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Anti-periodic solutions problem for inertial competitive neutral-type neural networks via Wirtinger inequality

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Abstract

By using the Wirtinger inequality and topology degree theory, we investigate the anti-periodic solutions problem for inertial competitive neutral-type neural networks and obtain the existence of anti-periodic solutions to the above system. Our results are completely new.

MSC: 34K20

Keywords: Wirtinger inequality; Anti-periodic solution; Existence

1 Introduction

The Wirtinger inequality was first used in Fourier analysis, then was used in 1904 to prove the isoperimetric inequality; see [1, 2]. Since the Wirtinger inequality has been recognized as a powerful tool to estimate the *prior* bounds of solutions, it has been used in many research areas, such as Hamiltonian system, delay equations, biomathematics, neural networks, partial differential equation; see [3–8] and the relevant references therein.

In this paper, we will obtain the existence of anti-periodic solutions for inertial competitive neutral-type neural networks by using the Wirtinger inequality and topology degree theory. In 1996, Meyer, Ohl and Scheich [9] first proposed the competitive neural network, the behavior of this network is characterized by an equation of neural activity as a fast phenomenon and an equation of synaptic modification as a slow part of the neural system. Since by a competitive neural network one can study the dynamics of complex neural networks including the aspects of long- and short-term memory, it has received great attention. Meyer, Pilyugin and Chen [10] studied global exponential stability of competitive neural networks with different time scales. They presented a new method of analyzing the dynamics of a biological relevant system with different time scales based on the theory of flow invariance. After that, Meyer, Reberts and Thmmle [11] further studied the local uniform stability of competitive neural networks with different time scales under vanishing perturbations. Gu, Jiang and Teng [12] obtained the existence and global exponential stability of a unique equilibrium point of competitive neural networks with different time scales and multiple delays by using a nonlinear Lipschitz measure (NLM) method and constructing a suitable Lyapunov functional. Liu *et al.* [13] obtained the existence of a pe-

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riodic solution for competitive neural networks with time-varying and distributed delays on time scales.

On the other hand, the anti-periodic solution problem for dynamic systems is an interesting topic which has been investigated by many researchers. Okochi [14] first considered the anti-periodic solution of nonlinear abstract parabolic equations. Then anti-periodic problems of neural networks have attracted much attention by many authors. Li, Yang and Wu [15] discussed an anti-periodic solution for impulsive BAM neural networks with time-varying leakage delays on time scales. Then Li *et al.* [16–18] further studied anti-periodic solutions for different types of neural networks. Xu, Chen and Guo [19] studied anti-periodic oscillations of bidirectional associative memory (BAM) neural networks with leakage delays. The existence and exponentially stability of anti-periodic solutions for neutral BAM neural networks with time-varying delays in the leakage terms have been obtained by Xu and Guo [20]. However, to the best of our knowledge there are only few results on the anti-periodic solutions for neutral-type competitive neural networks with inertial terms.

The above discussions constitute the motivation for the present paper. In this paper, we will study a kind of neutral-type competitive neural networks with inertial terms. Based on the Wirtinger inequality and topology degree theory, we prove the existence of anti-periodic solutions of the above neural network. We list the main contributions of this paper as follows:

(1) We propose a class of inertial competitive neutral-type neural networks which is different from the existing competitive neural networks; see [9–11, 13, 15, 19].
(2) Since the model of the present paper contains neutral terms, it is very difficult to estimate an a priori bound. In order to overcome this difficulty, we use the Wirtinger inequality and develop some new mathematical analysis techniques.
(3) A unified framework is established to handle competitive neural networks with neutral-type terms, time-varying delays and inertial terms.

The subsequent sections are organized as follows: In Sect. 2, the description of the model, some useful lemmas and notations are given. In Sect. 3, sufficient conditions are established for existence of anti-periodic solutions of considered neural networks. In Sect. 4, an example is given to show the feasibility of our results. Finally, Sect. 5 concludes the paper.

2 Model description and Lemmas

Consider the following generalized inertial competitive neutral-type neural networks with time-varying delays:

$$\begin{cases} \varepsilon(A_ix_i)''(t) = -\alpha_i(t)x_i'(t) - \beta_i(t)x_i(t) + \sum_{j=1}^n b_{ij}(t)f_j(x_j(t)) \\ \qquad\qquad + \sum_{j=1}^n c_{ij}(t)f_j(x_j(t-\gamma_{ij}(t))) + B_i(t)\sum_{j=1}^n d_j m_{ij}(t), \\ m_{ij}'(t) = -m_{ij}(t) + d_j f_i(x_i(t)), \end{cases} \tag{2.1}$$

where $i, j = 1, 2, \ldots, n$; $(A_ix_i)(t)$ is a neutral operator which is defined by

$$(A_ix_i)(t) = x_i(t) - c_ix_i(t-\sigma), \tag{2.2}$$

c_i and σ are constants with $|c_i| \neq 1$ and $\sigma > 0$; the second order derivative is an inertial term; $x_i(t)$ is the neuron current activity level; $\varepsilon > 0$ is a fast time scale; $\alpha_i(t) > 0$ is a variable coefficient; $\beta_i(t) > 0$ is a damping coefficient; $b_{ij}(t)$ and $c_{ij}(t)$ represent the connection weight and the synaptic weight of delayed synaptic efficiency; $m_{ij}(t)$ the is synaptic efficiency; d_j is the constant external stimulus; $B_i(t)$ is the strength of the external stimulus; $f_j(x_j(t))$ is the output of neurons; $\gamma_{ij}(t) > 0$ is a transmission delay.

The initial values of system (2.1) are given by

$$\begin{cases} x_i(s) = \phi_i(s), \qquad x_i'(s) = \theta_i(s), \\ m_{ij}(s) = \mu_{ij}(s), \end{cases} \tag{2.3}$$

where $i, j = 1, 2, \ldots, n$ and $s \in [-\tau, 0]$ with $\tau = \max_{1\le i,j\le n}\{\sigma, \gamma_{ij}(t), t \in \mathbb{R}\}$.

