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Log-Minkowski inequalities for the L_p-mixed quermassintegrals

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Abstract

Böröczky et al. proposed the log-Minkowski problem and established the plane log-Minkowski inequality for origin-symmetric convex bodies. Recently, Stancu proved the log-Minkowski inequality for mixed volumes; Wang, Xu, and Zhou gave the L_p version of Stancu's results. In this paper, we define the L_p-mixed quermassintegrals probability measure and obtain the log-Minkowski inequality for the L_p-mixed quermassintegrals. As its application, we establish the L_p-mixed affine isoperimetric inequality. In addition, we also consider the dual log-Minkowski inequalities for the L_p-dual mixed quermassintegrals.

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1 Introduction and main results

Let $\mathcal{K}^n$ denote a set of convex bodies (compact, convex subsets with non-empty interiors) in Euclidean space $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors and the set of origin-symmetric convex bodies in $\mathbb{R}^n$, we write $\mathcal{K}^n_o$ and $\mathcal{K}^n_{os}$, respectively. Let $\mathcal{F}^n_o$ denote the subset of $\mathcal{K}^n_o$ that has a positive continuous curvature function. Besides, let $\mathcal{S}^n_o$ denote the set of star bodies (with respect to the origin). Let S^{n-1} denote the unit sphere and $V(K)$ denote the n-dimensional volume of the convex body K. For the standard unit ball B, its volume is written by $V(B) = \omega_n$.

If $K \in \mathcal{K}^n$, then its support function, $h_K = h(K,\cdot) : \mathbb{R}^n \to \mathbb{R}$, is defined by (see [1, 2])

$$h(K,x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n, \tag{1.1}$$

where $x \cdot y$ denotes the standard inner product of x and y. From the definition of support function, we know that, for $\lambda > 0$,

$$h(\lambda K,x) = \lambda h(K,x). \tag{1.2}$$

The Brunn–Minkowski inequality is of utmost importance in the theory of convex geometric analysis. The well-known Brunn–Minkowski inequality can be stated as follows.

Brunn–Minkowski inequality *If $K, L \in \mathcal{K}^n$, then*

$$V(K+L)^{\frac{1}{n}} \geq V(K)^{\frac{1}{n}} + V(L)^{\frac{1}{n}},$$

with equality if and only if K and L are homothetic. Here $K + L = \{x + y : x \in K \text{ and } y \in L\}$ denotes the Minkowski sum of K and L.

As the first milestone of the Brunn–Minkowski theory, the Brunn–Minkowski inequality is a far-reaching generalization of the isoperimetric inequality. The Brunn–Minkowski inequality exposes the crucial log-concavity property of the volume functional because the Brunn–Minkowski inequality has an equivalent formulation as follows: *If $K, L \in \mathcal{K}^n$, real $\lambda \in [0,1]$, then*

$$V\big((1-\lambda)K + \lambda L\big) \geq V(K)^{1-\lambda} V(L)^{\lambda}, \tag{1.3}$$

with equality if and only if K and L are translates. Here, $(1-\lambda)K + \lambda L$ denotes the Minkowski combination of K and L with respect to λ, and

$$(1-\lambda)K + \lambda L = \bigcap_{u \in S^{n-1}} \big\{x \in \mathbb{R}^n : x \cdot u \leq (1-\lambda)h_K(u) + \lambda h_L(u)\big\}.$$

For more research on the classical Brunn–Minkowski inequality, see [1–3].

Similar to the definition of Minkowski combination, Böröczky et al. [4] gave the definition of log-Minkowski combination as follows: For $K, L \in \mathcal{K}_o^n$ and $0 \leq \lambda \leq 1$, the log-Minkowski combination, $(1-\lambda) \cdot K +_0 \lambda \cdot L$, of K and L is defined by

$$(1-\lambda) \cdot K +_0 \lambda \cdot L = \bigcap_{u \in S^{n-1}} \big\{x \in \mathbb{R}^n : x \cdot u \leq h_K(u)^{1-\lambda} h_L(u)^{\lambda}\big\}.$$

Meanwhile, according to the log-Minkowski combination, Böröczky et al. in [4] conjecture that, for origin-symmetric bodies, there is a stronger inequality than inequality (1.3), i.e., the following log-Brunn–Minkowski inequality.

The conjectured log-Brunn–Minkowski inequality *Let $K, L \in \mathcal{K}_{os}^n$, then for all $\lambda \in [0,1]$,*

$$V\big((1-\lambda) \cdot K +_0 \lambda \cdot L\big) \geq V(K)^{1-\lambda} V(L)^{\lambda}. \tag{1.4}$$

The case $n = 2$ of inequality (1.4) was proved by Böröczky et al. (see [4]). Afterwards, Saroglou [5] showed that the log-Brunn–Minkowski inequality (1.4) is valid when K and L are unconditional convex bodies with respect to the same orthonormal basis in $\mathbb{R}^n$. For further research on log-Brunn–Minkowski inequality, we also see [6–8].

Further, Böröczky et al. [4] proposed that log-Brunn–Minkowski inequality (1.4) is equivalent to the following log-Minkowski inequality.

The conjectured log-Minkowski inequality *Let $K, L \in \mathcal{K}_{os}^n$, then*

$$\int_{S^{n-1}} \ln\left(\frac{h_K}{h_L}\right) d\overline{V}_L \geq \frac{1}{n} \ln\left(\frac{V(K)}{V(L)}\right), \tag{1.5}$$

where $dV_L = \frac{1}{n}h_L(u)\,dS(L,u)$ *denotes cone-volume measure of* L *for any* $u \in S^{n-1}$, *and* $d\overline{V}_L = \frac{1}{V(L)}\,dV_L$ *denotes its normalization.*

For the log-Minkowski inequality (1.5), Böröczky et al. [4] proved that it is true when $n = 2$. In 2014, Zhu [9] solved the case of discrete measures and proved the log-Minkowski inequality (1.5) for polytopes in $\mathbb{R}^n$. Recently, Stancu [10] introduced the mixed cone-volume measure $dv_1(L,K)$ of $K, L \in \mathcal{K}^n_o$ by $dv_1(L,K) = \frac{1}{n}h(K,\cdot)\,dS(L,\cdot)$, and $d\overline{V}_1(L,K) = \frac{dv_1(L,K)}{V_1(L,K)}$ denotes its normalization, where $V_1(L,K)$ denotes the mixed volume of L and K, and $S(L,\cdot)$ is the surface area measure of L (see [1]). According to this notion, Stancu [10] proved a modified log-Minkowski inequality for n-dimensional convex bodies as follows.

Theorem 1.A (The log-Minkowski inequality for mixed volume) *If* $K, L \in \mathcal{K}^n_o$, *then*

$$\int_{S^{n-1}} \ln\left(\frac{h_K}{h_L}\right) d\overline{V}_1(L,K) \geq \ln\left(\frac{V_1(L,K)}{V(L)}\right) \geq \frac{1}{n}\ln\left(\frac{V(K)}{V(L)}\right), \tag{1.6}$$

with equality if and only if K *is homothetic to* L.

For $K, L \in \mathcal{K}^n_o$, $i = 0, \dots, n-1$, we write the mixed quermassintegrals $W_i(L,K)$ of L and K for the mixed volume $V(\underbrace{L,\dots,L}_{n-i-1}, K, \underbrace{B,\dots,B}_{i})$, where B is the standard unit ball. The mixed quermassintegrals $W_i(L,K)$ have the following integral representation (see [11]):

$$W_i(L,K) = \frac{1}{n}\int_{S^{n-1}} h(K,u)\,dS_i(L,u), \tag{1.7}$$

where $S_i(L,\cdot)$ denotes the mixed surface area measure of L. If $K = L$, then the quermassintegrals $W_i(L)$ of L are given by

$$W_i(L) = \frac{1}{n}\int_{S^{n-1}} h(L,u)\,dS_i(L,u). \tag{1.8}$$

Obviously, when $i = 0$, then $W_0(L) = V(L)$.

From (1.7), Wang and Feng [12] defined the mixed quermassintegral measure $dw_i(L,K)$ of L and K by

$$dw_i(L,K) = \frac{1}{n}h(K,\cdot)\,dS_i(L,\cdot). \tag{1.9}$$

Combining with (1.7) and (1.9), the mixed quermassintegral probability measure is given by

$$d\overline{W}_i(L,K) = \frac{1}{W_i(L,K)}\,dw_i(L,K). \tag{1.10}$$

Obviously, if $i = 0$ in (1.9) and (1.10), then $dw_i(L) = dV(L)$ and $d\overline{W}_i(L) = d\overline{V}(L)$.

If $K = L$, then (1.9) implies the quermassintegral measure $dw_i(L)$ by

$$dw_i(L) = \frac{1}{n}h(L,\cdot)\,dS_i(L,\cdot). \tag{1.11}$$

Equation (1.10) gives the quermassintegral probability measure $d\overline{W}_i(L)$ by

$$d\overline{W}_i(L) = \frac{1}{W_i(L)}\, dw_i(L). \tag{1.12}$$

Obviously, if $i = 0$ in (1.11) and (1.12), then $dw_i(L) = dV(L)$ and $d\overline{W}_i(L) = d\overline{V}(L)$.

