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On positive solutions for some second-order three-point boundary value problems with convection term

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Abstract

In this paper, a fixed point theorem in a cone and some inequalities of the associated Green's function are applied to obtain the existence of positive solutions of second-order three-point boundary value problem with dependence on the first-order derivative

$$x''(t) + f(t, x(t), x'(t)) = 0, \quad 0 < t < 1,$$
$$x(0) = 0, \qquad x(1) = \mu x(\eta),$$

where $f : [0, 1] \times [0, \infty) \times R \to [0, \infty)$ is a continuous function, $\mu > 0$, $\eta \in (0, 1)$, $\mu\eta < 1$. The interesting point is that the nonlinear term is dependent on the convection term.

MSC: 34B15; 34A08

Keywords: Three-point boundary value problem; Fixed point theorem; Positive solution

1 Introduction

In recent years, there has been much attention focused on questions of solutions of two-point, three-point, multi-point, and integral boundary value problems for nonlinear ordinary differential equations and fractional differential equations. For example, two-point boundary value problems [3, 15, 29, 39], beam equation problems [5, 13, 16, 36], boundary value problems at resonance [2, 6, 42, 43], fractional boundary value problems [8, 24], impulsive problems [4, 38], multi-point boundary value problems [10, 14, 20, 25, 26, 32, 33, 43], integral boundary value problems [7, 9, 17, 21, 22, 28, 37], p-Laplace problems [11, 13, 24, 27, 30, 31], delay problems [23, 34, 35], solitons [12], singular problems [3], Schrödinger problem [40, 41], etc.

Krasnosel'skii's fixed point theorem in a cone [18], the Leggett–Williams fixed point theorem [19], and five functional fixed point theorem [1] played an extremely important role in the research of the solvability of differential equation with boundary conditions.

However, most of the above works were done under the assumption that the first-order derivative is not involved explicitly in the nonlinear term [1–13, 16, 17, 20–30]. Krasnosel'skii's fixed point theorem in a cone [18] cannot concretely solve problems whose nonlinear terms involve the first-order derivative. In this paper, via a generalization of Krasnosel'skii's fixed point theorem in a cone [5] and some inequalities of the associated Green's function for the associated problem, the existence of positive solutions for the second-order three-point boundary value problem is studied

$$x''(t) + f\bigl(t, x(t), x'(t)\bigr) = 0, \quad 0 < t < 1, \tag{1.1}$$

$$x(0) = 0, \qquad x(1) = \mu x(\eta), \tag{1.2}$$

where $f : [0,1] \times [0,\infty) \times R \to [0,\infty)$ is a continuous function, $\mu > 0$, $\eta \in (0,1)$, $\mu\eta < 1$.

Readers may find that the concavity is crucial in giving some important estimates and in defining an appropriate cone, and the new fixed point theorem in a cone can be used to obtain positive solutions under more flexible conditions. Two examples are given to illustrate the main results.

2 Preliminaries and lemmas

In order to give the following lemma, let X be a Banach space and P be the cone in X. Assume that $\alpha, \beta : X \to R^+$ are two continuous nonnegative functionals that satisfy

$$\alpha(\lambda x) \le |\lambda| \alpha(x), \qquad \beta(\lambda x) \le |\lambda| \beta(x) \quad \text{for } x \in X, \lambda \in [0,1], \tag{2.1}$$

and

$$M_1 \max\bigl\{\alpha(x), \beta(x)\bigr\} \le \|x\| \le M_2 \max\bigl\{\alpha(x), \beta(x)\bigr\} \quad \text{for } x \in X, \tag{2.2}$$

where M_1, M_2 are two positive constants.