Let $\varepsilon = 1$ and $z_i(t) = \sum_{j=1}^n d_j m_{ij}(t)$, then (2.1) is transformed into

$$\begin{cases} (A_i x_i)''(t) = -\alpha_i(t)x_i'(t) - \beta_i(t)x_i(t) + \sum_{j=1}^n b_{ij}(t)f_j(x_j(t)) \\ \qquad\qquad + \sum_{j=1}^n c_{ij}(t)f_j(x_j(t-\gamma_{ij}(t))) + B_i(t)z_i(t), \\ z_i'(t) = -z_i(t) + df_i(x_i(t)), \end{cases} \tag{2.4}$$

where $d = \sum_{j=1}^n d_j^2 > 0$. Let

$$y_i(t) = (A_i x_i)'(t) + x_i(t), \quad i = 1, 2, \ldots, n, \tag{2.5}$$

then (2.4) can be written as follows:

$$\begin{cases} (A_i x_i)'(t) = -x_i(t) + y_i(t) := F_i(\cdot), \\ y_i'(t) = -(\alpha_i(t) - 1)x_i'(t) - \beta_i(t)x_i(t) + \sum_{j=1}^n b_{ij}(t)f_j(x_j(t)) \\ \qquad + \sum_{j=1}^n c_{ij}(t)f_j(x_j(t-\gamma_{ij}(t))) + B_i(t)z_i(t) := G_i(\cdot), \\ z_i'(t) = -z_i(t) + df_i(x_i(t)) := H_i(\cdot). \end{cases} \tag{2.6}$$

In view of the initial values of system (2.1), we can obtain the initial values of system (2.6)

$$\begin{cases} (A_i x_i)(s) = \phi_i(s) - c_i\phi_i(s-\sigma) := \varphi_i(s), \\ y_i(s) = \phi_i(s) + \theta_i(s) - c_i\theta_i(s-\sigma) := \upsilon_i(s), \\ z_i(s) = \sum_{j=1}^n d_j m_{ij}(s) := \omega_i(s), \end{cases}$$

where $i, j = 1, 2, \ldots, n$ and $s \in [-\tau, 0]$.

Now, we give the famous Wirtinger inequality.

Lemma 2.1 ([21, 22] Wirtinger inequality) *If u is a C^1 function such that $u(0) = u(T)$, then*

$$\|u - \bar{u}\|_{L_2} \le \frac{T}{2\pi}\|u'\|_{L_2},$$

where $\|u'\|_{L_2} = (\int_0^T |u(t)|^2\,dt)^{\frac{1}{2}}$ and $\bar{u} = \frac{1}{T}\int_0^T |u(t)|\,dt$.

Remark 2.1 When u is an anti-periodic function, i.e., $u(t+T) = -u(t)$, $\forall t \in \mathbb{R}$, then

$$u(0) = u(2T), \qquad \bar{u} = \frac{1}{T}\int_0^{2T} |u(t)|\,dt = 0,$$

and the Wirtinger inequality is given by

$$\left(\int_0^{2T}\left|u(t)\right|^2 dt\right)^{\frac{1}{2}} \le \frac{T}{\pi}\left(\int_0^{2T}\left|u'(t)\right|^2 dt\right)^{\frac{1}{2}}.$$

In this paper, we also need the following lemmas.

Lemma 2.2 ([22]) *Let X and Y be Banach spaces, and let $L : \operatorname{Dom} L \subset X \to Y$ be linear, $N : X \to Y$ be continuous. Assume that L is one-to-one and $K := L^{-1}N$ is compact. Furthermore, assume there exists a bounded and open subset $\Omega \subset X$ with $0 \in \Omega$ such that the equation $Lu = \lambda Nu$ has no solutions in $\partial\Omega \cup \operatorname{Dom} L$ for any $\lambda \in (0,1)$. Then the problem $Lu = Nu$ has at least one solution in $\bar{\Omega}$.*

Lemma 2.3 ([23, 24]) *Define A on C_T*

$$A : C_T \to C_T, \qquad [Ax](t) = x(t) - cx(t-\tau), \quad \forall t \in \mathbb{R},$$

where $C_T = \{x : x \in C(\mathbb{R},\mathbb{R}), x(t+T) \equiv x(t)\}$, c is constant. When $|c| \ne 1$, then A has a unique continuous bounded inverse A^{-1} satisfying

$$\left[A^{-1}f\right](t) = \begin{cases} \sum_{j\ge 0} c^j f(t-j\tau), & \text{if } |c| < 1, \forall f \in C_T, \\ -\sum_{j\ge 1} c^{-j} f(t+j\tau), & \text{if } |c| > 1, \forall f \in C_T. \end{cases}$$

Obviously, we have

(1) $\|A^{-1}\| \le \frac{1}{|1-|c||}$;
(2) $\int_0^T |[A^{-1}f](t)|\,dt \le \frac{1}{|1-|c||}\int_0^T |f(t)|\,dt$, $\forall f \in C_T$;
(3) $\int_0^T |[A^{-1}f](t)|^2\,dt \le \frac{1}{|1-|c||}\int_0^T |f(t)|^2\,dt$, $\forall f \in C_T$.

In what follows, for $u = (u_1, u_2, \ldots, u_{3n})^\top$, denote

$$\|u\| = \sum_{k=1}^{3n} |u_k|.$$

For $i, j = 1, 2, \ldots, n$, we list the following notations which will be used in this paper:

$$\alpha_i^- = \inf_{t\in\mathbb{R}} \alpha_i(t), \qquad \beta_i^+ = \sup_{t\in\mathbb{R}} \beta_i(t), \qquad b_{ij}^+ = \sup_{t\in\mathbb{R}}\left|b_{ij}(t)\right|,$$
$$b_{ij}^+ = \sup_{t\in\mathbb{R}}\left|b_{ij}(t)\right|, \qquad B_i^+ = \sup_{t\in\mathbb{R}}\left|B_i(t)\right|.$$

Throughout this paper, we need the following assumptions:

(H_1) For $i, j = 1, 2, \ldots, n$ and $t, x \in \mathbb{R}$, $\alpha_i, \beta_i, b_{ij}, c_{ij}, B_i, \gamma_{ij} \in C(\mathbb{R},\mathbb{R})$ with

$$\alpha_i(t+\omega) = \alpha_i(t), \qquad \beta_i(t+\omega) = \beta_i(t),$$
$$B_i(t+\omega) = B_i(t), \qquad \gamma_{ij}(t+\omega) = \gamma_{ij}(t),$$
$$b_{ij}(t+\omega)f_j(x) = -b_{ij}(t)f_j(-x), \qquad c_{ij}(t+\omega)f_j(x) = -c_{ij}(t)f_j(-x).$$

(H_2) For $j = 1,2,\dots,n$ and $x, y \in \mathbb{R}$, there exist positive constants $L_j > 0$

$$\bigl|f_j(x) - f_j(y)\bigr| \le L_j|x - y|.$$

(H_3) For $i, j = 1,2,\dots,n$

$$1 - \gamma_{ij}'(t) > 0, \quad t \in \mathbb{R}.$$

Remark 2.2 From $1 - \gamma_{ij}'(t) > 0$, $t \in \mathbb{R}$, it is easy to see that $t - \gamma_{ij}(t)$ has the inverse function Γ_{ij} for $i, j = 1,2,\dots,n$ and $t \in \mathbb{R}$. Hence, let $t - \gamma_{ij}(t) = u_{ij}$, then $t = \Gamma_{ij}(u_{ij})$.

