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Uniform attractors for the non-autonomous suspension bridge equation with time delay

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Abstract

In this paper, we investigate the existence of a uniform attractor for the non-autonomous suspension bridge equation with time delay by using the energy function and uniform asymptotic compactness of the process.

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1 Introduction

A one-dimensional simply supported beam suspended by hangers was modeled as a suspension bridge in [1] by Lazer and McKenna, which described the vibration of the roadbed in the vertical plane. In [1–3], the authors investigated the existence of periodic solutions, the property of traveling wave solutions, and the numerical mountain pass solutions for the suspension bridge equations. Ma and Zhong [4–6] showed the existence of both weak and strong solutions as well as the global attractors for the singled and coupled autonomous suspension bridge equations, respectively. For the non-autonomous case, Ma et al. [7, 8] studied their uniform attractors and pullback attractors. Existence of global attractors for the suspension bridge equations with nonlinear damping was achieved in [9].

However, the above-mentioned models have a drawback, that is, the torsional oscillations in suspension bridges were not described; see [10], for example, the collapse of Tacoma narrow bridge. In the last several decades, the spectacular collapse of the Tacoma narrow bridge has successfully attracted the attention of many engineers, physicists, and mathematicians. They tried their best to explain such an amazing event, and their explanations were mainly based on the aero elastic effects such as the frequency of the vortex shedding, parametric resonance, and flutter theory, but none of their explanations was universally accepted. In 2014, Arioli and Gazzola [11] gave a new explanation about this phenomenon, i.e., the appearance of torsional oscillations during the collapse of Tacoma narrow bridge led to the internal resonances, due to the bridge structure itself. Successively, Ferrero and Gazzola [12] suggested that one should regard a rectangular plate $\Omega = (0, \pi) \times (-l, l)$ ($l \ll \pi$) as a model of suspension bridges, and the plate was assumed

to be hinged on its vertical edges

$$u(0, y, t) = \partial_{xx}u(0, y, t) = u(\pi, y, t) = \partial_{xx}u(\pi, y, t) = 0, \quad y \in (-l, l), t > 0,$$

along with the free horizontal edges

$$\begin{aligned} u_{yy}(x, \pm l, t) + \sigma u_{xx}(x, \pm l, t) &= 0, \\ u_{yyy}(x, \pm l, t) + (2 - \sigma)u_{xxy}(x, \pm l, t) &= 0, \quad x \in (0, \pi), t > 0, \end{aligned}$$

where $0 < \sigma < \frac{1}{2}$ is the Poisson ratio. They introduced in [12] the following equation as the model of the nonlinear suspension bridges:

$$u_{tt}(x, y, t) + \delta u_t(x, y, t) + \Delta^2 u(x, y, t) + f(x, y, u) = g(x, y, t), \quad (x, y) \in \Omega, t > 0, \tag{1.1}$$

where $u(x, y, t)$ is the vertical displacement of the plate in the downward direction, $u_t(x, y, t)$ is a damped term, δ is a positive constant, $f(x, y, u)$ is a restoring force due to the hangers of the suspension bridge, $g(x, y, t)$ is the external force including both gravity and live as well as dead loads. More works related to the suspension bridge equations can be referred to [13–15]. For instance, Al-Gwaiz et al. [13] studied the bending and stretching energies about the rectangular plate model suggested in [12]. Berchio et al. [14] investigated the structural instability of nonlinear plate modeling suspension bridges. For more details on these suspension bridge equations, we refer to the new book [15] published by Gazzola. Besides, Messaoudi et al. [16] obtained the global attractor of nonlinear suspension bridges equations.

Time delays arise usually in physical, chemical, biological, thermal phenomena and so on. In recent years, the partial differential equations with time delay effects have become an active area of research, see [17–28] and the references therein. For example, in [19], the authors showed that a small delay in a boundary control could turn such a well-behaved hyperbolic system into a wild one, so the delay was a source leading to instability of the system. Nicasise and Pignotti in [20] studied the following wave equation with time delay in an open bounded set $\Omega \subset \mathbb{R}^n$:

$$u_{tt}(x, t) - \Delta u(x, t) + a_0 u_t(x, t) + a_1 u_t(x, t - \tau) = 0, \quad x \in \Omega, t > 0,$$

and they proved, under the assumptions that the weight of the feedback is larger than the weight of the delay ($a_1 < a_0$), that the energy is exponentially stable. However, they also produced a sequence of delays, which led to the instability of the corresponding solution. The same results were obtained for the case of boundary delay, see [21]. The time delay in a Timoshenko-type system of the form

$$\begin{cases} \rho_2 u_{tt} - K(u_x + \psi)_x = 0, & \text{in } (0, L) \times (0, \infty), \\ \rho_2 \psi_{tt} - b\psi_{xx} + K(u_x + \psi) + \mu_1 \psi_t + \mu_2 \psi_t(x, t - \tau) = 0, & \text{in } (0, L) \times (0, \infty), \end{cases}$$

was proposed by Said-Houari and Laskri in [22]. They arrived at an exponential decay result under the assumption $\mu_2 < \mu_1$. The authors in [23] considered the well-posedness and exponential stability of wave equation with a strong damping and a strong time delay.

Recently, Park [24] considered the suspension bridge equations with time delay in a bounded domain $\Omega \subset \mathbb{R}^2$

$$u_{tt}(x, t) + \Delta^2 u(x, t) + ku^+ + a_0 u_t(x, t) + a_1 u_t(x, t - \tau) + f(u) = g(x), \quad x \in \Omega, t > 0,$$

and obtained the existence of the finite dimensional global attractors under the conditions that $0 < |a_1| < a_0$. Also, we would like to recommend many more literature sources such as [25–28] and the references therein. In this paper, we consider the existence of uniform attractors for the following nonlinear plate modeling suspension bridges with time delay in $\Omega = (0, \pi) \times (-l, l)$:

$$\begin{aligned} \partial_{tt} u + \Delta^2 u + \gamma_1 \partial_t u + \gamma_2 \partial_t u(x, y, t - h) + f(u(x, y, t)) &= g(x, y, t), \\ (x, y) \in \Omega, t \geq \tau, \tau \in \mathbb{R}, \end{aligned} \tag{1.2}$$

with boundary conditions

$$\begin{cases} u(0, y, t) = \partial_{xx} u(0, y, t) = u(\pi, y, t) = \partial_{xx} u(\pi, y, t) = 0, & y \in (-l, l), t \geq \tau, \\ \partial_{yy} u(x, \pm l, t) + \sigma \partial_{xx} u(x, \pm l, t) = 0, & x \in (0, \pi), t \geq \tau, \\ \partial_{yyy} u(x, \pm l, t) + (2 - \sigma) \partial_{xxy} u(x, \pm l, t) = 0, & x \in (0, \pi), t \geq \tau, \end{cases} \tag{1.3}$$

and initial conditions

$$\begin{cases} \partial_t u(x, y, t - h) = \phi_0(x, y, t - h), & (x, y) \in \Omega, t \in [\tau, h], \\ u(x, y, \tau) = u_0^\tau(x, y), \quad \partial_t u(x, y, \tau) = u_1^\tau(x, y), & (x, y) \in \Omega, \tau \in \mathbb{R}, \end{cases} \tag{1.4}$$

where $\gamma_1 > 0$ is the damped coefficient, $\gamma_2 \in \mathbb{R}$, $\partial_t u(x, y, t - h)$ is the delay term, and $h > 0$ represents the time delay. $u_0^\tau, u_1^\tau, \phi_0$ are given initial data belonging to suitable space. g and f are external forces. Since we have in mind a long narrow rectangle, that is, $l \ll \pi$, it is reasonable to assume that the forcing term does not depend on y , see [12]. So, we now assume that $g(x, y, t) = g(x, t) \in L^2_{loc}(\mathbb{R}^+; L^2(\Omega))$.

