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Rank-one perturbation bounds for singular values of arbitrary matrices

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Abstract

Rank-one perturbation of arbitrary matrices has many practical applications. In this paper, based on the relationship between the singular values and the eigenvalues, we discuss singular value variations and present two-side bounds of the singular values for rank-one perturbation of arbitrary matrices. Numerical results confirm that the proposed perturbation bounds are sharper than some existing bounds.

MSC: 65F15; 65H10; 65H20

Keywords: Arbitrary matrix; Singular values; Eigenvalue gap; Rank-one update

1 Introduction

Let $\mathcal{C}^{m \times n}$ (\mathcal{C}^n) be the set of $m \times n$ complex matrices (n dimension vectors). The norm $\|\cdot\|_2$ denotes the two norm and the notation $\mathcal{H}(A) = \frac{1}{2}(A + A^*)$ stands for the Hermitian parts of a square matrix A . Let $A \in \mathcal{C}^{n \times n}$ have the singular value decomposition (SVD)

$$A = U\Sigma V^*,$$

where $U \in \mathcal{C}^{n \times n}$ and $V \in \mathcal{C}^{n \times n}$ are unitary, $\Sigma = \text{diag}(\sigma_1(A), \dots, \sigma_n(A))$, $\sigma_i(A)$, $i = 1, 2, \dots, n$, are the singular values of A with

$$\sigma_{\min}(A) = \sigma_n(A) \leq \sigma_{n-1}(A) \leq \dots \leq \sigma_1(A) = \sigma_{\max}(A), \quad (1)$$

and the superscript $*$ denotes the conjugate transpose.

Singular value variations for rank-one perturbation of arbitrary matrices have many applications, e.g., principal component analysis under a spiked covariance model, and pseudo arc length continuation methods for the solution of systems of nonlinear equations, see [1–4]. Some classical perturbation bounds for singular values can be found in [5], and low rank update of singular values has also been investigated in [6].

In the paper, motivated by the ideas in [7], we consider rank-one perturbation bounds for singular values of arbitrary matrices. Before providing the new bounds, we first introduce the associated results about eigenvalues of Hermitian matrices in [7].

Let $A \in \mathcal{C}^{n \times n}$ be Hermitian and have spectral decomposition

$$A = V\Lambda V^*,$$

where $V = (v_1, \dots, v_n) \in \mathbb{C}^{n \times n}$ is unitary, $\Lambda = \text{diag}(\lambda_1(A), \dots, \lambda_n(A))$ with the order of its eigenvalues

$$\lambda_{\min}(A) = \lambda_n(A) \leq \dots \leq \lambda_1(A) = \lambda_{\max}(A). \tag{2}$$

Define the projection of a vector $x \in \mathbb{C}^n$ onto the eigenvectors of a Hermitian matrix A :

$$x_{ij} \equiv (v_i, \dots, v_j)^* x, \quad i \leq j.$$

Let

$$l_1(x) = \frac{1}{2} \left(\|x_{1:2}\|_2^2 - gap_2 + \sqrt{(gap_2 + \|x_{1:2}\|_2^2)^2 - 4gap_2|x_2|^2} \right),$$

$$l_i(x) = \frac{1}{2} \left(gap_i + \|x_{1:i}\|_2^2 - \sqrt{(gap_i + \|x_{1:i}\|_2^2)^2 - 4gap_i|x_i|^2} \right), \quad i = 2, 3, \dots, n,$$

and for $i = 1, 2, \dots, n - 1$,

$$u_i(x) = \frac{1}{2} \left(\|x_{i:n}\|_2^2 - gap_{i+1} + \sqrt{(gap_{i+1} + \|x_{i:n}\|_2^2)^2 - 4gap_{i+1}\|x_{i+1:n}\|_2^2} \right),$$

$$u_n(x) = \frac{1}{2} \left(gap_n + \|x_{n-1:n}\|_2^2 - \sqrt{(gap_n + \|x_{n-1:n}\|_2^2)^2 - 4gap_n|x_n|^2} \right),$$