Lemma 2.1 ([5]) *Let $r_2 > r_1 > 0$, $L_2 > L_1 > 0$ be constants and*

$$\Omega_i = \bigl\{x \in X \mid \alpha(x) < r_i, \beta(x) < L_i\bigr\}, \quad i = 1, 2,$$

two open subsets in X such that $\theta \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. In addition, let

$$C_i = \bigl\{x \in X \mid \alpha(x) = r_i, \beta(x) \le L_i\bigr\}, \quad i = 1, 2;$$

$$D_i = \bigl\{x \in X \mid \alpha(x) \le r_i, \beta(x) = L_i\bigr\}, \quad i = 1, 2.$$

Assume that $T : P \to P$ is a completely continuous operator satisfying

(S_1) $\alpha(Tx) \le r_1$, $x \in C_1 \cap P$; $\beta(Tx) \le L_1$, $x \in D_1 \cap P$; $\alpha(Tx) \ge r_2$, $x \in C_2 \cap P$; $\beta(Tx) \ge L_2$, $x \in D_2 \cap P$;

or

(S_2) $\alpha(Tx) \ge r_1$, $x \in C_1 \cap P$; $\beta(Tx) \ge L_1$, $x \in D_1 \cap P$; $\alpha(Tx) \le r_2$, $x \in C_2 \cap P$; $\beta(Tx) \le L_2$, $x \in D_2 \cap P$;

then T has at least one fixed point in $(\overline{\Omega}_2 \setminus \Omega_1) \cap P$.

Lemma 2.2 *Let* $0 < \mu < \frac{1}{\eta}$, $\eta \in (0,1)$. *The Green's function of the following boundary value problem*:

$$-x''(t) = 0, \quad 0 < t < 1, \tag{2.3}$$

$$x(0) = 0, \qquad x(1) = \mu x(\eta), \tag{2.4}$$

is given by

$$G(t,s) = \begin{cases} s \in [0,\eta]: & \begin{cases} \frac{t}{1-\mu\eta}[(1-s) - \mu(\eta - s)]: & t \le s; \\ \frac{s}{1-\mu\eta}[(1-t) - \mu(\eta - t)]: & s \le t; \end{cases} \\ s \in [\eta,1]: & \begin{cases} \frac{1}{1-\mu\eta} t(1-s): & t \le s; \\ \frac{1}{1-\mu\eta}[s(1-t) + \mu\eta(t-s)]: & s \le t. \end{cases} \end{cases} \tag{2.5}$$

Moreover, for each $0 < s < 1$,

$$G(t,s) \ge \gamma \max_{0 \le t \le 1} G(t,s), \quad \eta \le t \le 1, \tag{2.6}$$

where $\gamma = \min\{\mu\eta, \frac{\mu(1-\eta)}{(1-\mu\eta)}, \eta\}$.

Proof The detailed formula of Green's function $G(t,s)$ was given in [14]. In the following proof, we focus on the existence of γ. It is clear that $G(t,s) \ge 0$ for $t \in [\eta,1]$, $s \in [0,1]$. Consider the relation of μ and η, we divide the range of μ into two cases.

Case 1: $0 < \mu \le 1$. With the definition of $G(t,s)$, there are

$$\min_{t \in [\eta,1]} G(t,s) = \begin{cases} \frac{s\mu(1-\eta)}{1-\mu\eta}, & s \in [0,\eta]; \\ \frac{\eta\mu(1-s)}{1-\mu\eta}, & s \in [\eta,1], \end{cases}$$

and

$$\max_{t \in [0,1]} G(t,s) = \begin{cases} \frac{s}{1-\mu\eta}[(1-s) - \mu(\eta - s)], & s \in [0,\eta]; \\ \frac{s(1-s)}{1-\mu\eta}, & s \in [\eta,1]. \end{cases}$$

Here we set $\gamma = \min\{\mu\eta, \frac{\mu(1-\eta)}{1-\mu\eta}\}$, and then it satisfies

$$G(t,s) \ge \gamma \max_{t \in [0,1]} G(t,s) \quad \text{for } t \in [\eta,1], s \in [0,1].$$

