g_2\circ\nabla v)(t)\\ &\leq -m\zeta(t)E(t) + C\frac{\lambda_1(t)}{\lambda_1(t)}\int_0^t\bigl(\zeta_1^\gamma(\tau)g_1^\gamma(\tau)\bigr)^{\frac{1}{\gamma}}\bigl\|\nabla u(t)-\nabla u(t-\tau)\bigr\|_2^2\,d\tau\\ &\quad + C\frac{\lambda_2(t)}{\lambda_2(t)}\int_0^t\bigl(\zeta_2^\gamma(\tau)g_2^\gamma(\tau)\bigr)^{\frac{1}{\gamma}}\bigl\|\nabla v(t)-\nabla v(t-\tau)\bigr\|_2^2\,d\tau.\end{aligned} \tag{5.38}$$

Exploiting Jensen's inequality, for the second term on the right-hand side of (5.38), with $G(y) = y^{\frac{1}{\gamma}}$, $y > 0$, $f(\tau) = \zeta^\gamma(\tau)g^\gamma(\tau)$ and $h(\tau) = \|\nabla u(t)-\nabla u(t-\tau)\|_2^2$, where we assume that $\lambda_1(t), \lambda_2(t) > 0$, otherwise we get $\|\nabla u(t)-\nabla u(t-\tau)\| = \|\nabla v(t)-\nabla v(t-\tau)\| = 0$, then by using (5.24) we deduce

$$E(t) \leq Ce^{-mt}.$$

Since $\zeta_1(t)$ and $\zeta_2(t)$ are nonincreasing, then for some $C_1 > 0$, estimate (5.38) becomes

$$\begin{aligned}\zeta(t)W'(t) &\leq -m\zeta(t)E(t) + C\lambda_1(t)\biggl[\frac{1}{\lambda_1(t)}\int_0^t\zeta_1^\gamma(\tau)g_1^\gamma(\tau)\bigl\|\nabla u(t)-\nabla u(t-\tau)\bigr\|_2^2\,d\tau\biggr]^{\frac{1}{\gamma}}\\ &\quad + C\lambda_2(t)\biggl[\frac{1}{\lambda_2(t)}\int_0^t\zeta_2^\gamma(\tau)g_2^\gamma(\tau)\bigl\|\nabla v(t)-\nabla v(t-\tau)\bigr\|_2^2\,d\tau\biggr]^{\frac{1}{\gamma}}\\ &\leq -m\zeta(t)E(t) + C\lambda_1^{\frac{\gamma-1}{\gamma}}(t)\biggl[\zeta_1^{\gamma-1}(0)\int_0^t\zeta_1(\tau)g_1^\gamma(\tau)\bigl\|\nabla u(t)-\nabla u(t-\tau)\bigr\|_2^2\,d\tau\biggr]^{\frac{1}{\gamma}}\\ &\quad + C\lambda_2^{\frac{\gamma-1}{\gamma}}(t)\biggl[\zeta_2^{\gamma-1}(0)\int_0^t\zeta_2(\tau)g_2^\gamma(\tau)\bigl\|\nabla v(t)-\nabla v(t-\tau)\bigr\|_2^2\,d\tau\biggr]^{\frac{1}{\gamma}}\\ &= -m\zeta(t)E(t) + C\bigl(\lambda_1\zeta_1(0)\bigr)^{\frac{\gamma-1}{\gamma}}\bigl(-g_1'\circ\nabla u\bigr)^{\frac{1}{\gamma}} + C\bigl(\lambda_2\zeta_2(0)\bigr)^{\frac{\gamma-1}{\gamma}}\bigl(-g_2'\circ\nabla v\bigr)^{\frac{1}{\gamma}}\\ &\leq -m\zeta(t)E(t) + C_1\bigl[-E'(t)\bigr]^{\frac{1}{\gamma}}.\end{aligned} \tag{5.39}$$