3 Main results

Let

$$X = \bigl\{u = (x,y,z) = (x_1,\dots,x_n,y_1,\dots,y_n,z_1,\dots,z_n,)^\top \in C\bigl(\mathbb{R},\mathbb{R}^{3n}\bigr), u(t+\omega) = -u(t)\bigr\}$$

with the norm

$$\|u\|_X = \sum_{i=1}^{n}\bigl(|x_i|_\infty + |y_i|_\infty + |z_i|_\infty\bigr), \qquad |f|_\infty = \sup_{t\in\mathbb{R}}\bigl|f(t)\bigr|.$$

Clearly, X is a Banach space. Let

$$L : D(L) \subset X \to X, \qquad Lu = \bigl((A_1x_1)',\dots,(A_nx_n)',y_1',\dots,y_n',z_1',\dots,z_n',\bigr)^\top, \tag{3.1}$$

where $D(L) = \{u : u \in X, (A_ix_i)', y_i', z_i' \in X\}$. Let there be a nonlinear operator $N : X \to X$:

$$(Nu)(t) = \bigl(F_1(\cdot),\dots,F_n(\cdot),G_1(\cdot),\dots,G_n(\cdot),H_1(\cdot),\dots,H_n(\cdot)\bigr)^\top. \tag{3.2}$$

Clearly,

$$\operatorname{Ker} L = \mathbb{R}^{3n}, \qquad \operatorname{Im} L = \biggl\{u : u \in X, \int_0^{2\omega} u(s)\,ds = \mathbf{0}\biggr\}.$$

For $u = (x_1,\dots,x_n,y_1,\dots,y_n,z_1,\dots,z_n)^\top \in \operatorname{Im} L$, let the inverse operator of L be L^{-1} as follows:

$$\begin{aligned}\bigl(L^{-1}u\bigr)(t) = {}&\bigl(\bigl(A_1^{-1}F_1x_1\bigr)(t),\dots,\bigl(A_n^{-1}F_nx_n\bigr)(t),(F_1y_1)(t),\dots,\\ &(F_ny_n)(t),(F_1z_1)(t),\dots,(F_nz_n)(t)\bigr)^\top,\end{aligned}$$

where

$$(F_iu_i)(t) = \int_0^T G(t,s)u_i(s)\,ds, \qquad G(t,s) = \begin{cases} \frac{s}{T}, & 0 \le s < t \le T,\\ \frac{s-T}{T}, & 0 \le t < s \le T.\end{cases}$$

Theorem 3.1 *Assume that the assumptions* (H_1)–(H_3) *hold. Furthermore, the following assumption holds*:

(H_4)

$$\begin{aligned}\Theta = \min_{1\le i\le n}\Bigg\{ 1 &- \sum_{i=1}^{n}\frac{\omega|1-\alpha_i^-|}{|1-|c_i||\pi} \\ &- \sum_{i=1}^{n}\Bigg[\beta_i^+ + \sum_{j=1}^{n}\Bigg(b_{ij}^+L_j + c_{ij}^+L_j\max_{s\in\mathbb{R}}\frac{1}{|1-\gamma_{ij}'(s)|}\Bigg)\Bigg]\frac{\omega^2}{|1-|c_i||\pi^2} \\ &- \sum_{i=1}^{n}\frac{B_i^+ dL_i\omega^3}{|1-|c_i||\pi^3}\Bigg\} > 0.\end{aligned}$$

Then system (2.1) *has at least one anti-periodic solution.*

Proof Consider the operator equation

$$Lu = \lambda Nu, \qquad u\in D(L), \lambda\in(0,1), \tag{3.3}$$

where L and N are defined by (3.1) and (3.2). Let $u\in D(L)$ be an arbitrary solution of (3.3), then

$$\begin{cases}(A_ix_i)'(t) = \lambda F_i(\cdot), \\ y_i'(t) = \lambda G_i(\cdot), \qquad i = 1,2,\ldots,n. \\ z_i'(t) = \lambda H_i(\cdot),\end{cases} \tag{3.4}$$

Multiplying by $y_i'(t)$ on both sides of the second equation of (3.4) and integrating it over $[0,2\omega]$, we have

$$\begin{aligned}&\int_0^{2\omega}\left|y_i'(t)\right|^2 dt \\ &\quad= \lambda\int_0^{2\omega}\Bigg[-\big(\alpha_i(t)-1\big)x_i'(t)y_i'(t) - \beta_i(t)x_i(t)y_i'(t) + \sum_{j=1}^{n}b_{ij}(t)f_j\big(x_j(t)\big)y_i'(t) \\ &\qquad+ \sum_{j=1}^{n}c_{ij}(t)f_j\big(x_j\big(t-\gamma_{ij}(t)\big)\big)y_i'(t) + B_i(t)z_i(t)y_i'(t)\Bigg] \\ &\quad= \lambda\int_0^{2\omega}\Bigg[-\big(\alpha_i(t)-1\big)x_i'(t)y_i'(t) - \beta_i(t)x_i(t)y_i'(t) + \sum_{j=1}^{n}b_{ij}(t)\big(f_j\big(x_j(t)\big)-f_j(0)\big)y_i'(t) \\ &\qquad+ \sum_{j=1}^{n}c_{ij}(t)\big(f_j\big(x_j\big(t-\gamma_{ij}(t)\big)\big)-f_j(0)\big)y_i'(t) \\ &\qquad+ \sum_{j=1}^{n}\big(b_{ij}(t)+c_{ij}(t)\big)f_j(0)y_i'(t) + B_i(t)z_i(t)y_i'(t)\Bigg] \\ &\quad\le \left|1-\alpha_i^-\right|\int_0^{2\omega}\left|x_i'(t)y_i'(t)\right| dt + \beta_i^+\int_0^{2\omega}\left|x_i(t)y_i'(t)\right| dt + \sum_{j=1}^{n}b_{ij}^+L_j\int_0^{2\omega}\left|x_j(t)y_i'(t)\right| dt\end{aligned}$$

$$+\sum_{j=1}^{n} c_{ij}^{+} L_j \int_0^{2\omega} \left| x_j\big(t-\gamma_{ij}(t)\big) y_i'(t) \right| dt + \sum_{j=1}^{n} \big(b_{ij}^{+} + c_{ij}^{+}\big) \left| f_j(0) \right| \int_0^{2\omega} \left| y_i'(t) \right| dt$$
$$+ B_i^{+} \int_0^{2\omega} \left| z_i(t) y_i'(t) \right| dt. \tag{3.5}$$