Case 2: $1 < \mu < 1/\eta$. With the definition of $G(t,s)$, there are

$$\min_{t \in [\eta,1]} G(t,s) = \begin{cases} \frac{s(1-\eta)}{1-\mu\eta}, & s \in [0,\eta]; \\ \frac{\eta(1-s)}{1-\mu\eta}, & s \in [\eta,1], \end{cases}$$

and

$$\max_{t \in [0,1]} G(t,s) = \begin{cases} \frac{s\mu(1-\eta)}{1-\mu\eta}, & s \in [0,\eta]; \\ \frac{s(1-s)}{1-\mu\eta} \max\{s, \mu\eta\}, & s \in [\eta,1]. \end{cases}$$

Set $\gamma = \min\{\eta, \frac{1}{\mu}\} = \eta$, then

$$G(t,s) \geq \gamma \max_{t\in[0,1]} G(t,s), \quad \text{for } t \in [\eta, 1], s \in [0,1].$$

The proof is complete. □

3 Existence results of positive solutions

In this section, by using Lemma 2.1 and Lemma 2.2, we obtain positive solutions of (1.1), (1.2).

If $x = x(t)$ satisfies the operator equation

$$x(t) = (Tx)(t) := \int_0^1 G(t,s) f\bigl(s, x(s), x'(s)\bigr)\,ds, \quad 0 \leq t \leq 1,$$

where $G(t,s) \geq 0$ is Green's function for boundary value problem (2.3), (2.4), then $x = x(t)$ is the solution of problem (1.1), (1.2).

Let X be a Banach space in $C^1[0,1]$, with

$$\|x\| = \max\Bigl\{\max_{0\leq t\leq 1}\bigl|x(t)\bigr|, \max_{0\leq t\leq 1}\bigl|x'(t)\bigr|\Bigr\}.$$

Define a cone P by

$$P = \Bigl\{x \in X \bigm| x(t) \geq 0, \text{ and } \min_{\eta\leq t\leq 1} x(t) \geq \gamma \max_{0\leq t\leq 1}\bigl|x(t)\bigr|\Bigr\},$$

and functionals

$$\alpha(x) = \max_{0\leq t\leq 1}\bigl|x(t)\bigr|, \qquad \beta(x) = \max_{0\leq t\leq 1}\bigl|x'(t)\bigr| \quad \text{for } x \in X.$$

By (2.1), (2.2), $\alpha, \beta : X \to R^+$ are two continuous nonnegative functionals such that $\|x\| = \max\{\alpha(x), \beta(x)\}$ and

$$P = \Bigl\{x \in X \bigm| x(t) \geq 0, \text{ and } \min_{\eta\leq t\leq 1} x(t) \geq \gamma\alpha(x)\Bigr\}.$$

Denote

$$M = \max_{0\leq t\leq 1}\int_0^1 G(t,s)\,ds, \qquad N = \max_{0\leq t\leq 1}\int_\eta^1 G(t,s)\,ds,$$

$$A = \int_\eta^1 (1-s)\,ds + \int_0^\eta (1 - s - \mu\eta + \mu s)\,ds,$$

$$\overline{A} = \int_\eta^{1-h} (1-s)\,ds + \int_h^\eta (1 - s - \mu\eta + \mu s)\,ds,$$

$$B = \frac{1}{1-\mu\eta}\max\Bigl\{\int_\eta^1 (1-s)\,ds + \int_0^\eta (1 - s - \mu\eta + \mu s)\,ds, \int_\eta^1 |\mu\eta - s|\,ds + \int_0^\eta s|\mu - 1|\,ds\Bigr\}.$$

$T: P \to P$ is completely continuous, and the following is a simple proof of that. In fact, if $x \in P$, there is

$$\begin{aligned}(Tx)(t) &= \int_0^1 G(t,s)f\bigl(s,x(s),x'(s)\bigr)\,ds \\ &\leq \int_0^1 \max_{0\leq t\leq 1} G(t,s)\cdot f\bigl(s,x(s),x'(s)\bigr)\,ds,\end{aligned}$$

so that

$$\alpha(Tx) = \max_{0\leq t\leq 1}(Tx)(t) \leq \int_0^1 \max_{0\leq t\leq 1} G(t,s)\cdot f\bigl(s,x(s),x'(s)\bigr)\,ds.$$