As introduced in [23–25], our equation can be regarded as a Kelvin–Voigt linear model for a viscoelastic material in the presence of a time delay response in connection with the stress-strain. Besides, the action of any external force $g(x, y, t)$ on the plate Ω is transmitted through hangers to the sustaining cables, and this may yield certain delay, see, for example, [25]. It is well known that the presence of delay may lead to instability of the system, and hence it could affect the existence of compact attractors. For these purposes, we will continue to focus on the well-posedness and existence of uniform attractors under the suitable conditions. We extend the results of [25] from the autonomous case to the non-autonomous case.

The paper is organized as follows. In Sect. 2, we present some fundamental and basic results. In Sect. 3, firstly, we show the well-posedness by using maximal monotone operator theory. Secondly, we obtain a uniformly absorbing set. Finally, we establish the existence of a uniform attractor for the process generated by the related problem to (1.2)–(1.4).

2 Preliminaries

We first review some notations about functions spaces and give hypothesis for problem (1.2)–(1.4). From now on, let $c_i, i = 0, 1, 2, \dots$, denote the different positive constants throughout the paper.

As in [12], let us introduce the space

$$H_*^2(\Omega) = \{w \in H^2(\Omega) : w(0, y) = w(\pi, y) = 0, \forall y \in (-l, l)\},$$

equipped with the inner product and norm

$$(u, v)_{H_*^2} = \int_{\Omega} [\Delta u \Delta v + (1 - \sigma)(2\partial_{xy}u\partial_{xy}v - \partial_{xx}u\partial_{yy}v - \partial_{yy}u\partial_{xx}v)] dx dy,$$

$$\|u\|_{H_*^2} = \left[\int_{\Omega} [(\Delta u)^2 + 2(1 - \sigma)((\partial_{xy}u)^2 - \partial_{xx}u\partial_{yy}u)] dx dy \right]^{\frac{1}{2}}.$$

It was proven that $\|\cdot\|_{H_*^2}$ is a norm on H_*^2 which is equivalent to the usual $H^2(\Omega)$ -norm. Moreover, H_*^2 is a Hilbert space when endowed with the scalar product $(\cdot, \cdot)_{H_*^2}$.

We define the phase space

$$\mathcal{H} = H_*^2(\Omega) \times L^2(\Omega) \times L^2((0, 1) \times \Omega)$$

equipped with the inner product and norm, respectively

$$(U, V)_{\mathcal{H}} = (u, \bar{u})_{H_*^2(\Omega)} + (v, \bar{v})_{L^2(\Omega)} + h|\gamma_2|(z, \bar{z})_{L^2((0,1)\times\Omega)},$$

$$(U, U)_{\mathcal{H}} = \|u\|_{H_*^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2 + h|\gamma_2|\|z\|_{L^2((0,1)\times\Omega)}^2,$$
(2.1)

where

$$U = (u, v, z)^T, \quad V = (\bar{u}, \bar{v}, \bar{z})^T \in \mathcal{H}.$$

With regard to problem (1.2)–(1.4), we impose the following assumptions:

(H1) Nonlinear term $f \in C^1(\mathbb{R})$ satisfies

$$\begin{cases} |f(s_1) - f(s_2)| \leq c_0(|s_1|^p + |s_2|^p)|s_1 - s_2|, & \forall s_1, s_2 \in \mathbb{R}, p > 0, \\ -c_1 \leq F(s) \leq sf(s), & \forall s \in \mathbb{R}, \end{cases}$$

where $F(s) = \int_0^s f(v) dv$.

(H2) The coefficients γ_1 and γ_2 satisfy $0 < c|\gamma_2| \leq \gamma_1$, where c is a proper positive constant.

As in [20], we introduce the function

$$z(\rho, x, y, t) = \partial_t u(x, y, t - h\rho), \quad \rho \in (0, 1), (x, y) \in \Omega, t \geq \tau,$$
(2.2)

then differentiation with respect to t , we can see that

$$h\partial_t z(\rho, x, y, t) + \partial_\rho z(\rho, x, y, t) = 0, \quad \rho \in (0, 1), (x, y) \in \Omega, t \geq \tau.$$

Thus, problem (1.2)–(1.4) is equivalent to

$$\begin{cases} \partial_{tt}u + \Delta^2u + \gamma_1 \partial_t u + \gamma_2 z(1) + f(u) = g(x, y, t), & \Omega \times (\tau, +\infty), \\ h\partial_t z(\rho, x, y, t) + \partial_\rho z(\rho, x, y, t) = 0, & (0, 1) \times \Omega \times (\tau, +\infty), \end{cases} \tag{2.3}$$

with boundary conditions

$$\begin{cases} u(0, y, t) = \partial_{xx}u(0, y, t) = u(\pi, y, t) = \partial_{xx}u(\pi, y, t) = 0, & y \in (-l, l), t \geq \tau, \\ \partial_{yy}u(x, \pm l, t) + \sigma \partial_{xx}u(x, \pm l, t) = 0, & x \in (0, \pi), t \geq \tau, \\ \partial_{yyy}u(x, \pm l, t) + (2 - \sigma)\partial_{xxy}u(x, \pm l, t) = 0, & x \in (0, \pi), t \geq \tau, \end{cases} \tag{2.4}$$

and initial conditions

$$\begin{cases} u(x, y, \tau) = u_0^\tau(x, y), & \partial_t u(x, y, \tau) = u_1^\tau(x, y), & (x, y) \in \Omega, \tau \in \mathbb{R}, \\ z(\rho, x, y, \tau) = \phi_0(x, y, \tau - \rho h), & & (\rho, x, y) \in (0, 1) \times \Omega. \end{cases} \tag{2.5}$$

Lemma 2.1 ([12]) *Let $u \in H_*^2(\Omega)$ and suppose $1 \leq p < +\infty$. Then there exists a positive constant $c_\lambda = c_\lambda(\Omega, p)$ such that*

$$\|u\|_{L^p(\Omega)} \leq c_\lambda \|u\|_{H_*^2(\Omega)}. \tag{2.6}$$

To prove the existence of uniform attractors corresponding to (2.3)–(2.5), we also need the following abstract results.

Let X be a complete metric space and Σ be a parameter set.

The operators $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$ are said to be a family of processes in X with symbol space Σ if, for any $\sigma \in \Sigma$,

$$\begin{aligned} U_\sigma(t, s) \circ U_\sigma(s, \tau) &= U_\sigma(t, \tau), \quad \forall t \geq s \geq \tau, \tau \in \mathbb{R}, \\ U_\sigma(\tau, \tau) &= \text{Id (identity)}, \quad \forall \tau \in \mathbb{R}. \end{aligned} \tag{2.7}$$

Let $\{T(r)|r \geq 0\}$ be the translation semigroup on Σ , we say that a family of processes $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$ satisfies the translation identity if

$$T(r)\Sigma = \Sigma, \tag{2.8}$$

and

$$U_\sigma(t + r, \tau + r) = U_{T(r)\sigma}(t, \tau), \quad \forall \sigma \in \Sigma, t \geq \tau, \tau \in \mathbb{R}, r \geq 0. \tag{2.9}$$

By $\mathcal{B}(E)$ we denote the collection of all bounded subsets of E .