From (3.5) and the Hölder inequality, we have

$$\begin{aligned}\int_0^{2\omega} \left|y_i'(t)\right|^2 dt \le{}& \left|1-\alpha_i^{-}\right| \left(\int_0^{2\omega} \left|x_i'(t)\right|^2 dt\right)^{\frac{1}{2}} \left(\int_0^{2\omega} \left|y_i'(t)\right|^2 dt\right)^{\frac{1}{2}} \\ &+ \beta_i^{+} \left(\int_0^{2\omega} \left|x_i(t)\right|^2 dt\right)^{\frac{1}{2}} \left(\int_0^{2\omega} \left|y_i'(t)\right|^2 dt\right)^{\frac{1}{2}} \\ &+ \sum_{j=1}^{n} b_{ij}^{+} L_j \left(\int_0^{2\omega} \left|x_j(t)\right|^2 dt\right)^{\frac{1}{2}} \left(\int_0^{2\omega} \left|y_i'(t)\right|^2 dt\right)^{\frac{1}{2}} \\ &+ \sum_{j=1}^{n} c_{ij}^{+} L_j \left(\int_0^{2\omega} \left|x_j\big(t-\gamma_{ij}(t)\big)\right|^2 dt\right)^{\frac{1}{2}} \left(\int_0^{2\omega} \left|y_i'(t)\right|^2 dt\right)^{\frac{1}{2}} \\ &+ \sum_{j=1}^{n} \sqrt{2\omega}\big(b_{ij}^{+} + c_{ij}^{+}\big) \left|f_j(0)\right| \left(\int_0^{2\omega} \left|y_i'(t)\right|^2 dt\right)^{\frac{1}{2}} \\ &+ B_i^{+} \left(\int_0^{2\omega} \left|z_i(t)\right|^2 dt\right)^{\frac{1}{2}} \left(\int_0^{2\omega} \left|y_i'(t)\right|^2 dt\right)^{\frac{1}{2}},\end{aligned}$$

which results in

$$\begin{aligned}&\left(\int_0^{2\omega} \left|y_i'(t)\right|^2 dt\right)^{\frac{1}{2}} \\ &\quad\le \left|1-\alpha_i^{-}\right| \left(\int_0^{2\omega} \left|x_i'(t)\right|^2 dt\right)^{\frac{1}{2}} + \beta_i^{+} \left(\int_0^{2\omega} \left|x_i(t)\right|^2 dt\right)^{\frac{1}{2}} \\ &\qquad+ \sum_{j=1}^{n} b_{ij}^{+} L_j \left(\int_0^{2\omega} \left|x_j(t)\right|^2 dt\right)^{\frac{1}{2}} + \sum_{j=1}^{n} c_{ij}^{+} L_j \left(\int_0^{2\omega} \left|x_j\big(t-\gamma_{ij}(t)\big)\right|^2 dt\right)^{\frac{1}{2}} \\ &\qquad+ \sum_{j=1}^{n} \sqrt{2\omega}\big(b_{ij}^{+} + c_{ij}^{+}\big) \left|f_j(0)\right| + B_i^{+} \left(\int_0^{2\omega} \left|z_i(t)\right|^2 dt\right)^{\frac{1}{2}}.\end{aligned} \tag{3.6}$$

Consider the fourth term $\sum_{j=1}^{n} c_{ij}^{+} L_j (\int_0^{2\omega} |x_j(t-\gamma_{ij}(t))|^2 dt)^{\frac{1}{2}}$ in (3.6). In view of Remark 2.2, we have

$$\begin{aligned}&\sum_{j=1}^{n} c_{ij}^{+} L_j \left(\int_0^{2\omega} \left|x_j\big(t-\gamma_{ij}(t)\big)\right|^2 dt\right)^{\frac{1}{2}} \\ &\quad= \sum_{j=1}^{n} c_{ij}^{+} L_j \left(\int_{-\gamma_{ij}(0)}^{2\omega-\gamma_{ij}(0)} \frac{|x_j(u_{ij}(t))|^2}{1-\gamma_{ij}'(\Gamma_{ij}(u_{ij}))} du_{ij}\right)^{\frac{1}{2}} \\ &\quad\le \sum_{j=1}^{n} c_{ij}^{+} L_j \max_{s\in\mathbb{R}} \frac{1}{|1-\gamma_{ij}'(s)|} \left(\int_0^{2\omega} \left|x_j(t)\right|^2 dt\right)^{\frac{1}{2}}.\end{aligned} \tag{3.7}$$

From (3.6) and (3.7), we have

$$\begin{aligned}
&\sum_{i=1}^{n}\left(\int_0^{2\omega}\left|y_i'(t)\right|^2 dt\right)^{\frac{1}{2}} \\
&\quad\leq \sum_{i=1}^{n}\left|1-\alpha_i^-\right|\left(\int_0^{2\omega}\left|x_i'(t)\right|^2 dt\right)^{\frac{1}{2}} + \sum_{i=1}^{n}\beta_i^+\left(\int_0^{2\omega}\left|x_i(t)\right|^2 dt\right)^{\frac{1}{2}} \\
&\qquad + \sum_{i=1}^{n}\sum_{j=1}^{n}\left(b_{ij}^+L_j + c_{ij}^+L_j\max_{s\in\mathbb{R}}\frac{1}{|1-\gamma_{ij}'(s)|}\right)\left(\int_0^{2\omega}\left|x_j(t)\right|^2 dt\right)^{\frac{1}{2}} \\
&\qquad + \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{2\omega}\left(b_{ij}^+ + c_{ij}^+\right)\left|f_j(0)\right| + \sum_{i=1}^{n}B_i^+\left(\int_0^{2\omega}\left|z_i(t)\right|^2 dt\right)^{\frac{1}{2}} \\
&\quad= \sum_{i=1}^{n}\left|1-\alpha_i^-\right|\left(\int_0^{2\omega}\left|x_i'(t)\right|^2 dt\right)^{\frac{1}{2}} \\
&\qquad + \sum_{i=1}^{n}\left[\beta_i^+ + \sum_{j=1}^{n}\left(b_{ij}^+L_j + c_{ij}^+L_j\max_{s\in\mathbb{R}}\frac{1}{|1-\gamma_{ij}'(s)|}\right)\right]\left(\int_0^{2\omega}\left|x_i(t)\right|^2 dt\right)^{\frac{1}{2}} \\
&\qquad + \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{2\omega}\left(b_{ij}^+ + c_{ij}^+\right)\left|f_j(0)\right| + \sum_{i=1}^{n}B_i^+\left(\int_0^{2\omega}\left|z_i(t)\right|^2 dt\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.8}$$