Combining this with (2.6), we get

$$\begin{aligned}\min_{\eta\leq t\leq 1}(Tx)(t) &= \min_{\eta\leq t\leq 1}\int_0^1 G(t,s)f\bigl(s,x(s),x'(s)\bigr)\,ds \\ &\geq \gamma\int_0^1 \max_{0\leq t\leq 1} G(t,s)\cdot f\bigl(s,x(s),x'(s)\bigr)\,ds \\ &\geq \gamma\cdot\alpha(Tx).\end{aligned}$$

Moreover, from the positivity of $G(t,s)$, there is $(Tx)(t)\geq 0$, $0\leq t\leq 1$, for $x\in P$. So we can get $T: P\to P$. Further, standard arguments yield that T is completely continuous.

Theorem 3.1 *Suppose that there are four constants* $r_2 > r_1 > 0$, $L_2 > L_1 > 0$ *such that* $\max\{\frac{r_1}{M}, \frac{L_1}{A}\} \leq \min\{\frac{r_2}{M}, \frac{L_2}{B}\}$ *and the following assumptions hold*:

(A_1) $f(t,u,v) \geq \max\{\frac{r_1}{M}, \frac{L_1}{A}\}$ *for* $(t,u,v)\in[0,1]\times[0,r_1]\times[-L_1,L_1]$;

(A_2) $f(t,u,v) \leq \min\{\frac{r_2}{M}, \frac{L_2}{B}\}$ *for* $(t,u,v)\in[0,1]\times[0,r_2]\times[-L_2,L_2]$.

Then problem (1.1), (1.2) *has at least one positive solution* $x(t)$ *such that*

$$r_1 \leq \max_{0\leq t\leq 1} x(t) \leq r_2 \quad \textit{or} \quad L_1 \leq \max_{0\leq t\leq 1}\bigl|x'(t)\bigr| \leq L_2.$$

Proof Take two bounded open subsets in X

$$\Omega_i = \bigl\{x\in X \mid \alpha(x) < r_i, \beta(x) < L_i\bigr\}, \quad i = 1,2.$$

In addition, let

$$\begin{aligned}C_i &= \bigl\{x\in X \mid \alpha(x) = r_i, \beta(x) \leq L_i\bigr\}, \quad i = 1,2; \\ D_i &= \bigl\{x\in X \mid \alpha(x) \leq r_i, \beta(x) = L_i\bigr\}, \quad i = 1,2.\end{aligned}$$

For $x\in C_1\cap P$, by (A_1), there is

$$\begin{aligned}\alpha(Tx) &= \max_{t\in[0,1]}\left|\int_0^1 G(t,s)f\bigl(s,x(s),x'(s)\bigr)\,ds\right| \\ &\geq \frac{r_1}{M}\cdot\max_{t\in[0,1]}\left|\int_0^1 G(t,s)\,ds\right| = r_1.\end{aligned}$$

Taking into account the continuity and properties of T, we have

$$(Tx)'(t) = -\int_0^t f\big(s,x(s),x'(s)\big)\,ds + \frac{1}{1-\mu\eta}\int_0^1 (1-s)f\big(s,x(s),x'(s)\big)\,ds$$
$$- \frac{\mu}{1-\mu\eta}\int_0^\eta (\eta-s)f\big(s,x(s),x'(s)\big)\,ds,$$
$$(Tx)''(t) = -f\big(t,x(t),x'(t)\big) \le 0, \quad 0 \le t \le 1.$$

Thus, $(Tx)(t)$ is concave on $[0,1]$ and

$$\max_{t\in[0,1]}\big|(Tx)'(t)\big| = \max\big\{\big|(Tx)'(0)\big|,\big|(Tx)'(1)\big|\big\}.$$