Definition 2.2 ([29, 30]) *A bounded set $B_0 \in \mathcal{B}(E)$ is said to be a bounded uniformly (w.r.t. $\sigma \in \Sigma$) absorbing set of $\{U_\sigma(t, \tau)\}$ ($\sigma \in \Sigma, t \geq \tau, \tau \in \mathbb{R}$) if, for any $\tau \in \mathbb{R}$ and $B \in \mathcal{B}(E)$, there exists a time $T_0 = T_0(B, \tau) \geq \tau$ such that*

$$\bigcup_{\sigma \in \Sigma} U_\sigma(t, \tau)B \subseteq B_0$$

for all $t \geq T_0$.

Definition 2.3 ([30, 31]) Let E be a Banach space and $B \in \mathcal{B}(E)$, Σ be a symbol space. We call a function $\psi(\cdot, \cdot, \cdot, \cdot)$ defined on $(E \times E) \times (\Sigma \times \Sigma)$ to be a contractive function on $B \times B$ if, for any sequence $\{x_n\}_{n=1}^\infty \subset B$ and any $\{\sigma_n\} \subset \Sigma$, there is a subsequence $\{x_{n_k}\}_{k=1}^\infty \subset \{x_n\}_{n=1}^\infty$ and $\{\sigma_{n_k}\}_{k=1}^\infty \subset \{\sigma_n\}_{n=1}^\infty$ such that

$$\lim_{k \rightarrow \infty} \lim_{l \rightarrow \infty} \psi(x_{n_k}, x_{n_l}; \sigma_{n_k}, \sigma_{n_l}) = 0.$$

We denote the set of all contractive functions on $B \times B$ by $\text{Contr}(B, \Sigma)$.

Lemma 2.4 ([32]) Let $\{U_\sigma(t, \tau)\}$ ($\sigma \in \Sigma, t \geq \tau, \tau \in \mathbb{R}$) be a family of processes satisfying the translation identity (2.8)–(2.9) on a Banach space E and having a bounded uniformly (w.r.t. $\sigma \in \Sigma$) absorbing set $B_0 \in \mathcal{B}(E)$. Moreover, assume that for any $\varepsilon > 0$ there exist $T = T(B_0, \varepsilon)$ and $\psi_T \in \text{Contr}(B_0, \Sigma)$ such that

$$\|U_{\sigma_1}(T, \tau)x - U_{\sigma_2}(T, \tau)y\| \leq \varepsilon + \psi(x, y; \sigma_1, \sigma_2), \quad \forall \sigma_1, \sigma_2 \in \Sigma, x, y \in B_0.$$

Then $\{U_\sigma(t, \tau)\}, \sigma \in \Sigma$ is uniformly (w.r.t. $\sigma \in \Sigma$) asymptotically compact in E .

Let X be a Banach space with spatial, the space $L^2_{\text{loc}}(\mathbb{R}^+; X)$ denotes all functions with spatial values in Banach space X and time variable locally 2-power integrable in the Bochner sense. $L^2_b(\mathbb{R}^+; X)$ is a set of all translation bound functions in $L^2_{\text{loc}}(\mathbb{R}^+; X)$ satisfying

$$\|\sigma\|_{L^2_b(\mathbb{R}^+; X)} = \sup_{t \in \mathbb{R}^+} \int_t^{t+1} \|\sigma(s)\|_X^2 ds < +\infty$$

for all $\sigma \in L^2_b(\mathbb{R}^+; X)$.

Theorem 2.5 ([30, 32]) Let E be a complete metric space, $\{U_\sigma(t, \tau)\}, \sigma \in \Sigma$ be a family of processes on E satisfying the translation identity (2.8)–(2.9). Then $\{U_\sigma(t, \tau)\}, \sigma \in \Sigma$ has a compactly uniform attractor (w.r.t. $\sigma \in \Sigma$) attractor \mathcal{A}_Σ in E and satisfies

$$\mathcal{A}_\Sigma = \omega_{0, \Sigma}(B_0) = \omega_{\tau, \Sigma}(B_0) = \bigcup_{B \in \mathcal{B}(E)} \omega_{\tau, \Sigma}(B), \quad \forall \tau \in \mathbb{R},$$

if and only if $\{U_\sigma(t, \tau)\}, \sigma \in \Sigma$

- (i) has a bounded uniformly (w.r.t. $\sigma \in \Sigma$) absorbing set B_0 ;
- (ii) is uniformly (w.r.t. $\sigma \in \Sigma$) asymptotically compact.

To describe the asymptotic behavior of the solutions to our problem, we set $g_0 \in L^2_b(\mathbb{R}^+; L^2(\Omega))$ and define the symbol space $\mathcal{H}(\sigma_0), \mathcal{H}(\sigma_0) = \mathcal{H}(g_0) = [g_0(x, s + r) | r \in \mathbb{R}^+]_{L^{2,w}_{\text{loc}}(\mathbb{R}^+; L^2(\Omega))}$, where $[\]$ denotes the closure of a set in a topological space $L^{2,w}_{\text{loc}}(\mathbb{R}^+; L^2(\Omega))$, and $L^{2,w}_{\text{loc}}(\mathbb{R}^+; L^2(\Omega))$ denotes the space $L^{2,w}_{\text{loc}}(\mathbb{R}^+; L^2(\Omega))$ endowed with local weak convergence topology. Therefore, for any $g(x, t) \in \mathcal{H}(g_0)$, (2.3)–(2.5) with g_0 instead of g possesses a corresponding process $\{U_g(t, \tau)\}$ acting on \mathcal{H} . The translation semigroup $\{T(r) | r \geq 0\}$ satisfies (2.8), (2.9), that is, $T(r)\mathcal{H}(g_0) = \mathcal{H}(g_0)$ and $U_g(t + r, \tau + r) = U_{T(r)g(t)}(t, \tau), \forall g \in \mathcal{H}(g_0), t \geq \tau, \tau \in \mathbb{R}, r \geq 0$.

Proposition 2.6 ([29]) If E is a reflexive separable Banach space, then

- (1) for all $g \in \mathcal{H}(g_0)$, $\|g\|_{L^2_b(\mathbb{R}^+; L^2(\Omega))} \leq \|g_0\|_{L^2_b(\mathbb{R}^+; L^2(\Omega))}$,
- (2) the translation group $T(t)$ is weakly continuous on $\mathcal{H}(g_0)$,
- (3) $T(t)\mathcal{H}(g_0) = \mathcal{H}(g_0)$ for all $t \in \mathbb{R}$.

3 Uniform attractors in \mathcal{H}

Firstly, we establish the well-posedness of problem (2.3)–(2.5).