In relation to the mixed quermassintegrals, Wang and Feng [12] established the following log-Minkowski inequality for mixed quermassintegrals, which is more general than Stancu's results.

Theorem 1.B (The log-Minkowski inequality for mixed quermassintegrals) *If $K, L \in \mathcal{K}_o^n$ and $i = 0, \ldots, n-1$, then*

$$\int_{S^{n-1}} \ln\left(\frac{h_K}{h_L}\right) d\overline{W}_i(L,K) \ge \ln\left(\frac{W_i(L,K)}{W_i(L)}\right) \ge \frac{1}{n-i} \ln\left(\frac{W_i(K)}{W_i(L)}\right),$$

with equality if and only if K is homothetic to L.

The log-Minkowski inequality belongs to log-Minkowski theory. For more research on log-Minkowski theory, we may refer to [13–22].

In 2017, Wang, Xu, and Zhou [23] proposed p-mixed cone-volume measure: For $K, L \in \mathcal{K}_o^n$, $p \ge 1$, the p-mixed cone-volume measure $dV_p(L,K)$ of L and K is defined by

$$dV_p(L,K) = \frac{1}{n} h(K,\cdot)^p h(L,\cdot)^{1-p}\, dS(L,\cdot),$$

and $d\overline{V}_p(L,K) = \frac{dV_p(L,K)}{V_p(L,K)}$ denotes its normalization, where $V_p(L,K)$ is the L_p-mixed volume of L and K (see [24]). Based on this notion, they [23] proved the log-Minkowski inequality for L_p-mixed volumes as follows.

Theorem 1.C (The log-Minkowski inequality for L_p-mixed volume) *If $K, L \in \mathcal{K}_o^n$, $p > 1$, then*

$$\int_{S^{n-1}} \ln\left(\frac{h_K}{h_L}\right) d\overline{V}_p(L,K) \ge \frac{1}{p} \ln\left(\frac{V_p(L,K)}{V(L)}\right) \ge \frac{1}{n} \ln\left(\frac{V(K)}{V(L)}\right), \tag{1.13}$$

with equality if and only if K and L are dilates.

In [11], Lutwak defined L_p-mixed quermassintegrals as follows: For $K, L \in \mathcal{K}_o^n$, $p \ge 1$, and $i = 0, 1, \ldots, n-1$, there exists a positive Borel measure $S_{p,i}(L,\cdot)$ on S^{n-1} such that L_p-mixed quermassintegral $W_{p,i}(L,K)$ has the following integral representation:

$$W_{p,i}(L,K) = \frac{1}{n} \int_{s^{n-1}} h(K,u)^p\, dS_{p,i}(L,\cdot). \tag{1.14}$$

It turns out that the measure $S_{p,i}(L,\cdot)$ (called the L_p-mixed surface area measure) on S^{n-1} has the Radon–Nikodym derivative

$$\frac{dS_{p,i}(L,\cdot)}{dS_i(L,\cdot)} = h(L,\cdot)^{1-p}. \tag{1.15}$$

In this paper, in relation to L_p-mixed quermassintegrals, we continuously study log-Minkowski inequality. First, according to (1.14) and (1.15), we define the L_p-mixed quermassintegral probability measure as follows:

For $K, L \in \mathcal{K}_o^n$, $p \geq 1$, and $i = 0, 1, \dots, n-1$, the L_p-mixed quermassintegral measure $dw_{p,i}(L,K)$ of L and K is defined by

$$dw_{p,i}(L,K) = \frac{1}{n} h(K,u)^p h(L,u)^{1-p}\, dS_i(L,u). \tag{1.16}$$

From this, the L_p-mixed quermassintegral probability measure is written by

$$d\overline{W}_{p,i}(L,K) = \frac{1}{W_{p,i}(L,K)}\, dw_{p,i}(L,K). \tag{1.17}$$

In particular, if $p = 1$ and $i = 0$ in (1.16) and (1.17), then $dw_{p,i}(L,K) = dv_1(L,K)$ and $d\overline{W}_{p,i}(L,K) = d\overline{V}_1(L,K)$.

Next, combined with the above L_p-mixed quermassintegral probability measure, we give a generalization of the log-Minkowski inequalities (1.6) and (1.13).

Theorem 1.1 (The log-Minkowski inequality for L_p-mixed quermassintegral) *If $K, L \in \mathcal{K}_o^n$, $p \geq 1$, and $i = 0, 1, 2, \dots, n-1$, then*

$$\int_{S^{n-1}} \ln\left(\frac{h_K}{h_L}\right) d\overline{W}_{p,i}(L,K) \geq \frac{1}{p} \ln\left(\frac{W_{p,i}(L,K)}{W_i(L)}\right) \geq \frac{1}{n-i} \ln\left(\frac{W_i(K)}{W_i(L)}\right), \tag{1.18}$$

with equality, for $p = 1$ if and only if K and L are homothetic, for $p > 1$ if and only if K and L are dilates.

Remark 1.1 If $p = 1$ and $i = 0$ in Theorem 1.1, then Theorem 1.A can be obtained. If $p > 1$ and $i = 0$, then Theorem 1.1 implies Theorem 1.C.

In addition, we also consider the log-Minkowski inequality for quermassintegrals. For convenience, let

$$\begin{aligned}
\left(\frac{h_K}{h_L}\right)_{p\text{-average}} &= \frac{\int_{S^{n-1}} (\frac{h_K}{h_L})^p\, d\omega_i(L)}{\int_{S^{n-1}} d\omega_i(L)} = \frac{W_{p,i}(L,K)}{W_i(L)};\\
\left(\frac{h_K}{h_L}\right)_{\max} &= \max_{u \in \operatorname{supp} \omega_i(L)} \frac{h_K}{h_L};\\
\left(\frac{h_K}{h_L}\right)_{\min} &= \min_{u \in \operatorname{supp} \omega_i(L)} \frac{h_K}{h_L},
\end{aligned}$$

where $\operatorname{supp} \omega_i(L)$ denotes the support of the quermassintegral measure of $\omega_i(L)$. Our result can be stated as follows.

Theorem 1.2 *If $K, L \in \mathcal{K}_o^n$ satisfy $L \subseteq K$, $p \geq 1$, and $i = 0, 1, 2, \dots, n-1$, then*

$$\int_{S^{n-1}} \ln\left(\frac{h_K}{h_L}\right) d\overline{W}_i(L) \geq \frac{(\frac{h_K}{h_L})_{p\text{-average}}}{(\frac{h_K}{h_L})^p_{\max}} \frac{1}{n-i} \ln\left(\frac{W_i(K)}{W_i(L)}\right), \tag{1.19}$$

with equality if and only if $K = L$.

In general, if $K, L \in \mathcal{K}_o^n$, then

$$\int_{S^{n-1}} \ln\left(\frac{h_K}{h_L}\right) d\overline{W}_i(L) \geq \frac{(\frac{h_K}{h_L})_{p\text{-}average}}{(\frac{h_K}{h_L})^p_{\max}} \frac{1}{n-i} \ln\left(\frac{W_i(K)}{W_i(L)}\right) + \ln\left[\left(\frac{h_K}{h_L}\right)_{\min}\right]\left[1 - \frac{(\frac{h_K}{h_L})_{p\text{-}average}}{(\frac{h_K}{h_L})^p_{\max}}\right], \tag{1.20}$$

with equality if and only if K and L are dilates.

Remark 1.2 The case of $p = 1$ and $i = 0$ is just Stancu's result (see [10]). If $p > 1$ and $i = 0$ in Theorem 1.2, then inequalities (1.19) and (1.20) can be found in [23].

In Sect. 2, we complete the proofs of Theorems 1.1–1.2 and obtain some results about the log-Minkowski inequality. In Sect. 3, we establish the dual form of Theorem 1.1 and obtain some related inequalities. Finally, as the application of Theorem 1.1, an L_p-mixed affine isoperimetric inequality is given in Sect. 4.

2 Proofs of theorems

In this part, we will give the proofs of Theorems 1.1–1.2. First, in order to prove Theorem 1.1, the following lemma is required.

Lemma 2.1 ([11]) *If $K, L \in \mathcal{K}_o^n$, $p \geq 1$ and $i = 0, 1, 2, \ldots, n-1$, then*

$$W_{p,i}(L,K) \geq W_i(L)^{\frac{n-p-i}{n-i}} W_i(K)^{\frac{p}{n-i}}, \tag{2.1}$$

with equality, for $p = 1$ and $0 \leq i < n-1$, if and only if L and K are homothetic; for $p > 1$, if and only if L and K are dilates; for $p = 1$ and $i = n-1$, (2.1) is identical.