Multiplying both sides of (5.39) by $\zeta^\alpha(t)E^\alpha(t)$, with $\alpha = \gamma - 1$, we deduce

$$\zeta^{\alpha+1}(t)E^\alpha(t)W'(t) \leq -m\zeta^{\alpha+1}(t)E^{\alpha+1}(t) + C_1\zeta^\alpha(t)E^\alpha(t)\bigl[-E'(t)\bigr]^{\frac{1}{\alpha+1}}. \tag{5.40}$$

Applying Young's inequality, with $q = 1+\alpha$, and $q' = \frac{1+\alpha}{\alpha}$ leads to

$$\begin{aligned}\zeta^{\alpha+1}(t)E^\alpha(t)W'(t) &\leq -m\zeta^{\alpha+1}(t)E^{\alpha+1}(t) + C_1\bigl(\sigma\zeta^{\alpha+1}(t)E^{\alpha+1}(t) - C_\sigma E'(t)\bigr)\\ &= -(m - C_1\sigma)\zeta^{\alpha+1}(t)E^{\alpha+1}(t) - CE'(t).\end{aligned} \tag{5.41}$$

Then, by taking $\sigma < \frac{m}{C_1}$ and recalling that $\zeta'(t) \leq 0$, $\zeta(t) > 0$ by (G2), $E'(t) \leq 0$ by (3.6), and $W(t) \sim E(t) \geq 0$ by (3.21) and (5.18), together with (5.41), we get, for some $C_2 > 0$,

$$\bigl(\zeta^{\alpha+1}E^\alpha W\bigr)'(t) = (\alpha+1)\zeta^\alpha(t)\zeta'(t)E^\alpha(t)W(t)$$

$$\begin{aligned}
&+ \alpha \zeta^{\alpha+1}(t)E^{\alpha-1}(t)E'(t)W(t) \\
&+ \zeta^{\alpha+1}(t)E^{\alpha}(t)W'(t) \\
\le{}& \zeta^{\alpha+1}(t)E^{\alpha}(t)W'(t) \\
\le{}& -C_2\zeta^{\alpha+1}(t)E^{\alpha+1}(t) - CE'(t),
\end{aligned} \tag{5.42}$$

which implies

$$\bigl(\zeta^{\alpha+1}E^{\alpha}W + CE\bigr)'(t) \le -C_2\zeta^{\alpha+1}(t)E^{\alpha+1}(t). \tag{5.43}$$

Let $L = \zeta^{\alpha+1}E^{\alpha}W + CE$, then clearly $L \sim E$, we obtain, for some $C_3 > 0$,

$$L'(t) \le -C_3\zeta^{\alpha+1}(t)L^{\alpha+1}(t), \quad \forall t \ge t_0. \tag{5.44}$$

Integrating (5.44) over (t_0, t) and recalling that $L \sim E$ and $\alpha = \gamma - 1$ yields

$$E(t) \le k_2\left[\frac{1}{1 + \int_{t_0}^{t} \zeta^{\gamma}(\tau)\,d\tau}\right]^{\frac{1}{\gamma-1}}, \quad \forall t \ge t_0. \tag{5.45}$$

This completes the proof. □

6 The blow-up result

In this section, we carry out the proof of the finite time blow-up result.

Theorem 6.1 *If* (G1) *and* (G3) *hold and the initial energy* $E(0) < 0$. *Assume that* g_i *satisfies*

$$\int_0^{\infty} g_i(\tau)\,d\tau < \frac{p+1}{p+2}, \quad i = 1, 2. \tag{6.1}$$

Then the solution of system (1.1) *blows up in finite time*.