For $x \in D_1 \cap P$, combine (A_1) and $f \ge 0$, there is

$$\begin{aligned}
\beta(Tx) &= \max_{t\in[0,1]}\big|(Tx)'(t)\big| \\
&= \max\big\{\big|(Tx)'(0)\big|,\big|(Tx)'(1)\big|\big\} \\
&\ge \big|(Tx)'(0)\big| \\
&= \frac{1}{1-\mu\eta}\bigg[\int_\eta^1 (1-s)f\big(s,x(s),x'(s)\big)\,ds \\
&\quad + \int_0^\eta (1-s-\mu\eta+\mu s)f\big(s,x(s),x'(s)\big)\,ds\bigg] \\
&\ge \frac{L_1}{A}\cdot\bigg[\int_\eta^1 (1-s)\,ds + \int_0^\eta (1-s-\mu\eta+\mu s)\,ds\bigg] \\
&= \frac{L_1}{A}\cdot A = L_1.
\end{aligned}$$

For $x \in C_2 \cap P$, by (A_2), there is

$$\begin{aligned}
\alpha(Tx) &= \max_{t\in[0,1]}\bigg|\int_0^1 G(t,s)f\big(s,x(s),x'(s)\big)\,ds\bigg| \\
&\le \max_{t\in[0,1]}\int_0^1 G(t,s)\cdot\frac{r_2}{M}\,ds \\
&= \frac{r_2}{M}\cdot\max_{t\in[0,1]}\int_0^1 G(t,s)\,ds = r_2.
\end{aligned}$$

For $x \in D_2 \cap P$, by (A_2), there is

$$\begin{aligned}
\beta(Tx) &= \max_{t\in[0,1]}\big|(Tx)'(t)\big| \\
&= \max\big\{\big|(Tx)'(0)\big|,\big|(Tx)'(1)\big|\big\} \\
&\le \frac{1}{1-\mu\eta}\max\bigg\{\int_\eta^1 (1-s)f\big(s,x(s),x'(s)\big)\,ds \\
&\quad + \int_0^\eta (1-s-\mu\eta+\mu s)f\big(s,x(s),x'(s)\big)\,ds,
\end{aligned}$$

$$\begin{aligned}
&\int_\eta^1 |\mu\eta - s|f\bigl(s,x(s),x'(s)\bigr)\,ds + \int_0^\eta s|\mu-1|f\bigl(s,x(s),x'(s)\bigr)\,ds\Bigr\} \\
&\leq \frac{L_2}{B}\cdot B = L_2.
\end{aligned}$$

Now, from Lemma 2.1 we can get that there is $x \in (\overline{\Omega}_2 \setminus \Omega_1) \cap P$ such that $x = Tx$. The above proof satisfies the condition of Lemma 2.1, so problem (1.1), (1.2) has at least one positive solution $x(t)$ such that

$$r_1 \leq \alpha(x) \leq r_2 \quad \text{or} \quad L_1 \leq \beta(x) \leq L_2,$$

i.e.,

$$r_1 \leq \max_{0\leq t\leq 1} x(t) \leq r_2 \quad \text{or} \quad L_1 \leq \max_{0\leq t\leq 1}\bigl|x'(t)\bigr| \leq L_2.$$

The proof is complete. □

Theorem 3.2 *Suppose that there are five constants* $0 < r_1 < r_2$, $0 < L_1 < L_2$, $0 \leq h \leq \min\{\eta, 1-\eta\}$ *such that* $\max\{\frac{r_1}{N}, \frac{L_1}{A}\} \leq \min\{\frac{r_2}{M}, \frac{L_2}{B}\}$ *and the following assumptions hold*:

(A_3) $f(t,u,v) \geq \frac{r_1}{N}$ *for* $(t,u,v) \in [\eta,1] \times [\gamma r_1, r_1] \times [-L_1, L_1]$;

(A_4) $f(t,u,v) \geq \frac{L_1}{A}$ *for* $(t,u,v) \in [h,1-h] \times [0,r_1] \times [-L_1, L_1]$;

(A_5) $f(t,u,v) \leq \min\{\frac{r_2}{M}, \frac{L_2}{B}\}$ *for* $(t,u,v) \in [0,1] \times [0,r_2] \times [-L_2, L_2]$.