Proof of Theorem 1.1 For $K, L \in \mathcal{K}_o^n$, $p \geq 1$, and $i = 0, 1, 2, \ldots, n-1$, by (1.11) and (1.16) we have that

$$\int_{S^{n-1}} \left(\frac{h_K}{h_L}\right)^p \ln\left(\frac{h_K}{h_L}\right) dw_i(L) = \int_{S^{n-1}} \ln\left(\frac{h_K}{h_L}\right) dw_{p,i}(L,K). \tag{2.2}$$

From Lebesgue's dominated convergence theorem, and combined with formula (1.11), (1.14), (1.16), and (2.2), we obtain if $t \to \infty$, then

$$\int_{S^{n-1}} \left(\frac{h_K}{h_L}\right)^{\frac{pt}{t+n}} dw_i(L) \to W_{p,i}(L,K)$$

and

$$\int_{S^{n-1}} \left(\frac{h_K}{h_L}\right)^{\frac{pt}{t+n}} \ln\left(\frac{h_K}{h_L}\right) dw_i(L) \to \int_{S^{n-1}} \ln\left(\frac{h_K}{h_L}\right) dw_{p,i}(L,K).$$

Considering the function $F_{K,L}(t) : [1, \infty) \to \mathbb{R}$ by

$$F_{K,L}(t) = \frac{1}{W_{p,i}(L,K)} \int_{S^{n-1}} \left(\frac{h_K}{h_L}\right)^{\frac{pt}{t+n}} dw_i(L)$$

and using L'Hôpital's rule, we have

$$\begin{aligned}
&\lim_{t\to\infty} \ln\bigl[F_{K,L}(t)^{t+n}\bigr] \\
&= \lim_{t\to\infty} \frac{\ln F_{K,L}(t)}{\frac{1}{t+n}} \\
&= \lim_{p\to\infty} \frac{F'_{K,L}(t)}{-\frac{F_{K,L}(t)}{(t+n)^2}} \\
&= \lim_{t\to\infty} \frac{\frac{1}{W_{p,i}(L,K)} \cdot \frac{pn}{(t+n)^2} \int_{S^{n-1}} (\frac{h_K}{h_L})^{\frac{pt}{t+n}} \ln(\frac{h_K}{h_L})\, dw_i(L)}{-\frac{F_{K,L}(t)}{(t+n)^2}} \\
&= \lim_{t\to\infty} -\frac{\frac{pn}{W_{p,i}(L,K)} \int_{S^{n-1}} (\frac{h_K}{h_L})^{\frac{pt}{t+n}} \ln(\frac{h_K}{h_L})\, dw_i(L)}{\frac{1}{W_{p,i}(L,K)} \int_{S^{n-1}} (\frac{h_K}{h_L})^{\frac{pt}{t+n}}\, dw_i(L)} \\
&= -\frac{pn}{W_{p,i}(L,K)} \int_{S^{n-1}} \left(\frac{h_K}{h_L}\right)^p \ln\left(\frac{h_K}{h_L}\right) dw_i(L).
\end{aligned} \tag{2.3}$$

Consequently, by (2.3) we obtain

$$\begin{aligned}
&\exp\left[-\frac{pn}{W_{p,i}(L,K)} \int_{S^{n-1}} \left(\frac{h_K}{h_L}\right)^p \ln\left(\frac{h_K}{h_L}\right) dw_i(L)\right] \\
&= \lim_{t\to\infty} F_{K,L}(t)^{t+n} \\
&= \lim_{t\to\infty} \left[\frac{1}{W_{p,i}(L,K)} \int_{S^{n-1}} \left(\frac{h_K}{h_L}\right)^{\frac{pt}{t+n}} dw_i(L)\right]^{t+n}.
\end{aligned} \tag{2.4}$$

By Hölder's inequality (see [25]), (1.9) and (1.14) deduce that

$$\begin{aligned}
&\left[\int_{S^{n-1}} \left(\frac{h_K}{h_L}\right)^{p\cdot\frac{t}{t+n}} dw_i(L)\right]^{\frac{t+n}{t}} \cdot \left[\int_{S^{n-1}} dw_i(L)\right]^{-\frac{n}{t}} \\
&\le \int_{S^{n-1}} \left(\frac{h_K}{h_L}\right)^p dw_i(L) \\
&= W_{p,i}(L,K),
\end{aligned}$$

i.e.,

$$\left[\frac{1}{W_{p,i}(L,K)} \int_{S^{n-1}} \left(\frac{h_K}{h_L}\right)^{\frac{pt}{t+n}} dw_i(L)\right]^{t+n} \le \left(\frac{W_i(L)}{W_{p,i}(L,K)}\right)^n. \tag{2.5}$$

By the equality condition of Hölder's inequality, we see that equality holds in (2.5) if and only if $(\frac{h_K}{h_L})^p$ is a constant, i.e., L and K are dilates.

From this, together (2.5) with (2.4), we obtain

$$\exp\left[-\frac{pn}{W_{p,i}(L,K)} \int_{S^{n-1}} \left(\frac{h_K}{h_L}\right)^p \ln\left(\frac{h_K}{h_L}\right) dw_i(L)\right] \le \left(\frac{W_i(L)}{W_{p,i}(L,K)}\right)^n,$$

i.e.,

$$\frac{p}{W_{p,i}(L,K)}\int_{S^{n-1}}\left(\frac{h_K}{h_L}\right)^p\ln\left(\frac{h_K}{h_L}\right)dw_i(L)\geq\ln\left(\frac{W_{p,i}(L,K)}{W_i(L)}\right). \tag{2.6}$$

Therefore, by (2.6), (1.9), (1.16), (1.17), and (2.1), we have

$$\begin{aligned}
&\int_{S^{n-1}}\ln\left(\frac{h_K}{h_L}\right)d\overline{W}_{p,i}(L,K)\\
&\quad\geq\frac{1}{p}\ln\left(\frac{W_{p,i}(L,K)}{W_i(L)}\right)\\
&\quad\geq\frac{1}{p}\ln\left(\frac{W_i(L)^{\frac{n-p-i}{n-i}}W_i(K)^{\frac{p}{n-i}}}{W_i(L)}\right)\\
&\quad\geq\frac{1}{n-i}\ln\left(\frac{W_i(K)}{W_i(L)}\right).
\end{aligned}$$

This gives the desired inequality (1.18).

The equality conditions of inequalities (2.1) and (2.5) imply that equality holds in inequality (1.18) for $p=1$ if and only if K is homothetic to L, for $p>1$ if and only if L and K are dilates. □

Remark 2.1 The case $p=1$ of Theorem 1.1 is just Theorem 1.B which is obtained by Wang and Feng (see [12]).

Using Theorem 1.1, we have the following result.

Corollary 2.1 *If $K,L\in\mathcal{K}_o^n$ with $L\subseteq K$, $p\geq 1$, and $i=0,1,2,\ldots,n-1$, then*

$$\int_{S^{n-1}}\left(\frac{h_K}{h_L}\right)^p\ln\left(\frac{h_K}{h_L}\right)d\overline{W}_i(L)\geq\frac{1}{n-i}\left(\frac{W_i(K)}{W_i(L)}\right)^{\frac{p}{n-i}}\ln\left(\frac{W_i(K)}{W_i(L)}\right),$$

with equality, for $p=1$ if and only if K and L are homothetic, for $p>1$ if and only if K and L are dilates.

Proof From (1.10), (2.2), (1.18), and (2.1), we have

$$\begin{aligned}
&\int_{S^{n-1}}\left(\frac{h_K}{h_L}\right)^p\ln\left(\frac{h_K}{h_L}\right)d\overline{W}_i(L)\\
&\quad=\frac{1}{W_i(L)}\int_{S^{n-1}}\left(\frac{h_K}{h_L}\right)^p\ln\left(\frac{h_K}{h_L}\right)dw_i(L)\\
&\quad=\frac{W_{p,i}(L,K)}{W_i(L)}\int_{S^{n-1}}\ln\left(\frac{h_K}{h_L}\right)d\overline{W}_{p,i}(L,K)\\
&\quad\geq\frac{W_{p,i}(L,K)}{W_i(L)}\cdot\frac{1}{p}\ln\left(\frac{W_{p,i}(L,K)}{W_i(L)}\right)\\
&\quad\geq\frac{1}{n-i}\left(\frac{W_i(K)}{W_i(L)}\right)^{\frac{p}{n-i}}\ln\left(\frac{W_i(K)}{W_i(L)}\right).
\end{aligned}$$

The equality conditions of (1.18) and (2.1) imply that the equality holds in Corollary 2.1 for $p = 1$ if and only if K and L are homothetic, for $p > 1$ if and only if K and L are dilates. □

Remark 2.2 If $p = 1$ and $i = 0$, then Corollary 2.1 can be found in [10].

Next, we give an improved version of the right-hand inequality of (1.18) in Theorem 1.1.

Theorem 2.1 *If $K, L \in \mathcal{K}_o^n$, $p \geq 1$, and $i = 0, 1, 2, \ldots, n-1$, then*

$$\frac{1}{p} \ln\left(\frac{W_{p,i}(L,K)}{W_i(L)}\right) \geq \int_{S^{n-1}} \ln\left(\frac{h_K}{h_L}\right) d\overline{W}_i(L), \tag{2.7}$$

with equality if and only if K and L are dilates.