where gap_i is the distance of an eigenvalue $\lambda_i(A)$ to its right neighbor, *i.e.*,

$$gap_i = \lambda_{i-1}(A) - \lambda_i(A), \quad i = 2, 3, \dots, n.$$

Note that $0 \leq l_1(x) \leq \|x_{1:2}\|_2^2$, $0 \leq l_i(x) \leq \|x_{i:n}\|_2^2$, $i = 2, \dots, n$, $0 \leq u_i(x) \leq \|x_{i:n}\|_2^2$, $i = 1, 2, \dots, n - 1$, and $0 \leq u_n(x) \leq gap_n$.

With the above notations, the results in [7] are given as follows.

Theorem 1.1 ([7]) *Let $A \in \mathbb{C}^{n \times n}$ be Hermitian and $x \in \mathbb{C}^n$. Then*

$$\lambda_i(A) + l_i(x) \leq \lambda_i(A + xx^*) \leq \lambda_i(A) + u_i(x), \quad i = 1, n,$$

and

$$\lambda_i(A) + l_i(x) \leq \lambda_i(A + xx^*) \leq \min\{\lambda_i(A) + u_i(x), \lambda_{i-1}(A)\}, \quad 2 \leq i \leq n - 1.$$

Notice that the above results improve Weyl's theorem, *i.e.*,

$$\lambda_i(A) \leq \lambda_i(A + xx^*) \leq \lambda_{i-1}(A), \quad i = 2, \dots, n,$$

$$\lambda_1(A) \leq \lambda_1(A + xx^*) \leq \lambda_1(A) + \|x\|_2^2.$$

In this paper we investigate a singular value case and obtain some singular value variations for rank-one perturbation of arbitrary matrices. The main proof technique is based on Theorem 1.1 and the following relationship:

$$\sigma_k^2(A + yx^*) = \lambda_k(A^*A + \|y\|_2^2xx^* + A^*yx^* + xy^*A), \tag{3}$$

where $A \in \mathbb{C}^{n \times n}$ and $x, y \in \mathbb{C}^n$. In addition, some existing results on singular values are also used to deduce the new bounds.

2 Singular value variations

In this section we present bounds of singular values for rank-one perturbation of arbitrary matrices. We always assume that the singular values and eigenvalues have the decreasing orders given by (1) and (2), respectively.

We first give some notations. Setting $\delta_{k+1} = \sigma_k^2(A) - \sigma_{k+1}^2(A)$. Let

$$\zeta_1 = \frac{1}{2} \left(\|y\|_2^2 \|x_{1:2}\|_2^2 - \delta_2 + \sqrt{(\delta_2 + \|y\|_2^2 \|x_{1:2}\|_2^2)^2 - 4\delta_2 \|y\|_2^2 |x_2|^2} \right), \tag{4}$$

$$\zeta_k = \frac{1}{2} \left(\delta_k + \|y\|_2^2 \|x_{1:k}\|_2^2 - \sqrt{(\delta_k + \|y\|_2^2 \|x_{1:k}\|_2^2)^2 - 4\delta_k \|y\|_2^2 |x_k|^2} \right) \tag{5}$$

with $2 \leq k \leq n$,

$$\phi_k = \frac{1}{2} \left(\|y\|_2^2 \|x_{k:n}\|_2^2 - \delta_{k+1} + \sqrt{(\delta_{k+1} + \|y\|_2^2 \|x_{k:n}\|_2^2)^2 - 4\delta_{k+1} \|y\|_2^2 \|x_{k+1:n}\|_2^2} \right) \tag{6}$$

with $1 \leq k \leq n - 1$,

$$\phi_n = \frac{1}{2} \left(\delta_n + \|y\|_2^2 \|x_{n-1:n}\|_2^2 - \sqrt{(\delta_n + \|y\|_2^2 \|x_{n-1:n}\|_2^2)^2 - 4\delta_n \|y\|_2^2 |x_n|^2} \right). \tag{7}$$

Note that the expressions of $\zeta_1, \zeta_k, \phi_k, \phi_n$ are similar to the ones of $l_1(x), l_k(x), u_k(x), u_n(x)$, respectively. For simplicity, in the rest of this paper, we also use the notations

$$\mathcal{S}_1 = A^*A + \|y\|_2^2 x x^*, \quad \mathcal{S}_2 = A^* y x^* + x y^* A. \tag{8}$$

In order to deduce our results, we give the following lemmas.