Multiplying by $A_i x_i'(t)$ on both sides of the first equation of (3.4) and integrating it over $[0,2\omega]$, we have

$$\begin{aligned}
\int_0^{2\omega}\left|A_i x_i'(t)\right|^2 dt &= -\lambda\int_0^{2\omega}x_i(t)A_i x_i'(t)\,dt + \lambda\int_0^{2\omega}y_i(t)A_i x_i'(t)\,dt \\
&= \lambda\int_0^{2\omega}y_i(t)A_i x_i'(t)\,dt \\
&\leq \left(\int_0^{2\omega}\left|y_i(t)\right|^2 dt\right)^{\frac{1}{2}}\left(\int_0^{2\omega}\left|A_i x_i'(t)\right|^2 dt\right)^{\frac{1}{2}},
\end{aligned}$$

which results in

$$\left(\int_0^{2\omega}\left|A_i x_i'(t)\right|^2 dt\right)^{\frac{1}{2}} \leq \left(\int_0^{2\omega}\left|y_i(t)\right|^2 dt\right)^{\frac{1}{2}}. \tag{3.9}$$

Using Lemma 2.3 and (3.9), we have

$$\begin{aligned}
\left(\int_0^{2\omega}\left|x_i'(t)\right|^2 dt\right)^{\frac{1}{2}} &= \left(\int_0^{2\omega}\left|A_i^{-1}A_i x_i'(t)\right|^2 dt\right)^{\frac{1}{2}} \\
&\leq \frac{1}{|1-\|c_i\||}\left(\int_0^{2\omega}\left|A_i x_i'(t)\right|^2 dt\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.10}$$

The Wirtinger inequality, (3.9) and (3.10) result in

$$\begin{aligned}\left(\int_0^{2\omega}\left|x_i'(t)\right|^2 dt\right)^{\frac{1}{2}} &\le \frac{1}{|1-c_i|}\left(\int_0^{2\omega}\left|y_i(t)\right|^2 dt\right)^{\frac{1}{2}}\\ &\le \frac{\omega}{|1-\|c_i\||\pi}\left(\int_0^{2\omega}\left|y_i'(t)\right|^2 dt\right)^{\frac{1}{2}}.\end{aligned} \tag{3.11}$$

Using again the Wirtinger inequality and (3.11), we have

$$\left(\int_0^{2\omega}\left|x_i(t)\right|^2 dt\right)^{\frac{1}{2}} \le \frac{\omega}{\pi}\left(\int_0^{2\omega}\left|x_i'(t)\right|^2 dt\right)^{\frac{1}{2}} \le \frac{\omega^2}{|1-\|c_i\||\pi^2}\left(\int_0^{2\omega}\left|y_i'(t)\right|^2 dt\right)^{\frac{1}{2}}. \tag{3.12}$$

Multiplying by $z_i'(t)$ on both sides of the third equation of (3.4) and integrating it over $[0, 2\omega]$, we have

$$\begin{aligned}\int_0^{2\omega}\left|z_i'(t)\right|^2 dt &= -\lambda\int_0^{2\omega} z_i(t)z_i'(t)\,dt + \lambda\int_0^{2\omega} df_i\big(x_i(t)\big)z_i'(t)\,dt\\ &\le \int_0^{2\omega} d\big|f_i\big(x_i(t)\big)-f_i(0)\big|\left|z_i'(t)\right| dt + \int_0^{2\omega} d\big|f_i(0)\big|\left|z_i'(t)\right| dt\\ &\le dL_i\left(\int_0^{2\omega}\left|x_i(t)\right|^2 dt\right)^{\frac{1}{2}}\left(\int_0^{2\omega}\left|z_i'(t)\right|^2 dt\right)^{\frac{1}{2}}\\ &\quad + d\big|f_i(0)\big|\sqrt{2\omega}\left(\int_0^{2\omega}\left|z_i'(t)\right|^2 dt\right)^{\frac{1}{2}},\end{aligned}$$

which results in

$$\left(\int_0^{2\omega}\left|z_i'(t)\right|^2 dt\right)^{\frac{1}{2}} \le dL_i\left(\int_0^{2\omega}\left|x_i(t)\right|^2 dt\right)^{\frac{1}{2}} + d\big|f_i(0)\big|\sqrt{2\omega}. \tag{3.13}$$

From (3.12) and (3.13), we have

$$\left(\int_0^{2\omega}\left|z_i'(t)\right|^2 dt\right)^{\frac{1}{2}} \le \frac{dL_i\omega^2}{|1-\|c_i\||\pi^2}\left(\int_0^{2\omega}\left|y_i'(t)\right|^2 dt\right)^{\frac{1}{2}} + d\big|f_i(0)\big|\sqrt{2\omega} \tag{3.14}$$

which together with the Wirtinger inequality results in

$$\left(\int_0^{2\omega}\left|z_i(t)\right|^2 dt\right)^{\frac{1}{2}} \le \frac{dL_i\omega^3}{|1-\|c_i\||\pi^3}\left(\int_0^{2\omega}\left|y_i'(t)\right|^2 dt\right)^{\frac{1}{2}} + \frac{\omega d|f_i(0)|\sqrt{2\omega}}{\pi}. \tag{3.15}$$