Set $U = (u, v, z)^T$, where $v = \partial_t u$, initial data $U_\tau = (u_0^\tau, u_1^\tau, \phi_0)^T$, then problem (2.3)–(2.5) is given by

$$\begin{cases} U_t + AU = F, \\ U(\tau) = U_\tau, \end{cases} \tag{3.1}$$

where

$$AU = \begin{pmatrix} -v \\ \Delta^2 u + \gamma_1 v + \gamma_2 z(1, \cdot) \\ \frac{1}{h} \rho_z \end{pmatrix}, \quad F(U) = \begin{pmatrix} 0 \\ -f(u) + g(t) \\ 0 \end{pmatrix}, \quad U_\tau = \begin{pmatrix} u_0^\tau \\ u_1^\tau \\ \phi_0 \end{pmatrix}.$$

The domain of the operator A is defined as follows:

$$D(A) = \{(u, v, z) \in \mathcal{H} \mid u \in H^4(\Omega), v \in H^2_*(\Omega), z(0, \cdot) = v, z_\rho \in L^2((0, 1) \times \Omega), \text{ and (3.2) holds}\},$$

where (3.2) is the following boundary conditions:

$$\begin{cases} \partial_{xx}u(0, y) = \partial_{xx}u(\pi, y) = 0, \\ \partial_{yy}u(x, \pm l) + \sigma \partial_{xx}u(x, \pm l) = 0, \\ \partial_{yyy}u(x, \pm l) + (2 - \sigma) \partial_{xxy}u(x, \pm l) = 0. \end{cases} \tag{3.2}$$

For convenience, we denote $d\theta = dx dy$.

Lemma 3.1 ([33]) *Suppose that $0 < |\gamma_2| \leq \gamma_1$, the operator $A : D(A) \subset \mathcal{H} \rightarrow \mathcal{H}$ is monotone and maximal.*

By the semigroup theory, see [33], we have the following result.

Theorem 3.2 *Assume that condition (H1) holds and $g \in L^2_b(\mathbb{R}_\tau; L^2(\Omega))$. Let $U_\tau \in \mathcal{H}$ be given and assume that $\gamma_1 > \frac{3}{2}|\gamma_2|$, then problem (3.1) has a unique global solution $U = (u, \partial_t u, \phi) \in C(\mathbb{R}_\tau; \mathcal{H})$, where $\mathbb{R}_\tau = \{t \in \mathbb{R}, t \geq \tau\}$.*

Proof Thanks to Lemma 3.1 and F satisfying locally Lipschitz, we can apply the Hille–Yosida theorem to get the existence of a unique weak local solution for problem (3.1),

$$U = (u, \partial_t u, \phi) \in C([\tau, T_m], \mathcal{H}), \quad \forall T_m > 0.$$

In order to prove that the solution is global and $U \in C([\tau, \infty], \mathcal{H})$, namely we need to prove that $\|U(t)\|_{\mathcal{H}}$ is uniformly bounded with respect to time.

Multiplying (2.3)₁ by $\partial_t u$ and integrating over Ω , we obtain

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{2} \|\partial_t u\|_{L^2(\Omega)}^2 + \frac{1}{2} \|u\|_{H_*^2(\Omega)}^2 + \int_{\Omega} F(u) d\theta \right) \\ & + \gamma_1 \|\partial_t u\|_{L^2(\Omega)}^2 + \gamma_2 (\partial_t u, z(1)) = (g(t), \partial_t u). \end{aligned} \tag{3.3}$$

Multiplying (2.3)₂ by $|\gamma_2|z$ and integrating over $(0, 1) \times \Omega$, we get

$$h|\gamma_2| \int_0^1 \int_{\Omega} \partial_t z z d\theta d\rho + |\gamma_2| \int_0^1 \int_{\Omega} \partial_{\rho} z z d\theta d\rho = 0. \tag{3.4}$$

Thus

$$\frac{d}{dt} \left(\frac{h|\gamma_2|}{2} \int_0^1 \int_{\Omega} |z|^2 d\theta d\rho \right) + \frac{|\gamma_2|}{2} \int_{\Omega} \int_0^1 \frac{d}{d\rho} |z|^2 d\rho d\theta = 0.$$

Furthermore, we have

$$\frac{d}{dt} \left(\frac{h|\gamma_2|}{2} \int_0^1 \int_{\Omega} |z|^2 d\theta d\rho \right) + \frac{|\gamma_2|}{2} \int_{\Omega} |z(1)|^2 d\theta - \frac{|\gamma_2|}{2} \int_{\Omega} |\partial_t u|^2 d\theta = 0. \tag{3.5}$$

Set

$$E_1(t) = \frac{1}{2} \|\partial_t u(t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|u(t)\|_{H_*^2(\Omega)}^2 + \int_{\Omega} F(u(t)) d\theta + \frac{1}{2} \|z(t)\|_{L^2((0,1) \times \Omega)}^2. \tag{3.6}$$

Combining (3.3)–(3.5), we get that

$$\begin{aligned} \frac{d}{dt} E_1(t) &= -\gamma_1 \|\partial_t u\|_{L^2(\Omega)}^2 - \gamma_2 (\partial_t u, z(1)) - \frac{|\gamma_2|}{2} \|z(1)\|_{L^2(\Omega)}^2 \\ &+ |\gamma_2| \|\partial_t u\|_{L^2(\Omega)}^2 + \frac{1}{2|\gamma_2|} \|g(t)\|_{L^2(\Omega)}^2. \end{aligned} \tag{3.7}$$

Exploiting Young’s inequality, we obtain

$$-\gamma_2 (\partial_t u, z(1)) \leq \frac{|\gamma_2|}{2} \|z(1)\|_{L^2(\Omega)}^2 + \frac{|\gamma_2|}{2} \|\partial_t u\|_{L^2(\Omega)}^2. \tag{3.8}$$

Substituting (3.8) into (3.7), we find that

$$\frac{d}{dt} E_1(t) \leq -\left(\gamma_1 - \frac{3}{2} |\gamma_2| \right) \|\partial_t u\|_{L^2(\Omega)}^2 + \frac{1}{2|\gamma_2|} \|g(t)\|_{L^2(\Omega)}^2. \tag{3.9}$$

Integrating (3.9) over $[\tau, t]$, $\gamma_1 > \frac{3}{2} |\gamma_2|$, we can deduce that

$$E_1(t) \leq E_1(\tau) + \frac{1}{2|\gamma_2|} \int_{\tau}^t \|g(s)\|_{L^2(\Omega)}^2 ds, \quad \forall t \geq \tau.$$

By Proposition 2.6, we know that $\|g\|_{L_b^2(\mathbb{R}_{\tau}; L^2(\Omega))}^2 \leq \|g_0\|_{L_b^2(\mathbb{R}_{\tau}; L^2(\Omega))}^2$, one sees that

$$E_1(t) \leq E_1(\tau) + \frac{1}{2|\gamma_2|} \|g_0\|_{L_b^2(\mathbb{R}_{\tau}; L^2(\Omega))}^2. \tag{3.10}$$

Applying (H1), $E_1(t)$ satisfies

$$E_1(t) \geq \frac{1}{2} \|\partial_t u(t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|u(t)\|_{H_*^2(\Omega)}^2 + \frac{1}{2} \|z(t)\|_{L^2((0,1)\times\Omega)}^2 - c_1|\Omega|,$$

namely

$$E_1(t) \geq c_2\|(u(t), \partial_t u(t), z(t))\|_{\mathcal{H}}^2 - c_3, \quad \forall t \geq \tau. \tag{3.11}$$

Using (3.10) and (3.11), it yields that

$$\|(u(t), \partial_t u(t), z(t))\|_{\mathcal{H}}^2 \leq \frac{1}{c_2} \left(E_1(\tau) + \frac{1}{2|\gamma_2|} \|g_0\|_{L_b^2(\mathbb{R}_\tau; L^2(\Omega))}^2 + c_3 \right) \leq c_4, \quad \forall t \geq \tau. \tag{3.12}$$

□

Owing to Theorem 3.2, we know that (2.3)–(2.5) is well-posed for all $g(s) \in \mathcal{H}(g_0)$ and generates a family of processes $\{U_g(t, \tau)\}, g \in \mathcal{H}(g_0)$ given by $U_g(t, \tau)U_\tau = U(t)$, where $U(t)$ is the solution of (2.3)–(2.5), and $\{U_g(t, \tau)\}, g \in \mathcal{H}(g_0)$ satisfies (2.7).