Lemma 2.1 ([5]) *Let $A, B \in \mathbb{C}^{n \times n}$ be given. Then*

$$\sigma_{i+j-1}(A + B) \leq \sigma_i(A) + \sigma_j(B)$$

for $1 \leq i, j \leq n$ and $i + j \leq n + 1$.

Lemma 2.2 *Let $A \in \mathbb{C}^{n \times n}$ and $x, y \in \mathbb{C}^n$. Then*

$$\sigma_i(yx^*) = 0, \quad i = 2, \dots, n.$$

Proof The result follows from the fact that $\text{rank}(yx^*) \leq 1$. □

Lemma 2.3 *Let $A \in \mathbb{C}^{n \times n}$ and $x, y \in \mathbb{C}^n$. Then*

$$\begin{aligned} \lambda_1(A^* y x^* + x y^* A) &\leq 2 \|A^* y x^*\|_2, \\ \lambda_n(A^* y x^* + x y^* A) &\geq -2 \|A^* y x^*\|_2. \end{aligned}$$

Proof It is easy to obtain

$$\lambda_1(A^*yx^* + xy^*A) \leq \sigma_1(A^*yx^* + xy^*A) \leq 2\|A^*yx^*\|_2.$$

The lower bound for $\lambda_n(*)$ can be obtained by an analogical way. □

Lemma 2.4 ([5]) *Let $A, B \in \mathbb{C}^{n \times n}$ be Hermitian. Then, for $i = 1, 2, \dots, n$, it holds that*

$$\lambda_k(A) + \lambda_n(B) \leq \lambda_k(A + B) \leq \lambda_k(A) + \lambda_1(B). \tag{9}$$

From Lemmas 2.1–2.4 and Theorem 1.1, we can obtain the following bounds of singular values.

Theorem 2.1 *Let $A \in \mathbb{C}^{n \times n}$ and $x, y \in \mathbb{C}^n$ be given. Then, for $i = 1, 2, \dots, n$, it holds that*

$$\max\{\sigma_{k+1}(A), Lb_1, Lb_2\} \leq \sigma_k(A + yx^*) \leq \min\{\sigma_{k-1}(A), Ub_1, Ub_2\}, \tag{10}$$

where the two lower bounds Lb_1 and Lb_2 are defined by

$$Lb_1 = \sigma_k(A) - \|x\|_2\|y\|_2, \quad Lb_2 = \sqrt{\max\{0, \sigma_k^2(A) + \zeta_k - 2\|A^*yx^*\|_2\}} \tag{11}$$

and the two upper bounds Ub_1 and Ub_2 are defined by

$$Ub_1 = \sigma_k(A) + \|x\|_2\|y\|_2, \quad Ub_2 = \sqrt{\sigma_k^2(A) + \phi_k + 2\|A^*yx^*\|_2}. \tag{12}$$

Here we define $\sigma_0(A) = +\infty, \sigma_{n+1}(A) = 0; \zeta_k, \phi_k$ are given by (4)–(5) and (6)–(7), respectively.

Proof We will complete the proof according to the following three different strategies.