Lemma 2.2 ([26, 27]) *Let $f(x)$ and $g(x)$ be the probability density functions on a measure space (X, v) for v-almost all $x \in X$, if $\int_X f(x)\,dv(x) = 1$, $\int_X g(x)\,dv(x) = 1$, then*

$$\int_X f(x) \ln f(x)\,dv(x) \geq \int_X f(x) \ln g(x)\,dv(x),$$

with equality if and only if $f(x) = g(x)$.

Proof of Theorem 2.1 For $K, L \in \mathcal{K}_o^n$, $p \geq 1$, $i = 0, 1, 2, \ldots, n-1$, and all $u \in S^{n-1}$, let

$$f(u) = \frac{1}{W_i(L)}\left(\frac{h_L}{h_K}\right)^p, \qquad g(u) = \frac{1}{W_{p,i}(L,K)}, \qquad dv(u) = dw_{p,i}(L,K),$$

then we have

$$\int_{S^{n-1}} f(u)\,dv(u) = 1, \qquad \int_{S^{n-1}} g(u)\,dv(u) = 1.$$

Thus by Lemma 2.2 we get

$$\begin{aligned}
&\int_{S^{n-1}} \frac{1}{W_i(L)}\left(\frac{h_L}{h_K}\right)^p \ln\left[\frac{1}{W_i(L)}\left(\frac{h_L}{h_K}\right)^p\right] dw_{p,i}(L,K) \\
&\quad \geq \int_{S^{n-1}} \frac{1}{W_i(L)}\left(\frac{h_L}{h_K}\right)^p \ln\left(\frac{1}{W_{p,i}(L,K)}\right) dw_{p,i}(L,K),
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\int_{S^{n-1}} \frac{1}{W_i(L)}\left(\frac{h_L}{h_K}\right)^p \ln\left(\frac{h_L}{h_K}\right)^p dw_{p,i}(L,K) \\
&\quad \geq \ln\left(\frac{W_i(L)}{W_{p,i}(L,K)}\right) \int_{S^{n-1}} \frac{1}{W_i(L)}\left(\frac{h_L}{h_K}\right)^p dw_{p,i}(L,K).
\end{aligned}$$

From this, and combined with (1.16), (1.11), and (1.12), we have

$$\int_{S^{n-1}} \ln\left(\frac{h_L}{h_K}\right)^p d\overline{W}_i(L) \geq \ln\left(\frac{W_i(L)}{W_{p,i}(L,K)}\right),$$

i.e.,

$$\int_{S^{n-1}} \ln\left(\frac{h_K}{h_L}\right) d\overline{W}_i(L) \leq \frac{1}{p} \ln\left(\frac{W_{p,i}(L,K)}{W_i(L)}\right).$$

According to Lemma 2.2, the equality holds in (2.7) if and only if $\frac{1}{W_i(L)}(\frac{h_L}{h_K})^p = \frac{1}{W_{p,i}(L,K)}$, i.e., h_K/h_L is a constant. Hence the equality holds in (2.7) if and only if K and L are dilates. □

Actually, applying Lemma 2.2, we may give another proof of the left-hand inequality of (1.18) in Theorem 1.1.

Another proof of the left-hand inequality of (1.18) Taking

$$f(u) = \frac{1}{W_{p,i}(L,K)}, \qquad g(u) = \frac{1}{W_i(L)}\left(\frac{h_L}{h_K}\right)^p, \qquad dv(u) = dw_{p,i}(L,K),$$

by Lemma 2.2 we get

$$\begin{aligned} &\int_{S^{n-1}} \frac{1}{W_{p,i}(L,K)} \ln\left(\frac{1}{W_{p,i}(L,K)}\right) dw_{p,i}(L,K) \\ &\quad \geq \int_{S^{n-1}} \frac{1}{W_{p,i}(L,K)} \ln\left[\frac{1}{W_i(L)}\left(\frac{h_L}{h_K}\right)^p\right] dw_{p,i}(L,K). \end{aligned}$$

This gives

$$\begin{aligned} &\ln\left(\frac{W_i(L)}{W_{p,i}(L,K)}\right) \int_{S^{n-1}} \frac{1}{W_{p,i}(L,K)} dw_{p,i}(L,K) \\ &\quad \geq \int_{S^{n-1}} \frac{1}{W_{p,i}(L,K)} \ln\left(\frac{h_L}{h_K}\right)^p dw_{p,i}(L,K). \end{aligned}$$

From this, combined with (1.17), we obtain

$$\ln\left(\frac{W_i(L)}{W_{p,i}(L,K)}\right) \geq \int_{S^{n-1}} \ln\left(\frac{h_L}{h_K}\right)^p d\overline{W}_{p,i}(L,K),$$

i.e.,

$$\frac{1}{p} \ln\left(\frac{W_{p,i}(L,K)}{W_i(L)}\right) \leq \int_{S^{n-1}} \ln\left(\frac{h_K}{h_L}\right) d\overline{W}_{p,i}(L,K). \tag{2.8}$$

This is just the left-hand inequality of (1.18).

By Lemma 2.2, equality holds in (2.8) if and only if $\frac{1}{W_{p,i}(L,K)} = \frac{1}{W_i(L)}(\frac{h_L}{h_K})^p$, i.e., h_K/h_L is a constant. This means that equality holds in the left-hand inequality of (1.18) if and only if K and L are dilates. □

From inequality (2.7) and the left-hand inequality of (1.18), we easily obtain the following.

Corollary 2.2 *If $K, L \in \mathcal{K}_o^n$, $p \geq 1$, and $i = 0, 1, 2, \ldots, n-1$, then*

$$\int_{S^{n-1}} \ln\left(\frac{h_K}{h_L}\right) d\overline{W}_{p,i}(L,K) \geq \frac{1}{p} \ln\left(\frac{W_{p,i}(L,K)}{W_i(L)}\right) \geq \int_{S^{n-1}} \ln\left(\frac{h_K}{h_L}\right) d\overline{W}_i(L). \tag{2.9}$$

In each case, equality holds if and only if K and L are dilates.

Corollary 2.2 shows that the log-Minkowski inequality for L_p-mixed quermassintegrals is stronger than the log-Minkowski inequality for quermassintegrals.

Now, we give the proof of Theorem 1.2, the following lemma is necessary.

Lemma 2.3 (Hadamard type inequality [28]) *Let f be a positive, log-convex function on $[a, b]$, then*

$$\frac{1}{b-a} \int_a^b f(t)\,dt \leq \frac{f(b) - f(a)}{\ln f(b) - \ln f(a)}. \tag{2.10}$$

Proof of Theorem 1.2 For $K, L \in \mathcal{K}_o^n$, if $L \subseteq K$, we define the function

$$G(q) = \int_{S^{n-1}} \left(\frac{h_K}{h_L}\right)^{pq} \ln\left(\frac{h_K}{h_L}\right) dw_i(L), \quad q \in \mathbb{R}, \tag{2.11}$$

then $G(q)$ is non-negative. If $u \to \ln(\frac{h_K}{h_L})(u)$ is zero on the support of the quermassintegral measure of L, then G is identically zero. If G is not identically zero, then by (2.9) we know $G(1) \geq G(0) > 0$. If $G(1) = G(0)$, then K must be equal to L. So assume $G(1) > G(0)$.

Here, we show that $G(q)$ is a log-convex function. In fact, for $\alpha \in (0, 1)$ and $\beta, \gamma \in \mathbb{R}$, by (2.11) and Hölder's inequality [25], we have

$$\begin{aligned}
&G\big((1-\alpha)\beta + \alpha\gamma\big) \\
&\quad = \int_{S^{n-1}} \left(\frac{h_K}{h_L}\right)^{p((1-\alpha)\beta+\alpha\gamma)} \ln\left(\frac{h_K}{h_L}\right) dw_i(L) \\
&\quad = \int_{S^{n-1}} \left(\frac{h_K}{h_L}\right)^{p\beta(1-\alpha)} \left(\frac{h_K}{h_L}\right)^{p\alpha\gamma} \ln\left(\frac{h_K}{h_L}\right) dw_i(L) \\
&\quad \leq \left(\int_{S^{n-1}} \left(\frac{h_K}{h_L}\right)^{p\beta} \ln\left(\frac{h_K}{h_L}\right) dw_i(L)\right)^{(1-\alpha)} \left(\int_{S^{n-1}} \left(\frac{h_K}{h_L}\right)^{p\gamma} \ln\left(\frac{h_K}{h_L}\right) dw_i(L)\right)^{\alpha} \\
&\quad = G(\beta)^{1-\alpha} G(\gamma)^{\alpha}.
\end{aligned}$$

From this, by (2.11) and Hadamard type inequality (2.10), we obtain

$$\frac{G(1) - G(0)}{\ln G(1) - \ln G(0)} \geq \int_0^1 \left[\int_{S^{n-1}} \left(\left(\frac{h_K}{h_L}\right)^p\right)^q \ln\left(\frac{h_K}{h_L}\right) dw_i(L)\right] dq. \tag{2.12}$$