Proof First, we define

$$H(t) = -E(t), \qquad G(t) = \int_{\Omega} F(u, v)\,dx, \tag{6.2}$$

where $E(t)$ is defined in (3.3). By (G1) and (3.6), we find that

$$\begin{aligned}
H'(t) = -E'(t) ={}& -\frac{1}{2}\bigl[\bigl(g_1' \circ \nabla u\bigr)(t) + \bigl(g_2' \circ \nabla v\bigr)(t)\bigr] + \|u_t\|_2^2 \\
&+ \|v_t\|_2^2 + \frac{g_1(t)}{2}\|\nabla u\|_2^2 + \frac{g_2(t)}{2}\|\nabla v\|_2^2 \\
\ge{}& \|u_t\|_2^2 + \|v_t\|_2^2 \\
\ge{}& 0.
\end{aligned} \tag{6.3}$$

Noting the assumption $E(0) < 0$, from (4.1), (6.2), and (6.3), we get

$$0 < H(0) \le H(t) \le G(t) \le c_1\bigl(\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}\bigr). \tag{6.4}$$

From (4.1), we see that

$$G(t) \geq c_0\big(\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}\big). \tag{6.5}$$

Let

$$L(t) = H^{1-\alpha}(t) + \epsilon \int_\Omega uu_t\,dx + \epsilon \int_\Omega vv_t\,dx, \tag{6.6}$$

where ϵ is a positive constant to be chosen later and

$$0 < \alpha < \min\left\{\frac{1}{2} - \frac{1}{2(p+2)}, \frac{p}{2p+2}\right\}. \tag{6.7}$$

Taking the derivative of $L(t)$ and using system (1.1) gives

$$\begin{aligned} L'(t) &= (1-\alpha)H^{-\alpha}(t)H'(t) + \epsilon \int_\Omega uu_{tt}\,dx + \epsilon \int_\Omega vv_{tt}\,dx + \epsilon\|u_t\|_2^2 + \epsilon\|v_t\|_2^2 \\ &= (1-\alpha)H^{-\alpha}(t)H'(t) + \epsilon \int_0^t g_1(t-\tau)\int_\Omega \nabla u(\tau)\nabla u(t)\,dx\,d\tau \\ &\quad + \epsilon \int_0^t g_2(t-\tau)\int_\Omega \nabla u(\tau)\nabla u(t)\,dx\,d\tau - \epsilon\|\nabla u\|_2^2 - \epsilon\|\nabla v\|_2^2 \\ &\quad - \epsilon \int_\Omega uu_t\,dx - \epsilon \int_\Omega vv_t\,dx + 2(p+2)\epsilon G(t). \end{aligned} \tag{6.8}$$

It follows from Young's inequality that

$$\begin{aligned} &\int_0^t g_1(t-\tau)\int_\Omega \nabla u(\tau)\nabla u(t)\,dx\,d\tau \\ &\quad= \int_0^t g_1(t-\tau)\int_\Omega \nabla u(t)\big(\nabla u(\tau) - \nabla u(t)\big)\,dx\,d\tau \\ &\qquad + \int_0^t g_1(\tau)\,d\tau\,\big\|\nabla u(t)\big\|_2^2 \\ &\quad\geq -\int_0^t g_1(\tau)\,d\tau\,\big\|\nabla u(t)\big\|_2^2 - \frac{1}{4}(g_1 \circ \nabla u)(t) \\ &\qquad + \int_0^t g_1(\tau)\,d\tau\,\big\|\nabla u(t)\big\|_2^2 \\ &\quad= -\frac{1}{4}(g_1 \circ \nabla u)(t). \end{aligned} \tag{6.9}$$

Similarly,

$$\int_0^t g_2(t-\tau)\int_\Omega \nabla v(\tau)\nabla v(t)\,dx\,d\tau \geq -\frac{1}{4}(g_2 \circ \nabla v)(t). \tag{6.10}$$

A combination of (6.8)–(6.10) leads to

$$L'(t) \geq (1-\alpha)H^{-\alpha}(t)H'(t) - \frac{\epsilon}{4}(g_1 \circ \nabla u)(t) - \frac{\epsilon}{4}(g_2 \circ \nabla v)(t) - \epsilon\|\nabla u\|_2^2$$

$$- \epsilon \|\nabla v\|_2^2 - \epsilon \int_\Omega u u_t \, dx - \epsilon \int_\Omega v v_t \, dx + \epsilon \|u_t\|_2^2 + \epsilon \|v_t\|_2^2 + 2(p+2)\epsilon G(t). \tag{6.11}$$