From (3.8), (3.11), (3.12) and (3.15), we have

$$\begin{aligned}&\sum_{i=1}^n\left(\int_0^{2\omega}\left|y_i'(t)\right|^2 dt\right)^{\frac{1}{2}}\\ &\quad\le \sum_{i=1}^n \frac{\omega|1-\alpha_i^-|}{|1-\|c_i\||\pi}\left(\int_0^{2\omega}\left|y_i'(t)\right|^2 dt\right)^{\frac{1}{2}}\end{aligned}$$

$$+\sum_{i=1}^{n}\left[\beta_i^+ + \sum_{j=1}^{n}\left(b_{ij}^+L_j + c_{ij}^+L_j \max_{s\in\mathbb{R}} \frac{1}{|1-\gamma_{ij}'(s)|}\right)\right]$$
$$\times \frac{\omega^2}{|1-\|c_i\||\pi^2}\left(\int_0^{2\omega}|y_i'(t)|^2\,dt\right)^{\frac{1}{2}}$$
$$+\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{2\omega}\left(b_{ij}^+ + c_{ij}^+\right)\left|f_j(0)\right| + \sum_{i=1}^{n}\frac{B_i^+ dL_i\omega^3}{|1-\|c_i\||\pi^3}\left(\int_0^{2\omega}|y_i'(t)|^2\,dt\right)^{\frac{1}{2}}$$
$$+\sum_{i=1}^{n}\frac{B_i^+\omega d|f_i(0)|\sqrt{2\omega}}{\pi}. \tag{3.16}$$

By using assumption (H_4) and (3.16), we obtain

$$\begin{aligned}&\sum_{i=1}^{n}\left(\int_0^{2\omega}|y_i'(t)|^2\,dt\right)^{\frac{1}{2}}\\ &\quad\le \frac{1}{\Theta}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{2\omega}\left(b_{ij}^+ + c_{ij}^+\right)\left|f_j(0)\right| + \sum_{i=1}^{n}\frac{B_i^+\omega d|f_i(0)|\sqrt{2\omega}}{\pi}\right)\\ &\quad:= M_1.\end{aligned} \tag{3.17}$$

Equations (3.11) and (3.17) result in

$$\sum_{i=1}^{n}\left(\int_0^{2\omega}|x_i'(t)|^2\,dt\right)^{\frac{1}{2}} \le \sum_{i=1}^{n}\frac{M_1\omega}{|1-\|c_i\||\pi} := M_2. \tag{3.18}$$

Equations (3.14) and (3.17) result in

$$\begin{aligned}\sum_{i=1}^{n}\left(\int_0^{2\omega}|z_i'(t)|^2\,dt\right)^{\frac{1}{2}} &\le \sum_{i=1}^{n}\frac{M_1 dL_i\omega^2}{|1-\|c_i\||\pi^2} + d\left|f_i(0)\right|\sqrt{2\omega}\\ &:= M_3.\end{aligned} \tag{3.19}$$

Since $u\in X$ is an ω-anti-periodic function, there exist $\xi_i,\eta_i,\zeta_i\in[0,2\omega]$ such that

$$x_i(\xi_i)=y_i(\eta_i)=z_i(\zeta_i)=0,$$

which results in

$$|x_i|_\infty \le \int_0^{2\omega}\left|x_i'(t)\right|dt \le \sqrt{2\omega}\left(\int_0^{2\omega}|x_i'(t)|^2\,dt\right)^{\frac{1}{2}}$$

and

$$\sum_{i=1}^{n}|x_i|_\infty \le \sqrt{2\omega}\sum_{i=1}^{n}\left(\int_0^{2\omega}|x_i'(t)|^2\,dt\right)^{\frac{1}{2}} := \widetilde{M}_1.$$

Similarly, using (3.17) and (3.19), we also find that there exist positive constants $\widetilde{M}_2$, $\widetilde{M}_3$ such that

$$\sum_{i=1}^{n} |y_i|_\infty \le \widetilde{M}_2, \tag{3.20}$$

$$\sum_{i=1}^{n} |z_i|_\infty \le \widetilde{M}_3. \tag{3.21}$$

From (3.19)–(3.21), we get

$$\|u\|_X = \sum_{i=1}^{n} \bigl(|x_i|_\infty + |y_i|_\infty + |z_i|_\infty\bigr) \le \widetilde{M}_1 + \widetilde{M}_2 + \widetilde{M}_3 := \widetilde{M}.$$

Let

$$\Omega = \bigl\{u \in X : \|u\|_X < \widetilde{M} + 1\bigr\}.$$

From Lemma 2.2, the operator equation $Lu = Nu$ has at least one ω-anti-periodic solution in X. Thus, system (2.1) has at least one ω-anti-periodic solution. □

Remark 3.1 We very much want to obtain the globally exponential stability of system (2.1) with initial values conditions (2.3). But transforming system (2.6) of system (2.1) contains a neutral term $A_i x_i$ which makes constructing the appropriate Lyapinov function very difficult. Hence, we wish that some authors will develop new methods to derive globally exponential stability of system (2.1) in the future.

4 A numerical example

In this section, a numerical example is given to illustrate the effectiveness of the results obtained in this paper.

Example 4.1 Consider the following inertial competitive neutral-type neural networks:

$$\begin{cases} (A_i x_i)''(t) = -\alpha_i(t)x_i'(t) - \beta_i(t)x_i(t) + \sum_{j=1}^{2} b_{ij}(t)f_j(x_j(t)) \\ \qquad\qquad + \sum_{j=1}^{2} c_{ij}(t)f_j(x_j(t-\gamma_{ij}(t))) + B_i(t)\sum_{j=1}^{n} d_j m_{ij}(t), \\ m_{ij}'(t) = -m_{ij}(t) + d_j f_i(x_i(t)), \end{cases} \tag{4.1}$$

where $i, j = 1, 2$,

$$\begin{bmatrix} A_1 x_1(t) \\ A_2 x_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t) - \frac{1}{2}x_1(t-\pi) \\ x_2(t) - \frac{1}{3}x_2(t-\pi) \end{bmatrix},$$

$$f_1(x) = f_2(x) = \frac{1}{5}x, \qquad d_1 = d_2 = \frac{1}{5},$$

$$\begin{bmatrix} \alpha_1(t) \\ \alpha_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{100}\cos 2t + \frac{101}{100} \\ \frac{1}{100}\sin^2 t + + \frac{101}{100} \end{bmatrix},$$

$$\begin{bmatrix} \beta_1(t) \\ \beta_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{25}\cos^2 t + \frac{1}{25} \\ \frac{1}{25}\sin^2 t + \frac{1}{25} \end{bmatrix},$$

$$\begin{bmatrix} b_{11}(t) & b_{12}(t) \\ b_{21}(t) & b_{22}(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{5}\cos 2t & \frac{1}{5}\cos^2 t \\ -\frac{1}{5}\cos^2 t & -\frac{1}{5}\cos^2 t \end{bmatrix},$$

$$\begin{bmatrix} c_{11}(t) & c_{12}(t) \\ c_{21}(t) & c_{22}(t) \end{bmatrix} = \begin{bmatrix} \cos^2 t - \frac{1}{4} & \frac{1}{2}\cos^2 t \\ \frac{1}{2}\cos^2 t - \frac{1}{4} & \frac{1}{4}\cos^2 t \end{bmatrix},$$

$$\begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sin^2 t \\ \frac{1}{2}\cos^2 t \end{bmatrix},$$

$$\begin{bmatrix} \gamma_{11}(t) & \gamma_{12}(t) \\ \gamma_{21}(t) & \gamma_{22}(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{5}\cos^2 t & \frac{1}{5}\sin^2 t \\ \frac{1}{8}\cos 2t + \frac{1}{8} & \frac{1}{10}\cos 2t + \frac{1}{10} \end{bmatrix}.$$