In what follows, we denote by $\{U_g(t, \tau)\}, g \in \mathcal{H}(g_0)$ the family of processes generated by (2.3)–(2.5).

Secondly, we prove the existence of a uniformly absorbing set in \mathcal{H} . We introduce a Lyapunov functional

$$L_1(t) = ME_1(t) + N\Psi_1(t) + \Phi_1(t),$$

where $M, N > 0$ are constants which will be determined later, and

$$\Psi_1(t) = (\partial_t u, u), \quad \Phi_1(t) = h \int_{\Omega} \int_0^1 |z|^2 e^{-\rho h} d\rho d\theta.$$

Lemma 3.3 *Assume that N is small enough and M is large enough, there exist $\delta_1, \delta_2 > 0$ such that*

$$\delta_1 \|(u, \partial_t u, z)\|_{\mathcal{H}}^2 - c_5 \leq L_1(t) \leq \delta_2 \|(u, \partial_t u, z)\|_{\mathcal{H}}^2 + Mc_6. \tag{3.13}$$

Proof On the one hand, by (H1)₂ and the above-mentioned functionals, choosing M large enough and N small enough such that $\frac{M-N}{2} > 0, \frac{M-Nc_\lambda}{2} > 0$, we can deduce that

$$\begin{aligned} L_1(t) &\geq \frac{M}{2} \|u\|_{H_*^2(\Omega)}^2 + \frac{M}{2} \|\partial_t u\|_{L^2(\Omega)}^2 + \frac{Mh|\gamma_2|}{2} \int_{\Omega} \int_0^1 |z(t)|^2 e^{-\rho h} d\rho d\theta - c_1|\Omega|M \\ &\quad - \frac{Nc_\lambda}{2} \|u\|_{H_*^2(\Omega)}^2 - \frac{N}{2} \|\partial_t u\|_{L^2(\Omega)}^2 + h \int_{\Omega} \int_0^1 |z(t)|^2 e^{-\rho h} d\rho d\theta \\ &\geq \left(\frac{M-Nc_\lambda}{2}\right) \|u\|_{H_*^2(\Omega)}^2 + \left(\frac{M-N}{2}\right) \|\partial_t u\|_{L^2(\Omega)}^2 \\ &\quad + \left(\frac{(M|\gamma_2|+2)h}{2}\right) \int_{\Omega} \int_0^1 |z(t)|^2 e^{-\rho h} d\rho d\theta - c_1|\Omega|M \\ &\geq \delta_1 (\|u\|_{H_*^2(\Omega)}^2 + \|\partial_t u\|_{L^2(\Omega)}^2 + \|z\|_{L^2((0,1)\times\Omega)}^2) - c_5. \end{aligned}$$

On the other hand, according to (H1)₁, we have

$$\begin{aligned} \int_{\Omega} F(u(t)) \, d\theta &\leq \int_{\Omega} |u| |f(u)| \, d\theta \leq \int_{\Omega} |u| |f(u) - f(0)| \, d\theta + \int_{\Omega} |u| |f(0)| \, d\theta \\ &\leq c_0 \int_{\Omega} |u|^2 |u|^p \, d\theta + \frac{1}{2} \int_{\Omega} |u|^2 \, d\theta + \frac{1}{2} |\Omega| |f(0)| \\ &\leq c_0 c_{\lambda} (\|u\|_{L^{\infty}(\Omega)}^p + 1) \|u\|_{H_*^2(\Omega)}^2 + c_6. \end{aligned} \tag{3.14}$$

Exploiting Young’s inequality, Sobolev’s embedding Theorem, and (3.14), we can get that

$$\begin{aligned} L_1(t) &\leq \frac{M}{2} \|u\|_{H_*^2(\Omega)}^2 + \frac{M}{2} \|\partial_t u\|_{L^2(\Omega)}^2 + \frac{Mh|\gamma_2|}{2} \int_{\Omega} \int_0^1 |z(t)|^2 e^{-\rho h} \, d\rho \, d\theta \\ &\quad + M \int_{\Omega} F(u(t)) \, d\theta + \frac{N}{2} \|\partial_t u\|_{L^2(\Omega)}^2 + \frac{Nc_{\lambda}}{2} \|u\|_{H_*^2(\Omega)}^2 + h \int_{\Omega} \int_0^1 |z(t)|^2 e^{-\rho h} \, d\rho \, d\theta \\ &\leq \left(\frac{M+N}{2}\right) \|\partial_t u\|_{L^2(\Omega)}^2 + \left(\frac{M+Nc_{\lambda}}{2}\right) \|u\|_{H_*^2(\Omega)}^2 \\ &\quad + \left(\frac{(M|\gamma_2|+2)h}{2}\right) \int_{\Omega} \int_0^1 |z(t)|^2 e^{-\rho h} \, d\rho \, d\theta + M \int_{\Omega} |u| |f(u)| \, d\theta \\ &\leq \left(\frac{M+N}{2}\right) \|\partial_t u\|_{L^2(\Omega)}^2 + \left(\frac{M+(N+2Mc_0[\|u\|_{L^{\infty}(\Omega)}^p+1])c_{\lambda}}{2}\right) \|u\|_{H_*^2(\Omega)}^2 \\ &\quad + \left(\frac{(M|\gamma_2|+2)h}{2}\right) \int_{\Omega} \int_0^1 |z(t)|^2 e^{-\rho h} \, d\rho \, d\theta \\ &\leq \delta_2 (\|u\|_{H_*^2(\Omega)}^2 + \|\partial_t u\|_{L^2(\Omega)}^2 + \|z\|_{L^2((0,1)\times\Omega)}^2) + Mc_6. \end{aligned} \quad \square$$

Lemma 3.4 *The function $\Psi_1(t) = (\partial_t u, u)$ satisfies*

$$\begin{aligned} \Psi_1'(t) &\leq \left(1 + \frac{\gamma_1}{2\xi}\right) \|\partial_t u\|_{L^2(\Omega)}^2 + \left[\left(\gamma_1\xi + |\gamma_2|\xi + |\gamma_2|\right) \frac{c_{\lambda}}{2} - 1\right] \|u\|_{H_*^2(\Omega)}^2 \\ &\quad + \frac{|\gamma_2|}{2\xi} \|z(1)\|_{L^2(\Omega)}^2 + \frac{1}{2|\gamma_2|} \|g(t)\|_{L^2(\Omega)}^2 + c_1 |\Omega|, \quad \forall \xi > 0. \end{aligned} \tag{3.15}$$

Proof According to (2.3)₁, we can see that

$$\begin{aligned} \Psi_1'(t) &= (\partial_{tt} u, u) + \|\partial_t u\|_{L^2(\Omega)}^2 \\ &= \|\partial_t u\|_{L^2(\Omega)}^2 - \|u\|_{H_*^2(\Omega)}^2 - \gamma_1 (\partial_t u, u) - \gamma_2 (u, z(1)) \\ &\quad - (f(u), u) + (g(t), u). \end{aligned} \tag{3.16}$$