Then problem (1.1), (1.2) *has at least one positive solution* $x(t)$ *such that*

$$r_1 \leq \max_{0\leq t\leq 1} x(t) \leq r_2 \quad \textit{or} \quad L_1 \leq \max_{0\leq t\leq 1}\bigl|x'(t)\bigr| \leq L_2.$$

Proof We just need to notice the difference between the following proof and the proof of Theorem 3.1.

For $x \in C_1 \cap P$, by the definition of P, there is

$$x(t) \geq \gamma\alpha(x) = \gamma r_1, \quad \text{for } t \in [\eta, 1].$$

By (A_3), there is

$$\begin{aligned}
\alpha(Tx) &= \max_{t\in[0,1]}\left|\int_0^1 G(t,s)f\bigl(s,x(s),x'(s)\bigr)\,ds\right| \\
&\geq \max_{t\in[0,1]}\left|\int_\eta^1 G(t,s)f\bigl(s,x(s),x'(s)\bigr)\,ds\right| \\
&\geq \max_{t\in[0,1]}\left|\int_\eta^1 G(t,s)\cdot\frac{r_1}{N}\,ds\right| \\
&= \frac{r_1}{N}\cdot\max_{t\in[0,1]}\left|\int_\eta^1 G(t,s)\,ds\right| = r_1.
\end{aligned}$$

For $x \in D_1 \cap P$, by (A_4), there is

$$\begin{aligned}
\beta(Tx) &= \max_{t\in[0,1]}\bigl|(Tx)'(t)\bigr| \\
&= \max\bigl\{\bigl|(Tx)'(0)\bigr|, \bigl|(Tx)'(1)\bigr|\bigr\}
\end{aligned}$$

$$
\begin{aligned}
&\geq \left|(Tx)'(0)\right| \\
&= \frac{1}{1-\mu\eta}\left[\int_\eta^1 (1-s)f\bigl(s,x(s),x'(s)\bigr)\,ds \right.\\
&\quad \left. + \int_0^\eta (1-s-\mu\eta+\mu s)f\bigl(s,x(s),x'(s)\bigr)\,ds\right] \\
&\geq \frac{1}{1-\mu\eta}\left[\int_\eta^{1-h} (1-s)f\bigl(s,x(s),x'(s)\bigr)\,ds \right.\\
&\quad \left. + \int_h^\eta (1-s-\mu\eta+\mu s)f\bigl(s,x(s),x'(s)\bigr)\,ds\right] \\
&\geq \frac{L_1}{\overline{\overline{A}}}\cdot\left[\int_\eta^{1-h}(1-s)\,ds + \int_h^\eta (1-s-\mu\eta+\mu s)\,ds\right] \\
&= \frac{L_1}{\overline{\overline{A}}}\cdot\overline{\overline{A}} = L_1.
\end{aligned}
$$

The rest of the proof is similar to that of Theorem 3.1 and is omitted. □

Remark 3.1 The conditions of our results are weaker than those of [14].

4 Examples

We present some examples to illustrate the main results.

Example 4.1 Consider the boundary value problem

$$x''(t) + f\bigl(t,x(t),x'(t)\bigr) = 0, \quad 0 < t < 1, \tag{4.1}$$

$$x(0) = 0, \qquad x(1) = x\left(\frac{1}{2}\right), \tag{4.2}$$

where

$$f(t,u,v) = \frac{u^2(\sin t)^2}{16} + \frac{v\cos t}{32} + \frac{65}{64}.$$

Direct computation shows that

$$M = \frac{7}{8}, \qquad N = 2, \qquad A = \frac{3}{8}, \qquad B = \frac{3}{4}.$$