After a simple calculation, we have

$$\begin{aligned}
&d = d_1^2 + d_2^2 = \frac{2}{25}, \qquad c_1 = \frac{1}{2}, \qquad c_2 = \frac{1}{3}, \qquad L_1 = L_2 = \frac{1}{5}, \\
&\alpha_1^- = \frac{1}{100}, \qquad \alpha_2^- = \frac{1}{100}, \\
&\beta_1^+ = \frac{2}{25}, \qquad \beta_2^+ = \frac{2}{25}, \qquad B_1^+ = B_2^+ = \frac{1}{2}, \qquad b_{11}^+ = b_{12}^+ = b_{21}^+ = b_{22}^+ = \frac{1}{5}, \\
&c_{11}^+ = c_{12}^+ = c_{21}^+ = c_{22}^+ = \frac{1}{4}, \qquad \omega = \pi, \\
&\gamma_{11}'(t) = -\frac{1}{5}\sin 2t, \qquad \gamma_{12}'(t) = -\frac{1}{5}\sin 2t, \\
&\gamma_{21}'(t) = -\frac{1}{8}\sin 2t, \qquad \gamma_{22}'(t) = -\frac{1}{10}\sin 2t.
\end{aligned}$$

Thus,

$$\begin{aligned}
&1 - \frac{\omega|1-\alpha_1^-|}{|1-|c_1||\pi} - \left[\beta_1^+ + \sum_{j=1}^{2}\left(b_{1j}^+ L_j + c_{1j}^+ L_j \max_{s\in\mathbb{R}} \frac{1}{|1-\gamma_{1j}'(s)|}\right)\right]\frac{\omega^2}{|1-|c_1||\pi^2} \\
&\quad - \frac{B_1^+ d L_1 \omega^3}{|1-|c_1||\pi^3} \approx 0.794 > 0
\end{aligned}$$

and

$$\begin{aligned}
&1 - \frac{\omega|1-\alpha_2^-|}{|1-|c_2||\pi} - \left[\beta_2^+ + \sum_{j=1}^{2}\left(b_{2j}^+ L_j + c_{2j}^+ L_j \max_{s\in\mathbb{R}} \frac{1}{|1-\gamma_{2j}'(s)|}\right)\right]\frac{\omega^2}{|1-|c_2||\pi^2} \\
&\quad - \frac{B_2^+ d L_2 \omega^3}{|1-|c_2||\pi^3} \approx 0.667 > 0.
\end{aligned}$$

Then

$$\Theta = \min\{0.794, 0.667\} > 0.$$

By Theorem 3.1, system (4.1) has at least a unique π-anti-periodic solution. For the trajectories of $x_i(t)$, $y_i(t)$, $z_i(t)$ and $m_{ij}(t)$ in system (4.1), see Figs. 1–3.

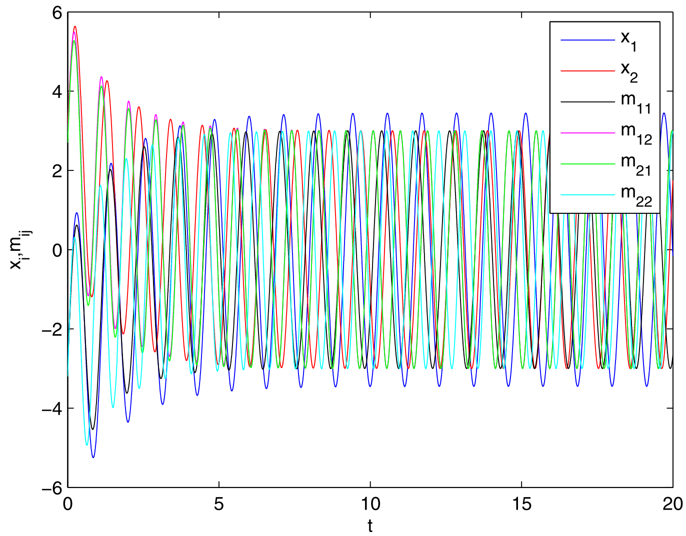

Figure 1 For $i,j = 1,2$, trajectories of $x_i(t)$ and $m_{ij}(t)$ in system (4.1)

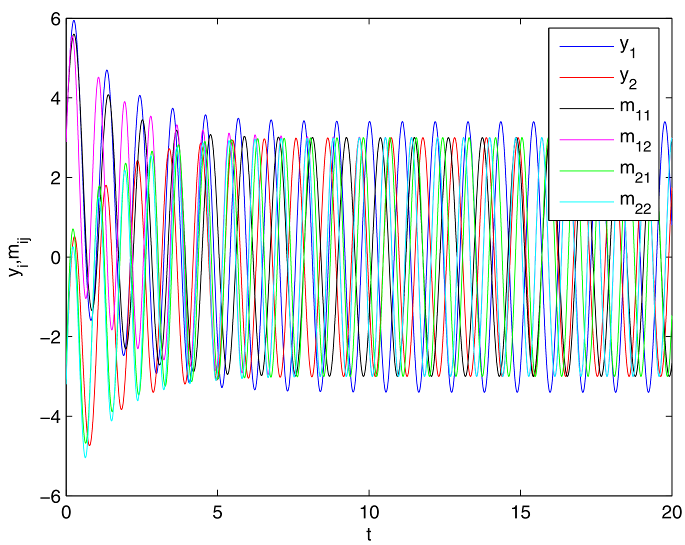

Figure 2 For $i,j = 1,2$, trajectories of $y_i(t)$ and $m_{ij}(t)$ in system (4.1)

5 Conclusions

In this paper, we study a class of he anti-periodic solutions problem for inertial competitive neutral-type neural networks. By employing the Wirtinger inequality, topology degree theory and some analytic techniques, we have presented some new sufficient criteria for the existence of anti-periodic for the above neural networks. These criteria possess adjustable parameters which are important in some applied fields. Finally, an example is given to demonstrate the effectiveness of the obtained theoretical results. However, there exist many problems for further study such as the problems of the stability and other dynamic properties of anti-periodic solutions to neutral-type neural networks.