Since $G(1) > G(0)$, combined with (2.12) and Fubini–Toneli's theorem, we have

$$G(0) \geq G(1) \cdot \exp\left[-\frac{G(1) - G(0)}{\int_0^1 \int_{S^{n-1}} (\frac{h_K}{h_L})^{pq} \ln(\frac{h_K}{h_L})\,dw_i(L)\,dq}\right]$$

$$
\begin{aligned}
&= G(1)\cdot\exp\biggl[-\frac{G(1)-G(0)}{\int_{S^{n-1}}[\int_0^1(\frac{h_K}{h_L})^{pq}\ln(\frac{h_K}{h_L})\,dq]\,dw_i(L)}\biggr]\\
&= G(1)\cdot\exp\biggl[-p\frac{G(1)-G(0)}{\int_{S^{n-1}}[(\frac{h_K}{h_L})^p-1]\,dw_i(L)}\biggr]. \qquad (2.13)
\end{aligned}
$$

In (2.13), note that

$$
\frac{G(1)-G(0)}{\int_{S^{n-1}}[(\frac{h_K}{h_L})^p-1]\,dw_i(L)}=\frac{\int_{S^{n-1}}\ln(\frac{h_K}{h_L})[(\frac{h_K}{h_L})^p-1]\,dw_i(L)}{\int_{S^{n-1}}[(\frac{h_K}{h_L})^p-1]\,dw_i(L)}\le\ln\left(\frac{h_K}{h_L}\right)_{\max}. \qquad (2.14)
$$

Thus, together with (2.13) and (2.14), we get

$$
\int_{S^{n-1}}\ln\left(\frac{h_K}{h_L}\right)dw_i(L)\ge\int_{S^{n-1}}\left(\frac{h_K}{h_L}\right)^p\ln\left(\frac{h_K}{h_L}\right)dw_i(L)\cdot\exp\left[-p\ln\left(\frac{h_K}{h_L}\right)_{\max}\right],
$$

namely,

$$
\begin{aligned}
&\frac{1}{W_i(L)}\int_{S^{n-1}}\ln\left(\frac{h_K}{h_L}\right)dw_i(L)\\
&\quad\ge\frac{W_{p,i}(L,K)}{W_i(L)}\cdot\frac{1}{W_{p,i}(L,K)}\cdot\left(\frac{h_K}{h_L}\right)_{\max}^{-p}\cdot\int_{S^{n-1}}\left(\frac{h_K}{h_L}\right)^p\ln\left(\frac{h_K}{h_L}\right)dw_i(L).
\end{aligned}
$$

This yields

$$
\begin{aligned}
\int_{S^{n-1}}\ln\left(\frac{h_K}{h_L}\right)d\overline{W}_i(L)&\ge\frac{W_{p,i}(L,K)}{W_i(L)}\cdot\left(\frac{h_K}{h_L}\right)_{\max}^{-p}\cdot\int_{S^{n-1}}\ln\left(\frac{h_K}{h_L}\right)d\overline{W}_{p,i}(L,K)\\
&\ge\frac{(\frac{h_K}{h_L})_{p\text{-average}}}{(\frac{h_K}{h_L})^p_{\max}}\int_{S^{n-1}}\ln\left(\frac{h_K}{h_L}\right)d\overline{W}_{p,i}(L,K)\\
&\ge\frac{(\frac{h_K}{h_L})_{p\text{-average}}}{(\frac{h_K}{h_L})^p_{\max}}\frac{1}{n-i}\ln\left(\frac{W_i(K)}{W_i(L)}\right).
\end{aligned}
$$

The last inequality is obtained by the left-hand inequality of (1.18). This is the desired inequality (1.19).

Assume that $G(q)$ is identically zero, then $h(K,u)=h(L,u)$ for almost all u in the support of the quermassintegral measure of L, or equivalently with respect to the L_p-surface area measure of L. This implies $W_{p,i}(L,K)=W_i(L)$. By (2.1), we can obtain $W_i(L)^{\frac{n-p-i}{n-i}}W_i(K)^{\frac{p}{n-i}}\le W_{p,i}(L,K)=W_i(L)$, i.e., $W_i(K)\le W_i(L)$, with $L\subseteq K$, we know that $W_i(K)=W_i(L)$. Since $L\subseteq K$ are convex bodies and with the equality condition of (2.1), thus $K=L$.

When L is not included in K, since $K,L\in\mathcal{K}_o^n$, thus there exist $\lambda\in\mathbb{R}$ and $0<\lambda<1$ such that $\lambda L\subseteq K$. From this, by (1.19) we have

$$
\int_{S^{n-1}}\ln\left(\frac{h_K}{h_{\lambda L}}\right)d\overline{W}_i(\lambda L)\ge\frac{(\frac{h_K}{h_{\lambda L}})_{p\text{-average}}}{(\frac{h_K}{h_{\lambda L}})^p_{\max}}\frac{1}{n-i}\ln\left(\frac{W_i(K)}{W_i(\lambda L)}\right).
$$

This and (1.2) give

$$\int_{S^{n-1}} \ln\left(\frac{h_K}{\lambda h_L}\right) d\overline{W}_i(L) \geq \frac{(\frac{h_K}{h_L})_{p\text{-average}}}{(\frac{h_K}{h_L})^p_{\max}} \frac{1}{n-i} \ln\left(\frac{W_i(K)}{\lambda^{n-i} W_i(L)}\right),$$

i.e.,

$$\begin{aligned} &\int_{S^{n-1}} \ln\left(\frac{h_K}{h_L}\right) d\overline{W}_i(L) \\ &\quad \geq \frac{(\frac{h_K}{h_L})_{p\text{-average}}}{(\frac{h_K}{h_L})^p_{\max}} \frac{1}{n-i} \ln\left(\frac{W_i(K)}{W_i(L)}\right) + \ln\lambda \cdot \left[1 - \frac{(\frac{h_K}{h_L})_{p\text{-average}}}{(\frac{h_K}{h_L})^p_{\max}}\right]. \end{aligned} \tag{2.15}$$

Now let $\lambda = (\frac{h_K}{h_L})_{\min}$ in (2.15), then (2.15) yields inequality (1.20). According to $\lambda L \subseteq K$ and the equality condition of inequality (1.19), we see that equality holds in (1.20) if and only if K and L are dilates. □

Remark 2.3 If $p = 1$ in Theorem 1.2, we can obtain Wang and Feng's result (see [12]).

Obviously, if there exists $\lambda \in [0, 1]$ such that $\lambda L \subseteq K$ is valid as well as $\lambda h_L(u) = h_K(u)$, $\frac{(\frac{h_K}{h_L})_{p\text{-average}}}{(\frac{h_K}{h_L})^p_{\max}} = 1$ in Theorem 1.2, the following result is obtained.

Corollary 2.3 *Let $K, L \in \mathcal{K}^n_o$ such that there exists a positive constant $\lambda > 0$ with $\lambda h_L(u) = h_K(u)$ for each u in the support of the quermassintegral measure of L. Then*

$$\int_{S^{n-1}} \ln\left(\frac{h_K}{h_L}\right) d\overline{W}_i(L) \geq \frac{1}{n-i} \ln\left(\frac{W_i(K)}{W_i(L)}\right),$$

with equality if and only if $K = \lambda L$.

Remark 2.4 If $i = 0$ in Corollary 2.3, then this inequality was firstly obtained by Gardner, Hug, and Weil (see [29]).

3 Dual log-Minkowski inequalities for the L_p-dual mixed quermassintegrals

Let K be a compact star-shaped (about the origin) set in $\mathbb{R}^n$, its radial function, $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \backslash \{0\} \to [0, +\infty)$, is defined by [1, 2]

$$\rho(K, x) = \max\{\lambda \geq 0 : \lambda x \in K\}, \quad x \in \mathbb{R}^n \backslash \{0\}. \tag{3.1}$$

If ρ_K is positive and continuous, K will be called a star body (about the origin).