Choose

$$r_1 = \frac{1}{2}, \qquad r_2 = 1, \qquad L_1 = \frac{1}{4}, \qquad L_2 = 1,$$

then $\max\{\frac{r_1}{M}, \frac{L_1}{A}\} = \frac{2}{3} \leq \min\{\frac{r_2}{M}, \frac{L_2}{B}\} = \frac{8}{7}$, and

$$
\begin{aligned}
&f(t,u,v) \geq \frac{169}{128} \geq \max\left\{\frac{r_1}{M}, \frac{L_1}{A}\right\} = \frac{2}{3}, \quad (t,u,v) \in [0,1]\times\left[0,\frac{1}{2}\right]\times\left[\frac{-1}{4},\frac{1}{4}\right],\\
&f(t,u,v) \leq \frac{71}{64} \leq \min\left\{\frac{r_2}{M}, \frac{L_2}{B}\right\} = \frac{8}{7}, \quad (t,u,v) \in [0,1]\times[0,1]\times[-1,1].
\end{aligned}
$$

That is to say, all the assumptions of Theorem 3.1 are satisfied, then problem (4.1), (4.2) has at least one positive solution x such that

$$\frac{1}{2} \leq \max_{0\leq t\leq 1} x(t) \leq 1, \quad \text{or} \quad \frac{1}{4} \leq \max_{0\leq t\leq 1} \left|x'(t)\right| \leq 1.$$

Example 4.2 Consider the boundary value problem

$$x''(t) + f\bigl(t, x(t), x'(t)\bigr) = 0, \quad 0 < t < 1, \tag{4.3}$$

$$x(0) = 0, \qquad x(1) = x\left(\frac{1}{3}\right), \tag{4.4}$$

where

$$f(t,u,v) = \left(\frac{t}{5} + 2\right)\left(\frac{1}{u+1} - \frac{v^2}{6}\right).$$

Direct computation shows

$$M = \frac{8}{9}, \qquad N = \frac{1}{2}, \qquad \bar{A} = \frac{4}{9}, \qquad B = \frac{2}{3}, \qquad \gamma = \frac{1}{3}.$$

Choose

$$r_1 = \frac{1}{2}, \qquad r_2 = 2, \qquad L_1 = \frac{1}{2}, \qquad L_2 = 2, \qquad h = \frac{1}{3},$$

then $\max\{\frac{r_1}{M}, \frac{L_1}{A}\} = \frac{9}{8} \leq \min\{\frac{r_2}{M}, \frac{L_2}{B}\} = \frac{9}{4}$, and

$$f(t,u,v) \geq \frac{31}{24} \approx 1.29 \geq \frac{r_1}{N} = 1, \quad (t,u,v) \in \left[\frac{1}{3}, 1\right] \times \left[\frac{1}{6}, \frac{1}{2}\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right],$$

$$f(t,u,v) \geq 1.29 \geq \frac{L_1}{A} = \frac{9}{8} \approx 1.125, \quad (t,u,v) \in \left[\frac{1}{3}, \frac{2}{3}\right] \times \left[0, \frac{1}{2}\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right],$$

$$f(t,u,v) \leq 2.2 \leq \min\left\{\frac{r_2}{M}, \frac{L_2}{B}\right\} = 2.25, \quad (t,u,v) \in [0,1] \times [0,2] \times [-2,2],$$

i.e., all the assumptions of Theorem 3.2 are satisfied, then problem (4.3), (4.4) has at least one positive solution x such that

$$\frac{1}{2} \leq \max_{0\leq t\leq 1} x(t) \leq 2, \quad \text{or} \quad \frac{1}{2} \leq \max_{0\leq t\leq 1} \left|x'(t)\right| \leq 2.$$

5 Conclusion

By the use of a fixed point theorem, some existence results for a class of second-order differential equations with three-point boundary value conditions are obtained. The interesting point is that the nonlinear term is dependent on the convection term.

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Abbreviations
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Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

Ethics approval and consent to participate
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Competing interests
The authors declare that they have no competing interests.

Consent for publication
All authors agree to publication of this paper in JIAP.

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