Using Young’s inequality, Hölder’s inequality, and (2.6), for any $\xi > 0$, we can obtain that

$$\begin{aligned} -\gamma_1 (\partial_t u, u) &\leq \frac{\gamma_1 \xi}{2} \|u\|_{L^2(\Omega)}^2 + \frac{\gamma_1}{2\xi} \|\partial_t u\|_{L^2(\Omega)}^2 \leq c_{\lambda} \frac{\gamma_1 \xi}{2} \|u\|_{H_*^2(\Omega)}^2 + \frac{\gamma_1}{2\xi} \|\partial_t u\|_{L^2(\Omega)}^2, \\ -\gamma_2 (u, z(1)) &\leq \frac{|\gamma_2| \xi}{2} \|u\|_{L^2(\Omega)}^2 + \frac{|\gamma_2|}{2\xi} \|z(1)\|_{L^2(\Omega)}^2 \leq c_{\lambda} \frac{|\gamma_2| \xi}{2} \|u\|_{H_*^2(\Omega)}^2 + \frac{|\gamma_2|}{2\xi} \|z(1)\|_{L^2(\Omega)}^2, \\ (g(t), u) &\leq \frac{1}{2|\gamma_2|} \|g(t)\|_{L^2(\Omega)}^2 + \frac{c_{\lambda} |\gamma_2|}{2} \|u\|_{H_*^2(\Omega)}^2. \end{aligned}$$

By assumption (H1), we have

$$-(f(u), u) \leq c_1|\Omega|.$$

Substituting the above inequalities into (3.16), we get (3.15). □

Lemma 3.5 *The functional $\Phi_1(t) = h \int_{\Omega} \int_0^1 |z|^2 e^{-\rho h} d\rho d\theta$ satisfies*

$$\Phi_1'(t) \leq -e^{-h} \left(\|z(1)\|_{L^2(\Omega)}^2 + h \int_{\Omega} \int_0^1 |z(t)|^2 d\rho d\theta \right) + \|\partial_t u\|_{L^2(\Omega)}^2. \tag{3.17}$$

Proof Direct differentiation (2.3)₂ leads to

$$\begin{aligned} \Phi_1'(t) &= 2h \int_{\Omega} \int_0^1 \partial_t z z e^{-\rho h} d\rho d\theta = -2 \int_{\Omega} \int_0^1 \partial_{\rho} z z e^{-\rho h} d\rho d\theta \\ &= - \int_{\Omega} \int_0^1 \frac{d}{d\rho} (|z|^2 e^{-\rho h}) d\rho d\theta - h \int_{\Omega} \int_0^1 |z|^2 e^{-\rho h} d\theta d\rho \\ &= -e^{-h} \|z(1)\|_{L^2(\Omega)}^2 + \|\partial_t u\|_{L^2(\Omega)}^2 - h \int_{\Omega} \int_0^1 |z|^2 e^{-\rho h} d\theta d\rho \\ &\leq -e^{-h} \left(\|z(1)\|_{L^2(\Omega)}^2 + h \int_{\Omega} \int_0^1 |z(t)|^2 d\rho d\theta \right) + \|\partial_t u\|_{L^2(\Omega)}^2. \end{aligned} \tag{3.18}$$

□

Theorem 3.6 (Uniformly (w.r.t. $g \in \mathcal{H}(g_0)$) absorbing set) *Under the assumption of Theorem 3.2, the family of processes $\{U_g(t, \tau)\}$, $g \in \mathcal{H}(g_0)$ of system (2.3)–(2.5) possesses a bounded uniformly (w.r.t. $g \in \mathcal{H}(g_0)$) absorbing set B in \mathcal{H} .*

Proof Exploiting (3.9), (3.15), (3.17), Lemma 3.4, and Lemma 3.5, direct computation yields

$$\begin{aligned} L_1'(t) &= ME_1'(t) + N\Psi_1'(t) + \Phi_1'(t) \\ &\leq -N \left(1 - (\gamma_1 \xi + |\gamma_2| \xi + |\gamma_2|) \frac{c_{\lambda}}{2} \right) \|u\|_{H_{*}^2(\Omega)}^2 \\ &\quad - \left(M \left(\gamma_1 - \frac{3}{2} |\gamma_2| \right) - N - \frac{\gamma_1 N}{2\xi} - 1 \right) \|\partial_t u\|_{L^2(\Omega)}^2 \\ &\quad - \left(e^{-h} - \frac{N|\gamma_2|}{2\xi} \right) \|z(1)\|_{L^2(\Omega)}^2 - e^{-h} h \|z\|_{L^2((0,1) \times \Omega)}^2 \\ &\quad + \frac{(M + N)}{2|\gamma_2|} \|g(t)\|_{L^2(\Omega)}^2 + Nc_1|\Omega|. \end{aligned} \tag{3.19}$$

Choosing ξ and N small enough, M large enough such that

$$\begin{aligned} 1 - (\gamma_1 \xi + |\gamma_2| \xi + |\gamma_2|) \frac{c_{\lambda}}{2} &> 0, & e^{-h} - \frac{N|\gamma_2|}{2\xi} &> 0, \\ M \left(\gamma_1 - \frac{3}{2} |\gamma_2| \right) - N - \frac{\gamma_1 N}{2\xi} - 1 &> 0. \end{aligned}$$

Thus, there exist positive constants $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta$ such that

$$\begin{aligned} \frac{dL_1(t)}{dt} &\leq -\beta_1 \|u\|_{H_x^2(\Omega)}^2 - \beta_2 \|\partial_t u\|_{L^2(\Omega)}^2 - \beta_3 \|z\|_{L^2((0,1)\times\Omega)}^2 + \beta_4 \|g(t)\|_{L^2(\Omega)}^2 + \beta_5 \\ &\leq -\beta (\|u\|_{H_x^2(\Omega)}^2 + \|\partial_t u\|_{L^2(\Omega)}^2 + \|z\|_{L^2((0,1)\times\Omega)}^2) + \beta_4 \|g(t)\|_{L^2(\Omega)}^2 + \beta_5. \end{aligned} \tag{3.20}$$

Applying Lemma 3.3, we can get

$$\frac{dL_1(t)}{dt} + \nu L_1(t) \leq c_7 \|g(t)\|_{L^2(\Omega)}^2 + c_8, \tag{3.21}$$

where $\nu = \frac{\beta}{\delta_2}$. Integrating (3.21) over $[\tau, t]$ and using (3.20), we can see that

$$\begin{aligned} L_1(t) &\leq L_1(\tau)e^{-\nu(t-\tau)} + c_7 \int_{\tau}^t e^{-\nu(t-s)} \|g(s)\|_{L^2(\Omega)}^2 ds + c_8 \int_{\tau}^t e^{-\nu(t-s)} ds \\ &\leq L_1(\tau)e^{-\nu(t-\tau)} + \frac{c_7}{1 - e^{-\nu}} \sup_{t \geq \tau} \int_t^{t+1} \|g(s)\|_{L^2(\Omega)}^2 ds + c_9 \\ &\leq L_1(\tau)e^{-\nu(t-\tau)} + \frac{c_{10}}{1 - e^{-\nu}} \|g_0\|_{L_b^2(\mathbb{R}_\tau; L^2(\Omega))} + c_9. \end{aligned} \tag{3.22}$$