Let $0 < r \le \min\{l, k\}$, then by the expression of $E(t)$ and $H(t)$, we obtain

$$-\left(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2\right) \ge \frac{2}{r} H(t) + \frac{1}{r}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) - \frac{2}{r} G(t) + \frac{1}{r}\left[(g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)\right]. \tag{6.12}$$

Inserting (6.12) into (6.11), we arrive at

$$L'(t) \ge (1-\alpha) H^{-\alpha}(t) H'(t) + \epsilon \frac{2}{r} H(t) + \epsilon \left(1 + \frac{1}{r}\right)\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + \epsilon \left(\frac{1}{r} - \frac{1}{4}\right)\left[(g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)\right] + \epsilon \left[2(p+2) - \frac{2}{r}\right] G(t) - \epsilon \int_\Omega u u_t \, dx - \epsilon \int_\Omega v v_t \, dx. \tag{6.13}$$

By using (G1), (6.1), and $r = \min\{l, k\}$, we infer that

$$2(p+2) - \frac{2}{r} > 0, \qquad \frac{1}{r} - \frac{1}{4} > 0. \tag{6.14}$$

For the last two terms of (6.13), by using Hölder's inequality and (6.3)–(6.5) yields

$$\begin{aligned} \int_\Omega u u_t \, dx &\le \|u\|_2 \|u_t\|_2 \\ &\le |\Omega|^{\frac{p+1}{2(p+2)}} \|u\|_{2(p+2)} \|u_t\|_2 \\ &\le c_0^{\frac{-1}{2(p+2)}} |\Omega|^{\frac{p+1}{2(p+2)}} G^{\frac{1}{2(p+2)}}(t) \|u_t\|_2 \\ &\le K_1 \|u_t\|_2 G^{\frac{1}{2}}(t) H^{\frac{1}{2(p+2)} - \frac{1}{2}}(t), \end{aligned} \tag{6.15}$$

where $K_1 = c_0^{\frac{-1}{2(p+2)}} |\Omega|^{\frac{p+1}{2(p+2)}}$.

By (6.7), we know $\frac{1}{2(p+2)} - \frac{1}{2} + \alpha < 0$, it follows from Young's inequality and (6.3) that

$$\begin{aligned} \int_\Omega u u_t \, dx &\le K_1 \|u_t\|_2 G^{\frac{1}{2}}(t) H^{\frac{1}{2(p+2)} - \frac{1}{2}}(t) \\ &\le H^{\frac{1}{2(p+2)} - \frac{1}{2}}(t)\left(\delta_1 G(t) + K_1^2 \delta_1^{-\frac{1}{2}} \|u_t\|_2^2\right) \\ &\le \delta_1 H^{\frac{1}{2(p+2)} - \frac{1}{2}}(0) G(t) + K_1^2 \delta_1^{-\frac{1}{2}} H^{\frac{1}{2(p+2)} - \frac{1}{2} + \alpha}(t) H^{-\alpha}(t) H'(t) \\ &\le \delta_1 H^{\frac{1}{2(p+2)} - \frac{1}{2}}(0) G(t) + K_1^2 \delta_1^{-\frac{1}{2}} H^{\frac{1}{2(p+2)} - \frac{1}{2} + \alpha}(0) H^{-\alpha}(t) H'(t), \quad \forall \delta_1 > 0. \end{aligned} \tag{6.16}$$

Likewise,

$$\int_\Omega v v_t \, dx \le \delta_2 H^{\frac{1}{2(p+2)} - \frac{1}{2}}(0) G(t) + K_1^2 \delta_1^{-\frac{1}{2}} H^{\frac{1}{2(p+2)} - \frac{1}{2} + \alpha}(0) H^{-\alpha}(t) H'(t), \quad \forall \delta_2 > 0. \tag{6.17}$$