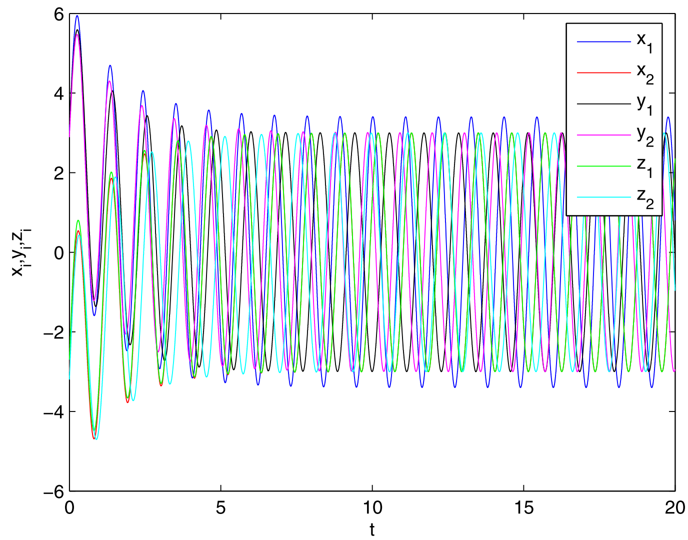

Figure 3 For $i,j = 1,2$, trajectories of $x_i(t)$, $y_i(t)$ and $z_i(t)$ in system (4.1)

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Authors' contributions
Only the author contributed to the writing of this paper. The author read and approved the final manuscript.

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References
1. Dym, H., McKean, H.: Fourier Series and Integrals. Academic Press, San Diego (1985)
2. Komkov, V.: Euler's buckling formula and Wirtinger's inequality. Int. J. Math. Educ. Sci. Technol. **14**, 661–668 (1983)
3. Coti-Zelati, V., Ekland, I., Sere, E.: A variational approach to homoclinic orbits in Hamiltonian systems. Math. Ann. **288**(1), 133–160 (1990)
4. Lv, Y., Tang, C.L.: Existence of even homoclinic orbits for second-order Hamiltonian systems. Nonlinear Anal. **67**(7), 2189–2198 (2007)
5. Du, B.: Stability analysis of periodic solution for a complex-valued neural networks with bounded and unbounded delays. Asian J. Control **20**, 1–12 (2018)
6. Du, B., Lian, X., Cheng, X.: Partial differential equation modeling with Dirichlet boundary conditions on social networks. Bound. Value Probl. **2018**, 50 (2018)
7. Xu, C., Li, P., Pang, Y.: Exponential stability of almost periodic solutions for memristor- based neural networks with distributed leakage delays. Neural Comput. **28**(12), 2726–2756 (2016)
8. Xu, C., Chen, L.: Effect of leakage delay on the almost periodic solutions of fuzzy cellular neural networks. J. Exp. Theor. Artif. Intell. **30**(6), 993–1011 (2018)
9. Meyer-Baese, A., Ohl, F., Scheich, H.: Singular perturbation analysis of competitive neural networks with different time scales. Neural Comput. **8**(8), 1731–1742 (1996)
10. Meyer-Baese, A., Pilyugin, S.S., Chen, Y.: Global exponential stability of competitive neural networks with different time scales. IEEE Trans. Neural Netw. **14**(3), 716–719 (2003)
11. Meyer-Baese, A., Roberts, R., Thmmler, V.: Local uniform stability of competitive neural networks with different time-scales under vanishing perturbations. Neurocomputing **73**, 770–775 (2010)

12. Gu, H., Jiang, H., Teng, Z.: Existence and global exponential stability of equilibrium of competitive neural networks with different time-scales and multiple delays. J. Franklin Inst. **347**(5), 719–731 (2010)
13. Liu, Y., Yang, Y., Liang, T., Li, L.: Existence of periodic solution for competitive neural networks with time-varying and distributed delays on time scales. Adv. Neural Netw. **22**, 181–188 (2013)
14. Okochi, H.: On the existence of periodic solutions to nonlinear abstract parabolic equations. J. Math. Soc. Jpn. **40**(1988), 541–553 (2006)
15. Li, Y., Yang, L., Wu, W.: Anti-periodic solution for impulsive BAM neural networks with time-varying leakage delays on time scales. Neurocomputing **149**, 536–545 (2015)
16. Li, Y., Yang, L.: Anti-periodic solutions for Cohen–Grossberg neural networks with bounded and unbounded delays. Commun. Nonlinear Sci. Numer. Simul. **14**(7), 3134–3140 (2009)
17. Li, Y., Xiang, J.: Existence and global exponential stability of anti-periodic solution for Clifford-valued inertial Cohen–Grossberg neural networks with delays. Neurocomputing **332**, 259–269 (2019)
18. Huo, N., Li, B., Li, Y.: Existence and exponential stability of anti-periodic solutions for inertial quaternion-valued high-order Hopfield neural networks with state-dependent delays. IEEE Access **7**, 60010–60019 (2019)
19. Xu, C., Chen, L., Guo, T.: Anti-periodic oscillations of bidirectional associative memory (BAM) neural networks with leakage delays. J. Inequal. Appl. **2018**, 68 (2018)
20. Xu, C., Li, P.: Existence and exponentially stability of anti-periodic solutions for neutral BAM neural networks with time-varying delays in the leakage terms. J. Nonlinear Sci. Appl. **9**(3), 1285–1305 (2016)
21. Mawhin, J., Willem, M.: Critical Point Theory and Hamiltonian Systems. Springer, Berlin (1989)
22. Amster, P.: Topological Methods in the Study of Boundary Valued Problems. Springer, New York (2013)
23. Zhang, M.: Periodic solutions of linear and quasilinear neutral functional differential equations. J. Math. Anal. Appl. **189**, 378–392 (1995)
24. Lu, S., Ge, W., Zheng, Z.: Periodic solutions to neutral differential equation with deviating arguments. Appl. Math. Comput. **152**, 17–27 (2004)

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