Now, for any bounded set $B \subseteq \mathcal{H}$ and for any $(u_0^\tau, u_1^\tau, \phi_0) \in B$, there exists a constant $C_B > 0$ such that $L_1(\tau) \leq C_B$, then for any $t \geq t_0$, we have

$$L_1(t) \leq C_B e^{-\nu(t-\tau)} + \frac{c_{10}}{1 - e^{-\nu}} \|g_0\|_{L_b^2(\mathbb{R}_\tau; L^2(\Omega))} + c_9. \tag{3.23}$$

From (3.23), it is easy to see that

$$\|(u(t), \partial_t u(t), z(t))\|_{\mathcal{H}}^2 \leq L_1(t) = r^2,$$

that is, $B = B(0, r) = \{(u(t), \partial_t u(t), z(t)) \in \mathcal{H} : \|(u(t), \partial_t u(t), z(t))\|_{\mathcal{H}}^2 \leq r^2\} \subseteq \mathcal{H}$ is a uniformly absorbing ball for any $g \in \mathcal{H}(g_0)$. The proof is complete. \square

Next, we will verify that the family of processes $\{U_g(t, \tau)\}$, $g \in \mathcal{H}(g_0)$ corresponding to (2.3)–(2.5) is uniformly (w.r.t. $g \in \mathcal{H}(g_0)$) asymptotically compact in \mathcal{H} .

Theorem 3.7 *Assume that condition (H1) holds and $g \in \mathcal{H}(g_0)$. Let $\gamma_1 > 2|\gamma_2|$, then the family of processes $\{U_g(t, \tau)\}$, $g \in \mathcal{H}(g_0)$ corresponding to (2.3)–(2.5) is uniformly (w.r.t. $g \in \mathcal{H}(g_0)$) asymptotically compact in \mathcal{H} .*

Proof We consider two symbols $g^1, g^2 \in \mathcal{H}(g_0)$, and let $(u^i, \partial_t u^i, z^i)$ be solutions of (2.3) corresponding to the initial data (u_0^i, u_1^i, ϕ_0^i) and the symbols g^1, g^2 , respectively.

Let $u = u^1 - u^2, z = z^1 - z^2, u_0^\tau = u_0^{1\tau} - u_0^{2\tau}, u_1^\tau = u_1^{1\tau} - u_1^{2\tau}, \phi_0 = \phi_0^1 - \phi_0^2$. Therefore, (u, z) satisfies

$$\begin{cases} \partial_{tt} u + \Delta^2 u + \gamma_1 \partial_t u + \gamma_2 z(1) + f(u^1) - f(u^2) \\ \quad = g^1(t) - g^2(t), & \Omega \times (\tau, +\infty), \\ h \partial_t z(\rho, x, y, t) + \partial_\rho z(\rho, x, y, t) = 0, & (0, 1) \times \Omega \times (\tau, +\infty), \end{cases} \tag{3.24}$$

with boundary conditions

$$\begin{cases} u(0, y, t) = \partial_{xx}u(0, y, t) = u(\pi, y, t) = \partial_{xx}u(\pi, y, t) = 0, & y \in (-l, l), t \geq \tau, \\ \partial_{yy}u(x, \pm l, t) + \sigma \partial_{xx}u(x, \pm l, t) = 0, & x \in (0, \pi), t \geq \tau, \\ \partial_{yyy}u(x, \pm l, t) + (2 - \sigma) \partial_{xxy}u(x, \pm l, t) = 0, & x \in (0, \pi), t \geq \tau. \end{cases}$$

We introduce the function

$$E_2(t) = \frac{1}{2} \|\partial_t u(t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|u(t)\|_{H_x^2(\Omega)}^2 + \frac{1}{2} \|z(t)\|_{L^2((0,1) \times \Omega)}^2, \tag{3.25}$$

and

$$L_2(t) = ME_2(t) + N\Psi_2(t) + \Phi_2(t),$$

where $\Psi_2(t) = (\partial_t u, u)$, $\Phi_2(t) = h \int_{\Omega} \int_0^1 |z|^2 e^{-\rho h} d\rho d\theta$.

It is easy to see that $L_2(t)$ and $E_2(t)$ are equivalent, so there exist two positive constants λ_1 and λ_2 depending on M, N such that

$$\lambda_1 E_2(t) \leq L_2(t) \leq \lambda_2 E_2(t), \tag{3.26}$$

where $M > 0$ large enough and $N > 0$ small enough.

Multiplying (3.24)₁ by $\partial_t u$ and (3.24)₂ by $|\gamma_2|z$, then integrating over Ω and $(0, 1) \times \Omega$, respectively, we obtain that

$$\begin{aligned} \frac{d}{dt} E_2(t) &\leq -\gamma_1 \|\partial_t u\|_{L^2(\Omega)}^2 + \frac{|\gamma_2|}{2} \|z(1)\|_{L^2(\Omega)}^2 + \frac{|\gamma_2|}{2} \|\partial_t u\|_{L^2(\Omega)}^2 \\ &\quad - \frac{|\gamma_2|}{2} \|z(1)\|_{L^2(\Omega)}^2 + \frac{|\gamma_2|}{2} \|\partial_t u\|_{L^2(\Omega)}^2 - \int_{\Omega} (f(u^1(t)) - f(u^2(t))) \partial_t u d\theta \\ &\quad + \int_{\Omega} (g^1(t) - g^2(t)) \partial_t u d\theta \\ &\leq c_{\gamma_2} \int_{\Omega} |f(u^1(t)) - f(u^2(t))|^2 d\theta - \left(\gamma_1 - \frac{3|\gamma_2|}{2}\right) \|\partial_t u\|_{L^2(\Omega)}^2 \\ &\quad + \int_{\Omega} (g^1(t) - g^2(t)) \partial_t u d\theta. \end{aligned} \tag{3.27}$$

Using (H1)₁, we find that

$$\int_{\Omega} |f(u^1(t)) - f(u^2(t))|^2 d\theta \leq c_0 \int_{\Omega} (|u^1|^{2p} + |u^2|^{2p}) |u|^2 d\theta \leq c_B \|u(\cdot, t)\|_{L^2(\Omega)}^2, \tag{3.28}$$

$$\int_{\Omega} (g^1(t) - g^2(t)) \partial_t u d\theta \leq c_{\gamma_2} \|g^1(t) - g^2(t)\|_{L^2(\Omega)}^2 + \frac{|\gamma_2|}{2} \|\partial_t u\|_{L^2(\Omega)}^2, \tag{3.29}$$

where c_B, c_{γ_2} are positive constants.