We pick

$$\delta_1 = \delta_2 = \frac{1}{4}\left[2(p+2) - \frac{2}{r}\right] H^{\frac{1}{2(p+2)} - \frac{1}{2}}(0), \tag{6.18}$$

then it follows from (6.13)–(6.18) that

$$\begin{aligned} L'(t) &\geq (1 - \alpha - \epsilon K_3) H^{-\alpha}(t) H'(t) + \epsilon \frac{2}{r} H(t) + \epsilon \left(1 + \frac{1}{r}\right)\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\ &\quad + \epsilon\left(\frac{1}{r} - \frac{1}{4}\right)\left[(g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)\right] + \frac{\epsilon}{2}\left[2(p+2) - \frac{2}{r}\right] G(t), \end{aligned} \tag{6.19}$$

where

$$\begin{aligned} K_3 &= K_1^2 \delta_1^{-\frac{1}{2}} H^{\frac{1}{2(p+2)} - \frac{1}{2} + \alpha}(0) + K_1^2 \delta_1^{-\frac{1}{2}} H^{\frac{1}{2(p+2)} - \frac{1}{2} + \alpha}(0) \\ &= 4\left[2(p+2) - \frac{2}{r}\right]^{-\frac{1}{2}} K_1^2 H^{\frac{1}{4(p+2)} - \frac{1}{4} + \alpha}(0). \end{aligned} \tag{6.20}$$

Now, we choose $0 < \epsilon < 1$ small enough such that

$$1 - \alpha - \epsilon K_3 \geq 0. \tag{6.21}$$

A combination of (6.14), (6.19), and (6.21) yields

$$\begin{aligned} L'(t) &\geq (1 - \alpha - \epsilon K_3) H^{-\alpha}(t) H'(t) + \epsilon \frac{2}{r} H(t) + \epsilon \left(1 + \frac{1}{r}\right)\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\ &\quad + \epsilon\left(\frac{1}{r} - \frac{1}{4}\right)\left[(g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)\right] + \frac{\epsilon}{2}\left[2(p+2) - \frac{2}{r}\right] G(t) \\ &\geq \epsilon C\left[H(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + (g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t) + G(t)\right] \\ &> 0, \end{aligned} \tag{6.22}$$

where C is a positive constant. It is clear that $L(t)$ is increasing on $[0, T)$ and

$$\begin{aligned} L(t) &= H^{1-\alpha}(t) + \epsilon \int_\Omega u u_t \, dx + \epsilon \int_\Omega v v_t \, dx \\ &\geq H^{1-\alpha}(0) + \epsilon \int_\Omega u_0 u_1 \, dx + \epsilon \int_\Omega v_0 v_1 \, dx. \end{aligned} \tag{6.23}$$

In the case of $\int_\Omega u_0 u_1 \, dx + \epsilon \int_\Omega v_0 v_1 \, dx \geq 0$, no further restriction on ϵ is needed. For the case $\int_\Omega u_0 u_1 \, dx + \epsilon \int_\Omega v_0 v_1 \, dx < 0$, we assume that

$$0 < \epsilon < -\frac{H^{1-\alpha}(0)}{2\int_\Omega u_0 u_1 \, dx + 2\int_\Omega v_0 v_1 \, dx}. \tag{6.24}$$

Therefore, in either case, we have

$$L(t) \geq \frac{1}{2} H^{1-\alpha}(0) > 0, \quad \forall t \in [0, T). \tag{6.25}$$

Now we prove that $L(t)$ satisfies the following inequality:

$$L'(t) \geq C\epsilon L^{\frac{1}{1-\alpha}}(t), \quad t \in [0, T), \tag{6.26}$$

where C is a positive constant, $1 < \frac{1}{1-\alpha} < 2$ assumed by (6.7). At this point, we distinguish two cases.

Case 1. $\int_\Omega uu_t\,dx + \int_\Omega vv_t\,dx \leq 0$ for some $t \in [0, T)$. For such t, we obtain

$$L^{\frac{1}{1-\alpha}}(t) = \left(H^{1-\alpha}(t) + \epsilon \int_\Omega uu_t\,dx + \epsilon \int_\Omega vv_t\,dx\right)^{\frac{1}{1-\alpha}} \leq H(t), \tag{6.27}$$

which together with (6.22), then (6.26) follows for all such t.