If $K \in \mathcal{K}^n_o$, the polar body K^* of K is defined by (see [1, 2])

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, y \in K\}.$$

From (1.1) and (3.1), it follows that if $K \in \mathcal{K}^n_o$, then

$$h(K^*, \cdot) = \frac{1}{\rho(K, \cdot)}, \qquad \rho(K^*, \cdot) = \frac{1}{h(K, \cdot)}. \tag{3.2}$$

The notion of dual quermassintegrals was given by Lutwak [30]. For $L \in \mathcal{S}_o^n$, i is any real, the dual quermassintegral $\widetilde{W}_i(L)$ of L is defined by

$$\widetilde{W}_i(L) = \frac{1}{n}\int_{S^{n-1}} \rho(L,u)^{n-i}\, dS(u). \tag{3.3}$$

Associated with (3.3), the dual quermassintegral measure $dw_i^{\rho}(L)$ of L is written as follows:

$$dw_i^{\rho}(L) = \frac{1}{n}\rho(L,u)^{n-i}\, dS(u). \tag{3.4}$$

From this, the dual quermassintegral probability measure is given by

$$d\widetilde{W}_i^{\rho}(L) = \frac{1}{\widetilde{W}_i(L)}\, dw_i^{\rho}(L). \tag{3.5}$$

Besides, Wang and Yan [31] gave the L_p-dual mixed quermassintegrals as follows: For $K, L \in \mathcal{S}_o^n$, $p \neq 0$, and real $i \neq n$, the L_p-dual mixed quermassintegral $\widetilde{W}_{p,i}(L,K)$ of L and K is defined by

$$\widetilde{W}_{p,i}(L,K) = \frac{1}{n}\int_{S^{n-1}} \rho(L,u)^{n-p-i}\rho(K,u)^p\, dS(u). \tag{3.6}$$

Based on (3.6), we define the L_p-dual mixed quermassintegral measure $dw_{p,i}^{\rho}(L,K)$ of L and K by

$$dw_{p,i}^{\rho}(L,K) = \frac{1}{n}\rho(L,u)^{n-p-i}\rho(K,u)^p\, dS(u). \tag{3.7}$$

According to (3.7), the L_p-dual mixed quermassintegral probability measure is written by

$$d\widetilde{W}_{p,i}^{\rho}(L,K) = \frac{1}{\widetilde{W}_{p,i}(L,K)}\, dw_{p,i}^{\rho}(L,K). \tag{3.8}$$

For the L_p-dual mixed quermassintegrals, Wang and Yan [31] established the L_p-dual Minkowski inequality as follows.

Lemma 3.1 *Let $K, L \in \mathcal{S}_o^n$, $p \neq 0$, and real $i \neq n$. If $p > 0$, then for $i < n - p$,*

$$\widetilde{W}_{p,i}(L,K) \leq \widetilde{W}_i(L)^{\frac{n-p-i}{n-i}}\, \widetilde{W}_i(K)^{\frac{p}{n-i}}; \tag{3.9}$$

for $n - p < i < n$ or $i > n$,

$$\widetilde{W}_{p,i}(L,K) \geq \widetilde{W}_i(L)^{\frac{n-p-i}{n-i}}\, \widetilde{W}_i(K)^{\frac{p}{n-i}}. \tag{3.10}$$

In each case, equality holds if and only if K and L are dilates. If $p < 0$, then for $i > n - p$, inequality (3.9) holds; for $i < n$ or $n < i < n - p$, inequality (3.10) holds. If $i = n - p$, (3.9) (or (3.10)) is identic.

Recently, the dual log-Minkowski inequality was established by Gardner et al. [32]. In 2017, Wang, Xu, and Zhou [23] obtained the dual log-Minkowski inequality for L_p-dual mixed volumes. In this part, we will establish the following dual log-Minkowski inequality for L_p-dual mixed quermassintegrals.

Theorem 3.1 *Let $K, L \in \mathcal{S}_o^n$, if $p > 0$ and $n - p < i < n$ or $i > n$, then*

$$\int_{S^{n-1}} \ln\left(\frac{\rho_K}{\rho_L}\right) d\widetilde{W}_{p,i}^{\rho}(L,K) \geq \frac{1}{p} \ln\left(\frac{\widetilde{W}_{p,i}(L,K)}{\widetilde{W}_i(L)}\right) \geq \frac{1}{n-i} \ln\left(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(L)}\right), \tag{3.11}$$

with equality if and only if K and L are dilates. If $p < 0$ and $i > n - p$, inequality (3.11) is reverse.

Proof of Theorem 3.1 For all $u \in S^{n-1}$, we take

$$f(u) = \frac{1}{\widetilde{W}_{p,i}(L,K)}, \qquad g(u) = \frac{1}{\widetilde{W}_i(L)}\left(\frac{\rho_L(u)}{\rho_K(u)}\right)^p, \qquad dv(u) = dw_{p,i}^{\rho}(L,K),$$

then we can obtain

$$\int_{S^{n-1}} f(u)\, dv(u) = 1, \qquad \int_{S^{n-1}} g(u)\, dv(u) = 1.$$

Thus, by Lemma 2.2, we have

$$\begin{aligned} &\int_{S^{n-1}} \frac{1}{\widetilde{W}_{p,i}(L,K)} \ln\left(\frac{1}{\widetilde{W}_{p,i}(L,K)}\right) dw_{p,i}^{\rho}(L,K) \\ &\quad \geq \int_{S^{n-1}} \frac{1}{\widetilde{W}_{p,i}(L,K)} \ln\left[\frac{1}{\widetilde{W}_i(L)}\left(\frac{\rho_L}{\rho_K}\right)^p\right] dw_{p,i}^{\rho}(L,K), \end{aligned}$$

i.e.,

$$\begin{aligned} &\ln\left(\frac{\widetilde{W}_i(L)}{\widetilde{W}_{p,i}(L,K)}\right) \int_{S^{n-1}} \frac{1}{\widetilde{W}_{p,i}(L,K)}\, dw_{p,i}^{\rho}(L,K) \\ &\quad \geq \int_{S^{n-1}} \frac{1}{\widetilde{W}_{p,i}(L,K)} \ln\left(\frac{\rho_L}{\rho_K}\right)^p dw_{p,i}^{\rho}(L,K). \end{aligned} \tag{3.12}$$

According to the equality condition of Lemma 2.2, we see that equality holds in (3.12) if and only if $\frac{1}{\widetilde{W}_{p,i}(L,K)} = \frac{1}{\widetilde{W}_i(L)}(\frac{\rho_L}{\rho_K})^p$, i.e., $\frac{\rho_L}{\rho_K}$ is a constant. Thus K and L are dilates.

From (3.8), inequality (3.12) can be written as follows:

$$\int_{S^{n-1}} \ln\left(\frac{\rho_K}{\rho_L}\right)^p d\widetilde{W}_{p,i}^{\rho}(L,K) \geq \ln\left(\frac{\widetilde{W}_{p,i}(L,K)}{\widetilde{W}_i(L)}\right).$$

By (3.10), if $p > 0$ and $n - p < i < n$ or $i > n$, then

$$\begin{aligned} \int_{S^{n-1}} \ln\left(\frac{\rho_K}{\rho_L}\right) d\widetilde{W}_{p,i}^{\rho}(L,K) &\geq \frac{1}{p} \ln\left(\frac{\widetilde{W}_{p,i}(L,K)}{\widetilde{W}_i(L)}\right) \\ &\geq \frac{1}{p} \ln\left(\frac{\widetilde{W}_i(L)^{\frac{n-p-i}{n-i}}\, \widetilde{W}_i(K)^{\frac{p}{n-i}}}{\widetilde{W}_i(L)}\right) \\ &\geq \frac{1}{n-i} \ln\left(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(L)}\right). \end{aligned}$$

If $p < 0$ and $i > n - p$, then by (3.9) and (3.12), we obtain inequality (3.11) is reverse.

According to the equality conditions of inequalities (3.10) and (3.12), we see that equality holds in (3.11) if and only if K and L are dilates. □

In addition, the left-hand inequality in (3.11) can be written as follows.

Corollary 3.1 *Let $K, L \in \mathcal{S}_o^n$, if $p > 0$ and $n - p < i < n$ or $i > n$, then*

$$\int_{S^{n-1}} \left(\frac{\rho_K}{\rho_L}\right)^p \ln\left(\frac{\rho_K}{\rho_L}\right) d\widetilde{W}_i^{\rho}(L) \geq \frac{1}{n-i}\left(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(L)}\right)^{\frac{p}{n-i}} \ln\left(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(L)}\right),$$

with equality if and only if K and L are dilates. If $p < 0$ and $i > n - p$, the inequality is reverse.

Proof From (3.5), (3.4), (3.7), (3.8), (3.11), and (3.10), we can obtain

$$\begin{aligned}
&\int_{S^{n-1}} \left(\frac{\rho_K}{\rho_L}\right)^p \ln\left(\frac{\rho_K}{\rho_L}\right) d\widetilde{W}_i^{\rho}(L) \\
&\quad = \frac{1}{\widetilde{W}_i(L)} \int_{S^{n-1}} \left(\frac{\rho_K}{\rho_L}\right)^p \ln\left(\frac{\rho_K}{\rho_L}\right) dw_i^{\rho}(L) \\
&\quad = \frac{\widetilde{W}_{p,i}(L,K)}{\widetilde{W}_i(L)} \int_{S^{n-1}} \ln\left(\frac{\rho_K}{\rho_L}\right) d\widetilde{W}_{p,i}^{\rho}(L,K) \\
&\quad \geq \frac{\widetilde{W}_{p,i}(L,K)}{\widetilde{W}_i(L)} \cdot \frac{1}{p} \ln\left(\frac{\widetilde{W}_{p,i}(L,K)}{\widetilde{W}_i(L)}\right) \\
&\quad \geq \frac{1}{n-i}\left(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(L)}\right)^{\frac{p}{n-i}} \ln\left(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(L)}\right).
\end{aligned}$$

If $p < 0$ and $i > n - p$, then by (3.9) and (3.11), we obtain this inequality is reverse.