Substituting (3.28), (3.29) into (3.27), we get

$$\frac{d}{dt} E_2(t) \leq c_B \|u(\cdot, t)\|_{L^2(\Omega)}^2 - (\gamma_1 - 2|\gamma_2|) \|\partial_t u\|_{L^2(\Omega)}^2 + c_{\delta_2} \|g^1(t) - g^2(t)\|_{L^2(\Omega)}^2. \tag{3.30}$$

Similar to the proof of (3.15) and (3.17), we have

$$\begin{aligned} \Psi_2'(t) &\leq \left(1 + \frac{\gamma_1}{2\xi}\right) \|\partial_t u\|_{L^2(\Omega)}^2 + \left((\gamma_1\xi + |\gamma_2|\xi + 3|\gamma_2|)\frac{c_\lambda}{2} - 1\right) \|u\|_{H_*^2(\Omega)}^2 \\ &\quad + \frac{|\gamma_2|}{2\xi} \|z(1)\|_{L^2(\Omega)}^2 + c_{\gamma_2} \|g^1(t) - g^2(t)\|_{L^2(\Omega)}^2 + c_B \|u(\cdot, t)\|_{L^2(\Omega)}^2, \end{aligned} \tag{3.31}$$

and

$$\Phi_2'(t) \leq -e^{-h} \left(\|z(1)\|_{L^2(\Omega)}^2 + h \int_{\Omega} \int_0^1 |z(t)|^2 d\rho d\theta \right) + \|\partial_t u\|_{L^2(\Omega)}^2. \tag{3.32}$$

Using (3.30)–(3.32), direct computation

$$\begin{aligned} L_2'(t) &= ME_2'(t) + N\Psi_2'(t) + \Phi_2'(t) \\ &\leq -N \left(1 - (\gamma_1\xi + |\gamma_2|\xi + 3|\gamma_2|)\frac{c_\lambda}{2}\right) \|u\|_{H_*^2(\Omega)}^2 \\ &\quad - \left(M(\gamma_1 - 2|\gamma_2|) - N - \frac{\gamma_1 N}{2\xi} - 1\right) \|\partial_t u\|_{L^2(\Omega)}^2 \\ &\quad - \left(e^{-h} - \frac{N|\gamma_2|}{2\xi}\right) \|z(1)\|_{L^2(\Omega)}^2 - e^{-h} h \|z\|_{L^2((0,1)\times\Omega)}^2 \\ &\quad + (M + N)c_{\gamma_2} \|g^1(t) - g^2(t)\|_{L^2(\Omega)}^2 + (M + N)c_B \|u(\cdot, t)\|_{L^2(\Omega)}^2. \end{aligned} \tag{3.33}$$

Choose ξ and N small enough, M large enough such that

$$\begin{aligned} 1 - (\gamma_1\xi + |\gamma_2|\xi + 3|\gamma_2|)\frac{c_\lambda}{2} &> 0, \quad e^{-h} - \frac{N|\gamma_2|}{2\xi} > 0, \\ M(\gamma_1 - 2|\gamma_2|) - N - \frac{\gamma_1 N}{2\xi} - 1 &> 0. \end{aligned}$$

Then there exists $\varpi > 0$ such that

$$\begin{aligned} \frac{dL_2(t)}{dt} &\leq -\varpi E_2(t) + (M + N)c_{\gamma_2} \|g^1(t) - g^2(t)\|_{L^2(\Omega)}^2 \\ &\quad + (M + N)c_B \|u(\cdot, t)\|_{L^2(\Omega)}^2. \end{aligned} \tag{3.34}$$

Exploiting (3.26), from (3.34), we obtain

$$\frac{dL_2(t)}{dt} + \varrho L_2(t) \leq (M + N)c_{\gamma_2} \|g^1(t) - g^2(t)\|_{L^2(\Omega)}^2 + (M + N)c_B \|u(\cdot, t)\|_{L^2(\Omega)}^2, \tag{3.35}$$

where $\varrho = \frac{\varpi}{\lambda_2}$. Integrating (3.35) over $[\tau, t]$, we can deduce that

$$\begin{aligned} L_2(t) &\leq L_2(\tau)e^{-\varrho(t-\tau)} + (M + N)c_{\gamma_2} \int_{\tau}^t e^{-\varrho(t-s)} \|g^1(s) - g^2(s)\|_{L^2(\Omega)}^2 ds \\ &\quad + (M + N)c_B \int_{\tau}^t e^{-\varrho(t-s)} \|u(\cdot, s)\|_{L^2(\Omega)}^2 ds. \end{aligned} \tag{3.36}$$

Let $T > \tau$ large enough such that

$$L_2(\tau)e^{-\varrho(T-\tau)} \leq \varepsilon. \tag{3.37}$$

Together with (3.26), (3.36), and (3.37), it yields

$$\begin{aligned} E_2(t) &\leq \varepsilon + c_{11} \int_{\tau}^t \|g^1(s) - g^2(s)\|_{L^2(\Omega)}^2 ds + c_{12} \int_{\tau}^t \|u(\cdot, s)\|_{L^2(\Omega)}^2 ds \\ &:= \varepsilon + \psi_T((u_0^{1\tau}, u_1^{1\tau}, \phi_0^1), (u_0^{2\tau}, u_1^{2\tau}, \phi_0^2); g^1, g^2). \end{aligned}$$

Subsequently, we will show $\psi_T(\cdot, \cdot; \cdot, \cdot) \in \text{Contr}(B, \mathcal{H}(g_0))$ for every fixed $T > \tau$. Because of the existence of a uniformly absorbing set, we can obtain that, for any fixed $T > \tau$ and any bounded set B depending on T ,

$$\bigcup_{g \in \mathcal{H}(g_0)} \bigcup_{t \in [\tau, T]} U_g(t, \tau)B$$

is bounded in \mathcal{H} . Let the sequence $(u_{0n}^\tau, u_{1n}^\tau, \phi_{0n}) \in B, g_n \in \mathcal{H}(g_0), n = 1, 2, \dots$. Since B is bounded, the corresponding sequence of solutions $(u_n(t), \partial_t u_n(t), z_n(t))$ of system (2.3)–(2.5) is uniformly bounded in \mathcal{H} . Hence, $(u_n, \partial_t u_n, z_n)$ is bounded in $C([\tau, T]; \mathcal{H})$, which implies that u_n is bounded in $C([\tau, T]; H^2(\Omega))$.

It follows that from the compact embedding of $C([\tau, T]; H^2(\Omega)) \cap C^1([\tau, T]; L^2(\Omega))$ into $C([\tau, T]; L^2(\Omega))$, there exists a subsequence $\{u_{n_k}\}$ that converges in $C([\tau, T]; L^2(\Omega))$, $\forall T > \tau$. So

$$\lim_{j \rightarrow \infty} \limsup_{k \rightarrow \infty} \int_{\tau}^t \|u_{n_j}(s) - u_{n_k}(s)\|^2 ds = 0. \tag{3.38}$$

On the other hand, by $g^n, g^m \in \mathcal{H}(g_0)$, we arrive at

$$\int_{\tau}^t \|g^n(s) - g^m(s)\|_{L^2(\Omega)}^2 ds \rightarrow 0, \quad m, n \rightarrow +\infty. \tag{3.39}$$

Hence, combining with (3.38)–(3.39), we get that $\psi_T \in \text{Contr}(B, \mathcal{H}(g_0))$, and then this completes the proof of Theorem 3.7. □

Theorem 3.8 (Uniform attractor) *Assume that conditions (H1)–(H2) hold and $g \in \mathcal{H}(g_0)$. Then the family of processes $\{U_g(t, \tau)\}, g \in \mathcal{H}(g_0)$ corresponding to (2.3)–(2.5) has a compactly uniform attractor (w.r.t. $g \in \mathcal{H}(g_0)$) \mathcal{A}_Σ in \mathcal{H} .*

Proof From Theorem 3.6 and Theorem 3.7 we know that the conditions of Theorem 2.5 are all satisfied. □

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Availability of data and materials

All data generated or analysed during this study are included in this published article.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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