Strategy 1: We use Theorem 1.1 and Lemma 2.3 to deduce the new bounds of $\sigma_k(A + yx^*)$. In fact, from (3) and (9), it follows that

$$\lambda_k(\mathcal{S}_1) + \lambda_n(\mathcal{S}_2) \leq \sigma_k^2(A + yx^*) \leq \lambda_k(\mathcal{S}_1) + \lambda_1(\mathcal{S}_2), \tag{13}$$

where \mathcal{S}_1 and \mathcal{S}_2 are defined by (8). Applying Theorem 1.1 to \mathcal{S}_1 gives

$$\sigma_k^2(A) + \zeta_k \leq \lambda_k(\mathcal{S}_1) \leq \sigma_k^2(A) + \phi_k,$$

which, together with (13) and Lemma 2.3, yields

$$\sigma_k^2(A) + \zeta_k - 2\|A^*yx^*\|_2 \leq \sigma_k^2(A + yx^*) \leq \sigma_k^2(A) + \phi_k + 2\|A^*yx^*\|_2. \tag{14}$$

Strategy 2: We use Lemmas 2.1–2.2 to obtain the new bounds of $\sigma_k(A + yx^*)$. In fact, by Lemma 2.2, we have

$$\sigma_{k_1}(yx^*) = 0, \quad k_1 = 2, 3, \dots, n.$$

It follows from Lemma 2.1 that

$$\sigma_{k+1}(A) \leq \sigma_k(A + yx^*) + \sigma_2(-yx^*) = \sigma_k(A + yx^*), \quad 1 \leq k \leq n - 1, \tag{15}$$

and

$$\sigma_k(A + yx^*) \leq \min_{2 \leq k_1 \leq k} [\sigma_{k+1-k_1}(A) + \sigma_{k_1}(yx^*)] = \sigma_{k-1}(A), \quad 2 \leq k \leq n. \tag{16}$$

Note that inequalities (15) and (16) also hold for the cases $k = n$ and $k = 1$, respectively, based on the definitions $\sigma_{n+1}(A) = 0$ and $\sigma_0(A) = +\infty$. Combining (15) and (16) gives

$$\sigma_{k+1}(A) \leq \sigma_k(A + yx^*) \leq \sigma_{k-1}(A). \tag{17}$$

Strategy 3: We use Lemma 2.1 and the definition of the two norm to give the new bounds of $\sigma_k(A + yx^*)$. Actually, from Lemma 2.1 and the fact that

$$\sigma_1(yx^*) = \sigma_1(-yx^*) = \|x\|_2 \|y\|_2,$$

we have

$$\begin{aligned} \sigma_k(A) &\leq \sigma_k(A + yx^*) + \sigma_1(-yx^*) = \sigma_k(A + yx^*) + \|x\|_2 \|y\|_2, \\ \sigma_k(A + yx^*) &\leq \sigma_k(A) + \sigma_1(yx^*) = \sigma_k(A) + \|x\|_2 \|y\|_2, \end{aligned}$$

which implies

$$\sigma_k(A) - \|x\|_2 \|y\|_2 \leq \sigma_k(A + yx^*) \leq \sigma_k(A) + \|x\|_2 \|y\|_2. \tag{18}$$

Now combining bounds (14), (17), (18) and $\sigma_k(A + yx^*) \geq 0$, we obtain bounds (10). \square

Remark 2.1 By Theorem 2.1, the classical bound

$$\|A + B\|_2 \leq \|A\|_2 + \|B\|_2$$

can be improved provided B is of rank one. In fact, from the upper bound of (10), it follows that

$$\|A + B\|_2 \leq \sqrt{\|A\|_2^2 + \phi_1 + 2\|A^*B\|_2},$$

which is always sharper than the existing bound $\|A\|_2 + \|B\|_2$ because $\phi_1 \leq \|B\|_2^2$.

If we do a restriction on A^*yx^* , then the lower or upper bound of (10) can be further simplified.