The equality conditions of (3.10) and (3.11) imply the equality holds in Corollary 3.1 if and only if K and L are dilates. □

In the following, we obtain a more general form than the dual log-Minkowski inequality.

Theorem 3.2 *Let $K, L \in \mathcal{S}_o^n$ and $\varphi(x) : (0,\infty) \to (0,\infty)$ be a monotonous convex function. If $p > 0$, real $i \neq n$ and $i < n - p$, then*

$$\varphi\left[\int_{S^{n-1}} \varphi^{-1}\left(\left(\frac{\rho_K}{\rho_L}\right)^p\right) d\widetilde{W}_i^{\rho}(L)\right] \leq \left(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(L)}\right)^{\frac{p}{n-i}}, \tag{3.13}$$

with equality if and only if K and L are dilates.

Lemma 3.2 (Jessen's inequality [25]) *Suppose that μ is a probability measure on a space X and $g : X \to I \subset \mathbb{R}$ is a μ-integrable function, where I is a possibly infinite interval. If $\varphi : X \to I \subset \mathbb{R}$ is a strictly convex function, then*

$$\int_X \varphi\big(g(x)\big)\, d\mu(x) \geq \varphi\left(\int_X g(x)\, d\mu(x)\right), \tag{3.14}$$

with equality if and only if $g(x)$ is a constant for μ-almost all $x \in X$.

Proof of Theorem 3.2 For $K,L\in\mathcal{S}_o^n$ and $\varphi(x):(0,\infty)\to(0,\infty)$ is a monotonous convex function. From (3.6), (3.4), (3.5), and (3.14), we have

$$\begin{aligned}\widetilde{W}_{p,i}(L,K)&=\frac{1}{n}\int_{S^{n-1}}\rho(L,u)^{n-p-i}\rho(K,u)^p\,dS(u)\\&=\frac{1}{n}\int_{S^{n-1}}\rho(L,u)^{n-i}\left(\frac{\rho(K,u)}{\rho(L,u)}\right)^p dS(u)\\&=\widetilde{W}_i(L)\int_{S^{n-1}}\left(\frac{\rho(K,u)}{\rho(L,u)}\right)^p d\widetilde{W}_i^{\rho}(L)\\&=\widetilde{W}_i(L)\int_{S^{n-1}}\varphi\left[\varphi^{-1}\left(\left(\frac{\rho(K,u)}{\rho(L,u)}\right)^p\right)\right]d\widetilde{W}_i^{\rho}(L)\\&\geq\widetilde{W}_i(L)\varphi\left[\int_{S^{n-1}}\varphi^{-1}\left(\left(\frac{\rho(K,u)}{\rho(L,u)}\right)^p\right)d\widetilde{W}_i^{\rho}(L)\right],\end{aligned}$$

i.e.,

$$\frac{\widetilde{W}_{p,i}(L,K)}{\widetilde{W}_i(L)}\geq\varphi\left[\int_{S^{n-1}}\varphi^{-1}\left(\left(\frac{\rho(K,u)}{\rho(L,u)}\right)^p\right)d\widetilde{W}_i^{\rho}(L)\right].\tag{3.15}$$

By the equality condition of inequality (3.14), we know that the equality holds in (3.15) if and only if $\varphi^{-1}((\frac{\rho(K,u)}{\rho(L,u)})^p)$ is a constant, i.e., $\frac{\rho(K,u)}{\rho(L,u)}=\lambda$ is a constant, then K and L are dilates.

Combining with (3.15) and Minkowski inequality (3.9), we obtain

$$\varphi\left[\int_{S^{n-1}}\varphi^{-1}\left(\left(\frac{\rho(K,u)}{\rho(L,u)}\right)^p\right)d\widetilde{W}_i^{\rho}(L)\right]\leq\frac{\widetilde{W}_{p,i}(L,K)}{\widetilde{W}_i(L)}\leq\left(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(L)}\right)^{\frac{p}{n-i}}.$$

This yields inequality (3.13).

According to the equality conditions of (3.9) and (3.15), we can know that the equality holds in (3.13) if and only if K and L are dilates. □

Theorem 3.3 *Let $K,L\in\mathcal{S}_o^n$ and $\varphi(x):(0,\infty)\to(0,\infty)$ be a monotonous convex function. If $p>0$, real $i\neq n$, and $i<n-p$, then*

$$\varphi\left[\int_{S^{n-1}}\varphi^{-1}\left(\left(\frac{\rho_L}{\rho_K}\right)^{n-p-i}\right)d\widetilde{W}_i^{\rho}(K)\right]\leq\left(\frac{\widetilde{W}_i(L)}{\widetilde{W}_i(K)}\right)^{\frac{n-p-i}{n-i}},\tag{3.16}$$

with equality if and only if K and L are dilates.

Proof of Theorem 3.3 Since $\varphi(x):(0,\infty)\to(0,\infty)$ is a monotonous convex function, thus by (3.6), (3.4), (3.5), and (3.14), we obtain

$$\begin{aligned}\widetilde{W}_{p,i}(L,K)&=\frac{1}{n}\int_{S^{n-1}}\rho(L,u)^{n-p-i}\rho(K,u)^p\,dS(u)\\&=\frac{1}{n}\int_{S^{n-1}}\rho(K,u)^{n-i}\left(\frac{\rho(L,u)}{\rho(K,u)}\right)^{n-p-i}dS(u)\\&=\widetilde{W}_i(K)\int_{S^{n-1}}\left(\frac{\rho(L,u)}{\rho(K,u)}\right)^{n-p-i}d\widetilde{W}_i^{\rho}(K)\end{aligned}$$

$$= \widetilde{W}_i(K)\int_{S^{n-1}}\varphi\left[\varphi^{-1}\left(\left(\frac{\rho(L,u)}{\rho(K,u)}\right)^{n-p-i}\right)\right]d\widetilde{W}_i^{\rho}(K)$$
$$\geq \widetilde{W}_i(K)\varphi\left[\int_{S^{n-1}}\varphi^{-1}\left(\left(\frac{\rho(L,u)}{\rho(K,u)}\right)^{n-p-i}\right)d\widetilde{W}_i^{\rho}(K)\right],$$

equivalently,

$$\frac{\widetilde{W}_{p,i}(L,K)}{\widetilde{W}_i(K)} \geq \varphi\left[\int_{S^{n-1}}\varphi^{-1}\left(\left(\frac{\rho(L,u)}{\rho(K,u)}\right)^{n-p-i}\right)d\widetilde{W}_i^{\rho}(K)\right]. \tag{3.17}$$

By the equality condition of inequality (3.14), we see that the equality holds in (3.17) if and only if $\varphi^{-1}((\frac{\rho(L,u)}{\rho(K,u)})^{n-p-i})$ is constant, i.e., $\frac{\rho(L,u)}{\rho(K,u)} = \lambda$ is constant, then K and L are dilates.

Together with (3.17) and Minkowski inequality (3.9), we get

$$\varphi\left[\int_{S^{n-1}}\varphi^{-1}\left(\left(\frac{\rho(L,u)}{\rho(K,u)}\right)^{n-p-i}\right)d\widetilde{W}_i^{\rho}(K)\right] \leq \frac{\widetilde{W}_{p,i}(L,K)}{\widetilde{W}_i(K)} \leq \left(\frac{\widetilde{W}_i(L)}{\widetilde{W}_i(K)}\right)^{\frac{n-p-i}{n-i}}.$$

This is the desired inequality (3.16).

According to the equality conditions of (3.9) and (3.17), we know that the equality holds in (3.16) if and only if K and L are dilates. □

In particular, let $\varphi(x) = \exp(x)$ in Theorem 3.2 and Theorem 3.3, then we can obtain the following dual log-Minkowski inequality of dual quermassintegrals probability measure.

Corollary 3.2 *If $K, L \in \mathcal{S}_o^n$, $p > 0$, and real $i \neq n$, then*

$$\int_{S^{n-1}}\ln\left(\frac{\rho_K}{\rho_L}\right)d\widetilde{W}_i^{\rho}(K) \geq \frac{1}{n-i}\ln\left(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(L)}\right) \geq \int_{S^{n-1}}\ln\left(\frac{\rho_K}{\rho_L}\right)d\widetilde{W}_i^{\rho}(L),$$

with equality if and only if K and L are dilates.

Remark 3.1 If $i = 0$ in Corollary 3.2, then this inequality firstly was obtain by Gardner et al. (see [32]).

4 L_p-Mixed affine isoperimetric inequality

For $p \geq 1$ and $i = 0, 1, \ldots, n-1$. A convex body $K \in \mathcal{K}_o^n$ is said to have a generalized L_p-curvature function (see [33]) $f_{p,i}(K,\cdot) : S^{n-1} \to \mathbb{R}$ if measure $S_{p,i}(K,\cdot)$ is absolutely continuous with respect to spherical Lebesgue measure S, and

$$\frac{dS_{p,i}(K,\cdot)}{dS} = f_{p,i}(K,\cdot). \tag{4.1}$$

Obviously, $f_{p,0}(K,\cdot) = f_p(K,\cdot)$. Here $f_p(K,\cdot)$ is the L_p-curvature function of $K \in \mathcal{K}_o^n$ (see [24]). Let $\mathcal{F}_o^n$ denote the subset of $\mathcal{K}_o^n$ that has a positive continuous curvature function.