Case 2. $\int_\Omega uu_t\,dx + \int_\Omega vv_t\,dx \geq 0$ for all $t \in [0, T)$. Since $0 < \alpha < \frac{1}{2}$, $1 < \frac{1}{1-\alpha} < 2$, and $0 < \epsilon < 1$, then we deduce

$$L^{\frac{1}{1-\alpha}}(t) \leq 2^{\frac{1}{1-\alpha}-1}\left(H(t) + \left|\int_\Omega uu_t\,dx + \epsilon \int_\Omega vv_t\,dx\right|^{\frac{1}{1-\alpha}}\right). \tag{6.28}$$

Exploiting Hölder's inequality and Young's inequality, we see that

$$\begin{aligned} \left|\int_\Omega uu_t\,dx + \epsilon \int_\Omega vv_t\,dx\right|^{\frac{1}{1-\alpha}} &\leq C\left(\|u\|_2^{\frac{1}{1-\alpha}}\|u_t\|_2^{\frac{1}{1-\alpha}} + \|v\|_2^{\frac{1}{1-\alpha}}\|v_t\|_2^{\frac{1}{1-\alpha}}\right) \\ &\leq C\left(\|u\|_{2p+2}^{\frac{1}{1-\alpha}}\|u_t\|_2^{\frac{1}{1-\alpha}} + \|v\|_{2p+2}^{\frac{1}{1-\alpha}}\|v_t\|_2^{\frac{1}{1-\alpha}}\right) \\ &\leq C\left(\|u\|_{2p+2}^{\frac{2}{1-2\alpha}} + \|u_t\|_2^2 + \|v\|_{2p+2}^{\frac{2}{1-2\alpha}} + \|v_t\|_2^2\right). \end{aligned} \tag{6.29}$$

From (6.7), we get that $\frac{1}{(1-2\alpha)(p+1)} - 1 < 0$. Then, by (6.3) and (6.5), we have

$$\begin{aligned} \|u\|_{2p+2}^{\frac{2}{1-2\alpha}} &= \left(\|u\|_{2p+2}^{2p+2}\right)^{\frac{1}{(1-2\alpha)(p+1)}} \\ &\leq CG^{\frac{1}{(1-2\alpha)(p+1)}}(t) \\ &\leq CG^{\frac{1}{(1-2\alpha)(p+1)}-1}(t)G(t) \\ &\leq CH^{\frac{1}{(1-2\alpha)(p+1)}-1}(0)G(t) \\ &\leq CG(t). \end{aligned} \tag{6.30}$$

Similarly,

$$\|u\|_{2p+2}^{\frac{2}{1-2\alpha}} \leq CG(t). \tag{6.31}$$

By combining (6.28)–(6.31), we deduce that

$$L^{\frac{1}{1-\alpha}}(t) \leq C\left(H(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + G(t)\right). \tag{6.32}$$

It follows from (6.22) and (6.32) that

$$L'(t) \geq C\epsilon\left(H(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + G(t)\right) \geq C\epsilon L^{\frac{1}{1-\alpha}}(t), \tag{6.33}$$

which shows (6.26) follows for Case 2.

Integrating (6.26) over $(0,t)$ yields

$$L^{\frac{\alpha}{1-\alpha}}(t) \geq \frac{1}{L^{\frac{-\alpha}{1-\alpha}}(0) - \frac{\alpha}{1-\alpha} C\epsilon t}. \tag{6.34}$$

This shows that $G(t)$ blows up in finite time

$$T^* \leq \frac{1-\alpha}{\alpha C \epsilon L^{\frac{\sigma}{1-\sigma}}(0)}, \quad \forall t \geq 0. \tag{6.35}$$

Therefore, the solution of system (1.1) blows up in finite time. □

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