Corollary 2.1 *Let $A \in \mathbb{C}^{n \times n}$ and $x, y \in \mathbb{C}^n$.*

(I) *If $\mathcal{H}(A^*yx^*)$ is positive semidefinite, then the lower bound of (10) is simplified as*

$$\sqrt{\sigma_k^2(A) + \zeta_k} \leq \sigma_k(A + yx^*). \tag{19}$$

(II) If $\mathcal{H}(A^*yx^*)$ is negative semidefinite, then the upper bound of (10) is given by

$$\sigma_k(A + yx^*) \leq \min \left\{ \sigma_{k-1}(A), \sigma_k(A) + \|x\|_2 \|y\|_2, \sqrt{\sigma_k^2(A) + \phi_k} \right\}. \tag{20}$$

In particular, for the case that $1 \leq k \leq n - 1$, the above bound can further be simplified as

$$\sigma_k(A + yx^*) \leq \min \left\{ \sigma_{k-1}(A), \sqrt{\sigma_k^2(A) + \phi_k} \right\}. \tag{21}$$

Proof If $\mathcal{H}(A^*yx^*)$ is positive semidefinite, then

$$\lambda_n(S_2) = \lambda_{\min}(A^*yx^* + xy^*A) \geq 0.$$

It follows that the lower bound of (13) is simplified as

$$\sigma_k^2(A + yx^*) \geq \lambda_k(S_1).$$

Hence

$$Lb_2 = \sqrt{\sigma_k^2(A) + \zeta_k}.$$

Obviously, for the case, it is easy to check that

$$\max \{ \sigma_{k+1}(A), Lb_1, Lb_2 \} = Lb_2,$$

from which one may deduce bound (19).

If $\mathcal{H}(A^*yx^*)$ is negative semidefinite, then

$$\lambda_1(S_2) = \lambda_{\max}(A^*yx^* + xy^*A) \leq 0.$$

It follows that the upper bound of (13) is simplified as

$$\sigma_k^2(A + yx^*) \leq \lambda_k(S_1).$$

Hence

$$Ub_2 = \sqrt{\sigma_k^2(A) + \phi_k},$$

which, together with the bounds of (10), gives bound (20). It is noted that $0 \leq \phi_k \leq \|y\|_2^2 \|x_{k:n}\|_2^2$ ($k = 1, 2, \dots, n - 1$). For this case, it is easy to check that

$$\sqrt{\sigma_k^2(A) + \phi_k} \leq \sigma_k(A) + \|x\|_2 \|y\|_2. \tag{22}$$

Therefore, we obtain (21) instead of the upper bound of (10) in Theorem 2.1. This completes the proof. □

3 Numerical examples

In this section we give some numerical examples to test the proposed bounds (10) in Theorem 2.1. Numerical examples are carried out in MATLAB R2014b, with machine epsilon $\epsilon \approx 2.2 \times 10^{-10}$.

The first example is randomly generated by MATLAB.

Example 1 Let A be a random matrix of order n and x, y be random vectors of dimension n , which can be generated by MATLAB command “ $A = \text{randn}(n, n) + 1i * \text{randn}(n, n)$, $x, y = \text{randn}(n, 1) + 1i * \text{randn}(n, 1)$ ”. Test bounds (10) in Theorem 2.1 according to the three cases $2 \leq k \leq n - 1$, $k = n$, and $k = 1$.

The second example comes from the aero engine fault diagnosis. It is significant to deduce two-side bounds of the singular values for rank-one update of A , where A is an available inter-segment attractor reconstruction matrix.

Example 2 Assume that the test signal of the mechanical system is the following numerical sequence: s_i ($i = 1, 2, \dots, 2n - 1$), then available inter-segment attractor reconstruction matrix A is

$$A = \begin{pmatrix} s_1 & s_2 & \cdots & s_n \\ s_2 & s_3 & \cdots & s_{n+1} \\ \vdots & \vdots & & \vdots \\ s_n & s_{n+1} & \cdots & s_{2n-1} \end{pmatrix};$$

see [8, 9] for more details. The following simulation signals are frequently used in aero engine fault diagnosis. At initial time t_0 the test signals are described as

$$s(t_0) = s_1 + \sigma e(t_0).$$

After t time, the test signals with engine fault are described as

$$s(t) = s_1 + s_2 + \sigma e(t),$$

where s_1, s_2 are random signals and s_i ($i = 3, 4, \dots, 2n - 1$) are generated in terms of the expression of $s(t)$ at random times; $\sigma e(t)$ is Gaussian white noise with a mean 0 and variance of 1 with $\sigma = 1$. Let x, y be random vectors of dimension n and yx^* be rank-one update of A . Test bounds (10) in Theorem 2.1 according to the three cases $2 \leq k \leq n - 1$, $k = n$, and $k = 1$.