For $K \in \mathcal{F}_o^n$, $p \geq 1$, and $i = 0, 1, \ldots, n-1$, the L_p-mixed curvature image $\Lambda_{p,i}K \in \mathcal{S}_o^n$ of K is defined by (see [33])

$$\rho(\Lambda_{p,i}K,\cdot)^{n+p-i} = \frac{\widetilde{W}_i(\Lambda_{p,i}K)}{\omega_n}f_{p,i}(K,\cdot). \tag{4.2}$$

In relation to the L_p-dual mixed quermassintegrals, Li and Wang [34] gave the notion of L_p-mixed affine surface area $\Omega_{p,i}(K)$ of K and obtained the following result: If $K \in \mathcal{F}_o^n$, $p \geq 1$, and $i = 0, 1, \ldots, n-1$, then

$$\Omega_{p,i}(K)^{n+p-i} = n^{n+p-i}\omega_n^{n-i}\widetilde{W}_i(\Lambda_{p,i}K)^p. \tag{4.3}$$

By (3.3), (4.2), and (4.3), we have the following integral formula of L_p-mixed affine surface area $\Omega_{p,i}(K)$: If $K \in \mathcal{F}_o^n$, $p \geq 1$, and $i = 0, 1, \ldots, n-1$, then

$$\Omega_{p,i}(K) = \int_{S^{n-1}} f_{p,i}(K,u)^{\frac{n-i}{n+p-i}}\, dS(u). \tag{4.4}$$

Obviously, let $i = 0$ in (4.4), then $\Omega_{p,0}(K)$ is just the L_p-affine surface area $\Omega_p(K)$ (see [24]).

As the application of Theorem 1.1, associated with the L_p-mixed affine surface areas, we establish the following L_p-mixed affine isoperimetric inequality.

Theorem 4.1 *If $K \in \mathcal{K}_o^n$, $L \in \mathcal{F}_o^n$, $p > 1$, and $i = 0, 1, \ldots, n-1$, then*

$$\bigl[W_i(K)W_i(K^*)\bigr]^p \geq \frac{1}{n^{n+p-i}H(K,L,p)}\frac{\Omega_{p,i}(L)^{n+p-i}}{W_i(L)^{n-p-i}}\left(\frac{W_i(K)}{W_i(L)}\right)^p, \tag{4.5}$$

with equality if and only if K and L are dilate balls centered at the origin. Here $H(K,L,p) = [\exp(\int_{S^{n-1}} \ln(\frac{h_K}{h_L})^p\, d\overline{W}_{p,i}(L,K))]^{n-i}$.

Proof Suppose that K and L are distinct, according to the left-hand inequality of (1.18), (1.14), (1.15), and (4.1), we have

$$\begin{aligned}
&\exp\left(\int_{S^{n-1}} \ln\left(\frac{h_K}{h_L}\right)^p d\overline{W}_{p,i}(L,K)\right)\\
&\quad\geq \frac{W_{p,i}(L,K)}{W_i(L)}\\
&\quad= \frac{1}{n}\cdot\frac{1}{W_i(L)}\cdot\int_{S^{n-1}} h(K,u)^p\, dS_{p,i}(L,u)\\
&\quad= \frac{1}{n}\cdot\frac{1}{W_i(L)}\cdot\int_{S^{n-1}} h(K,u)^p f_{p,i}(L,u)\, dS(u).
\end{aligned} \tag{4.6}$$

By the equality conditions of inequality (1.18), we see that equality holds in (4.6) for $p > 1$ if and only if K and L are dilates.

In fact, for $p > 1$ and $i = 0, 1, \ldots, n-1$, since $-\frac{n-i}{p} < 0$, thus by Hölder's inequality (see [25]), (3.2), (3.3), and (4.4), we have

$$\begin{aligned}
&\int_{S^{n-1}} h(K,u)^p f_{p,i}(L,u)\, dS(u)\\
&\quad\geq \left(\int_{S^{n-1}} \bigl[h(K,u)^p\bigr]^{-\frac{n-i}{p}}\, dS(u)\right)^{-\frac{p}{n-i}}\left(\int_{S^{n-1}} f_{p,i}(L,u)^{\frac{n-i}{n+p-i}}\, dS(u)\right)^{\frac{n+p-i}{n-i}}\\
&\quad= \left(\int_{S^{n-1}} h(K,u)^{-(n-i)}\, dS(u)\right)^{-\frac{p}{n-i}}\left(\int_{S^{n-1}} f_{p,i}(L,u)^{\frac{n-i}{n+p-i}}\, dS(u)\right)^{\frac{n+p-i}{n-i}}\\
&\quad= \bigl(n\widetilde{W}_i(K^*)\bigr)^{-\frac{p}{n-i}}\Omega_{p,i}(L)^{\frac{n+p-i}{n-i}}.
\end{aligned} \tag{4.7}$$

From the equality condition of Hölder's inequality, we see that equality holds in (4.7) if and only if $f_{p,i}(L,u) = \lambda h(K,u)^{-(n+p-i)}$ for all $u \in S^{n-1}$, where λ is a constant.

Therefore, by (4.6) and (4.7), we obtain

$$\begin{aligned}
&\exp\left(\int_{S^{n-1}} \ln\left(\frac{h_K}{h_L}\right)^p d\overline{W}_{p,i}(L,K)\right) \\
&\quad \ge \frac{1}{nW_i(L)} \big(n\widetilde{W}_i(K^*)\big)^{-\frac{p}{n-i}} \Omega_{p,i}(L)^{\frac{n+p-i}{n-i}} \\
&\quad = \frac{1}{W_i(L)} \frac{\Omega_{p,i}(L)^{\frac{n+p-i}{n-i}}}{n^{\frac{n+p-i}{n-i}} \widetilde{W}_i(K^*)^{\frac{p}{n-i}}}.
\end{aligned}$$

Equivalently,

$$\begin{aligned}
&\left[\exp\left(\int_{S^{n-1}} \ln\left(\frac{h_K}{h_L}\right)^p d\overline{W}_{p,i}(L,K)\right)\right]^{n-i} \\
&\quad \ge \left(\frac{1}{W_i(L)}\right)^{n-i} \frac{\Omega_{p,i}(L)^{n+p-i}}{n^{n+p-i}\widetilde{W}_i(K^*)^p} \\
&\quad = \frac{1}{n^{n+p-i}} \frac{\Omega_{p,i}(L)^{n+p-i}}{W_i(L)^{n-p-i}} \left(\frac{W_i(K)}{W_i(L)}\right)^p \left(\frac{1}{W_i(K)\widetilde{W}_i(K^*)}\right)^p.
\end{aligned} \tag{4.8}$$

In addition, for $K \in \mathcal{K}_o^n$ and $i = 1,2,\ldots,n-1$, it is well known that (see [1, 2])

$$\widetilde{W}_i(K) \le W_i(K), \tag{4.9}$$

with equality if and only if K is a ball centered at the origin.

Let $H(K,L,p) = [\exp(\int_{S^{n-1}} \ln(\frac{h_K}{h_L})^p \, d\overline{W}_{p,i}(L,K))]^{n-i}$ in (4.8), then by (4.9) we obtain

$$\frac{1}{n^{n+p-i}H(K,L,p)} \frac{\Omega_{p,i}(L)^{n+p-i}}{W_i(L)^{n-p-i}} \left(\frac{W_i(K)}{W_i(L)}\right)^p \le \big[W_i(K)\widetilde{W}_i\big(K^*\big)\big]^p \le \big[W_i(K)W_i\big(K^*\big)\big]^p.$$

This yields inequality (4.5).

Because of the equality conditions of inequalities (4.6) and (4.9), we deduce that K and L are dilate balls centered at the origin, these imply the equality condition of (4.7): $f_{p,i}(L,u) = \lambda h(K,u)^{-(n+p-i)}$. Hence, we see that equality holds in (4.5) if and only if K and L are dilate balls centered at the origin. □

Specially, if $K = L$ in Theorem 4.1, then

$$H(K,L,p) = \left[\exp\left(\int_{S^{n-1}} \ln\left(\frac{h_K}{h_L}\right)^p d\overline{W}_{p,i}(L,K)\right)\right]^{(n-i)} = 1.$$

Thus, we immediately obtain the following result.

Corollary 4.1 *If $K \in \mathcal{F}_o^n$, $p > 1$, and $i = 0,1,\ldots,n-1$, then*

$$\Omega_{p,i}(K)^{n+p-i} \le n^{n+p-i} W_i(K)^{n-i} W_i\big(K^*\big)^p,$$

with equality if and only if K is a ball centered at the origin.

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