In Tables 1–6, we give the lower and upper bounds determined by (10) for the above three cases, which are emphasized by the black text. Note that $\sigma_{n+1}(A) = 0$ and $\sigma_0(A) = +\infty$. Thus we have omitted the lower bound $\sigma_{n+1}(A)$ in Tables 2 and 5 and the upper bound $\sigma_0(A)$ in Tables 3 and 6. For comparison, we also list the exact values of $\sigma_k(A + yx^*)$. In addition, for the case of the largest singular values, the existing bounds $\sigma_1(A) + \sigma_1(yx^*)$ are given. The notations Lb_i and Ub_i ($i = 1, 2$) are defined by (11) and (12), respectively.

From Tables 1–6 we have the following observations and remarks:

Table 1 Comparison of the bounds in (10) for Example 1 ($2 \leq k \leq n - 1$): the bold face numbers confirm our sharp perturbation bounds

		$n = 40$		
		$k = 8$	$k = 18$	$k = 28$
Lower bounds	$\sigma_{k+1}(A)$	27.46	16.34	8.41
	Lb_1	28.52	16.97	8.85
	Lb_2	25.60	17.64	7.98
Upper bounds	$\sigma_{k-1}(A)$	66.30	45.10	21.46
	Ub_1	55.24	40.65	17.46
	Ub_2	46.32	39.14	19.45
Exact values	$\sigma_k(A + yx^*)$	35.23	27.35	11.46

		$n = 80$		
		$k = 38$	$k = 52$	$k = 66$
Lower bounds	$\sigma_{k+1}(A)$	29.38	26.19	14.57
	Lb_1	36.48	24.58	13.90
	Lb_2	38.20	27.30	15.04
Upper bounds	$\sigma_{k-1}(A)$	82.47	70.06	43.35
	Ub_1	81.69	71.59	42.53
	Ub_2	82.41	68.33	45.63
Exact values	$\sigma_k(A + yx^*)$	51.24	43.45	28.24

		$n = 100$		
		$k = 16$	$k = 44$	$k = 72$
Lower bounds	$\sigma_{k+1}(A)$	54.26	35.34	15.85
	Lb_1	56.17	31.95	10.24
	Lb_2	52.46	36.35	16.32
Upper bounds	$\sigma_{k-1}(A)$	93.14	75.35	44.52
	Ub_1	86.20	70.28	43.13
	Ub_2	89.35	78.01	48.76
Exact values	$\sigma_k(A + yx^*)$	70.56	45.15	30.42

Table 2 Comparison of the bounds in (10) for smallest singular values of Example 1: the bold face numbers confirm our sharp perturbation bounds

		n		
		60	90	120
Lower bounds	Lb_1	11.63	20.14	18.90
	Lb_2	12.76	20.62	14.23
Upper bounds	$\sigma_{n-1}(A)$	21.05	41.64	38.60
	Ub_1	20.83	39.83	34.53
	Ub_2	24.47	43.08	33.64
Exact values	$\sigma_n(A + yx^*)$	18.92	29.27	21.92

Table 3 Comparison results of the bounds for largest singular values of Example 1: the bold face numbers confirm our sharp perturbation bounds

		n		
		20	40	60
Lower bounds	$\sigma_2(A)$	18.46	45.26	37.64
	Lb_1	19.68	23.89	32.70
	Lb_2	21.84	46.90	40.02
Upper bounds	Ub_1	50.68	72.39	81.06
	Ub_2	42.04	65.79	73.80
	$\sigma_1(A) + \sigma_1(yx^*)$	58.53	86.48	96.46
Exact values	$\sigma_1(A + yx^*)$	30.58	69.57	48.19

Table 4 Comparison of the bounds in (10) for Example 2 ($2 \leq k \leq n - 1$): the bold face numbers confirm our sharp perturbation bounds

		$n = 30$		
		$k = 8$	$k = 16$	$k = 24$
Lower bounds	$\sigma_{k+1}(A)$	16.25	12.75	26.81
	Lb_1	13.96	12.91	28.54
	Lb_2	17.14	10.27	19.72
Upper bounds	$\sigma_{k-1}(A)$	53.45	36.08	51.08
	Ub_1	45.26	34.77	48.76
	Ub_2	40.18	34.82	50.73
Exact values	$\sigma_k(A + yx^*)$	34.03	27.59	48.50

		$n = 60$		
		$k = 22$	$k = 40$	$k = 58$
Lower bounds	$\sigma_{k+1}(A)$	30.13	23.26	10.26
	Lb_1	32.53	25.11	14.96
	Lb_2	36.18	10.04	15.60
Upper bounds	$\sigma_{k-1}(A)$	68.26	60.29	49.07
	Ub_1	70.24	58.65	47.43
	Ub_2	68.01	54.71	45.48
Exact values	$\sigma_k(A + yx^*)$	56.02	41.59	28.56

Table 5 Comparison of the bounds in (10) for smallest singular values of Example 2: the bold face numbers confirm our sharp perturbation bounds

		n		
		28	62	96
Lower bounds	Lb_1	24.14	16.84	11.48
	Lb_2	22.03	17.06	10.05
Upper bounds	$\sigma_{n-1}(A)$	38.14	39.23	34.29
	Ub_1	32.80	38.10	30.40
	Ub_2	34.28	37.95	37.22
Exact values	$\sigma_n(A + yx^*)$	28.55	27.83	17.74

Table 6 Comparison results of the bounds for largest singular values of Example 2: the bold face numbers confirm our sharp perturbation bounds

		n		
		20	40	60
Lower bounds	$\sigma_2(A)$	16.18	40.04	32.54
	Lb_1	17.05	28.60	30.11
	Lb_2	20.53	40.08	36.29
Upper bounds	Ub_1	46.13	68.25	70.57
	Ub_2	40.16	60.27	76.90
	$\sigma_1(A) + \sigma_1(yx^*)$	48.79	69.47	84.18
Exact values	$\sigma_1(A + yx^*)$	27.57	53.13	49.27

- (1) From comparison results with the exact values, the proposed bounds (10) are feasible and effective.
- (2) For Examples 1–2, the lower and the upper bounds given by (10) are chosen from Lb_1, Lb_2 and Ub_1, Ub_2 instead of $\sigma_{k+1}(A)$ and $\sigma_{k-1}(A)$, respectively. In other words, the bounds Lb_i ($i = 1, 2$) and Ub_i ($i = 1, 2$) are sharper than $\sigma_{k+1}(A)$ and $\sigma_{k-1}(A)$,

respectively. Note that $\sigma_{k+1}(A)$ and $\sigma_{k-1}(A)$ are essentially the existing bounds given by Lemma 2.1. This further verifies the advantage of the proposed bounds.

- (3) In particular, from Tables 3 and 6, the proposed upper bounds of (10) are always tighter than the existing upper bounds $\sigma_1(A) + \sigma_1(yx^*)$, which agrees with Remark 2.1.

4 Conclusions

In this paper, by making use of different strategies, we present the two-side bounds of singular values for rank-one perturbation of arbitrary matrices. In particular, the proposed upper bounds are proved to be always sharper than the classical bound $\|A + B\|_2 \leq \|A\|_2 + \|B\|_2$. Numerical examples further demonstrate the feasibility and effectiveness of the new perturbation bounds, which are tighter than some existing perturbation bounds.

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Availability of data and materials

The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

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