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Linear convergence of the relaxed gradient projection algorithm for solving the split equality problems in Hilbert spaces

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Abstract

In this paper, we consider the relaxed gradient projection algorithm to solve the split equality problem in Hilbert spaces, and we investigate its linear convergence. In particular, we use the concept of the bounded linear regularity property for the split equality problem to prove the linear convergence property for the above algorithm. Furthermore, we conclude the linear convergence rate of the relaxed gradient projection algorithm. Finally, some numerical experiments are given to test the validity of our results.

Keywords: Linear convergence; Split equality problem; Bounded linear regularity; Relaxed gradient projection algorithm

1 Introduction

Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, and let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators, H_3 is also a real Hilbert spaces. The split equality problem (SEP for short), as an important extension of the split feasibility problem, was first presented by Moudafi [1]. It can be mathematically characterized by finding points $x \in C$ and $y \in Q$ that satisfy the property

$$Ax = By, \tag{1.1}$$

which allows for asymmetric and partial relations between the variables x and y.

The split equality problem has received plenty attention due to its extraordinary practicality and wide applicability in many fields of applied mathematics; examples of such problems include decomposition methods for partial differential equations, applications in game theory and intensity-modulated radiation therapy (IMRT for short), for which comprehensive references are available [2, 3]. In fact, various algorithms have been used in studies extensively to find a solution to the split equality problem. One of the most original and important algorithms is the alternating CQ algorithm (ACQA for short), which was proposed by Moudafi [1], and it has the following iterative form:

$$(\text{ACQA}) \quad \begin{cases} x_{k+1} = P_C(x_k - \gamma_k A^*(Ax_k - By_k)), \\ y_{k+1} = P_Q(y_k + \gamma_k B^*(Ax_{k+1} - By_k)). \end{cases} \tag{1.2}$$

Then one proved the weak convergence of ACQA (1.2) provided that the solution set of SEP (1.1) is nonempty.

Since the ACQ algorithm involves two projections P_C and P_Q, it might be difficult to calculate in the case where one of them does not have a closed-form expression. To solve this problem, Moudafi [4] proposed the relaxed alternating CQ algorithm (RACQA for short) by using orthogonal projections onto half-spaces to replace the original closed convex sets, and it has the following iterative form:

$$\text{(RACQA)}\quad \begin{cases} x_{k+1} = P_{C_k}(x_k - \gamma A^*(Ax_k - By_k)), \\ y_{k+1} = P_{Q_k}(y_k + \beta B^*(Ax_{k+1} - By_k)). \end{cases} \tag{1.3}$$

Meanwhile, one proved that the above algorithm can converge weakly to a solution of SEP (1.1).

In the RACQA, the step size parameters do not vary. Then, as a quite important generalization of the RACQA, Moudafi [5] presented the relaxed simultaneous iterative algorithm (RSSEA for short), whose parameters are allowed to vary, and obtained a weak convergence result:

$$\text{(RSSEA)}\quad \begin{cases} x_{k+1} = P_{C_k}(x_k - \gamma_k A^*(Ax_k - By_k)), \\ y_{k+1} = P_{Q_k}(y_k + \gamma_k B^*(Ax_k - By_k)). \end{cases}$$

Moreover, in order to obtain a strong convergence result, Shi et al. [6] improved Moudafi's algorithms and proposed the following algorithm:

$$\begin{cases} x_{k+1} = P_C[(1-\alpha_k)(x_k - \gamma_k A^*(Ax_k - By_k))], \\ y_{k+1} = P_Q[(1-\alpha_k)(y_k + \gamma_k B^*(Ax_k - By_k))]. \end{cases}$$

The above basic methods for solving the split equality problem are well known. For more information with regard to methods solving the split equality problem, see [7–9]. However, the convergence results of the above algorithms are not good enough and the convergence rate of these algorithms have not been explicitly estimated.

Recently, Shi et al. [10] presented the varying step size gradient projection algorithm for solving the SEP and obtained a linear convergence result. In particular, they conclude the linear convergent rate of the varying step size gradient projection algorithm. However, this algorithm is not easy to implement.

Let $S = C \times Q \subseteq H_1 \times H_2 =: H$, $G = [A, -B] : H \to H_3$, and the adjoint operator of G is denoted by G^*. Then the problem (1.1) can be reformulated as to find $w = (x, y) \in S$ which satisfies $Gw = 0$. And then the relaxed simultaneous iterative algorithm (RSSEA) reduces to the following relaxed gradient projection algorithm (in short, RGPA):

$$\text{(RGPA)}\quad w_{k+1} = P_{S_k}\big(w_k - \gamma_k G^* G w_k\big).$$

Recall that the RGPA is an easily implementable algorithm that uses orthogonal projection onto half-spaces at each step. In this paper, what attracts us is to study RGPA for solving the split equality problem. In particular, to the best of our knowledge, in order to get the linear convergence property for the CQ algorithm which is to solve the split feasibility problem, Wang et al. [11] presented the linear regularity property for the split feasibility problem. Motivated and inspired by their work, we devote our work to proving the linear convergence property for the RGPA with the bounded linear regularity property for SEP (1.1). For this purpose, we introduce the notion of the bounded linear regularity property for SEP (1.1), and use some suitable types of step sizes to prove the linear convergence property for the RGPA. In addition, we conclude the linear convergence rate of RGPA. Finally, some numerical experiments are given to test the validity of our results.

2 Preliminaries

For convenience, we introducing several notations. Throughout the whole paper, we assume that H is a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. I denotes the identity operator on H. Let S be a nonempty subset of H, the relative interior of S is denoted by $\operatorname{ri} S$. T^* is the adjoint operators of T. We denote by $\mathbf{B}$ and $\overline{\mathbf{B}}$ the unit open ball and the unit closed ball with center at the origin, respectively, that is,

$$\mathbf{B} := \{v \in H : \|v\| < 1\} \quad \text{and} \quad \overline{\mathbf{B}} := \{v \in H : \|v\| \leq 1\}.$$

There are several definitions and basic results that will be used in the proofs of our main results.

Definition 2.1 ([12]) A mapping $T : H \to H$ goes by the name of

(i) non-expansive, if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in H;$$

(ii) firmly non-expansive, if

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle, \quad \forall x, y \in H.$$

For an element $w \in H$ and a set $S \subset H$, the distance of w onto S and the orthogonal projection from w onto S, denoted by $d_S(w)$ and $P_S(w)$, respectively, are defined by

$$d_S(w) = \inf_{v \in S} \|w - v\| \quad \text{and} \quad P_S(w) = \{v \in S : d(w, S) = \|w - v\|\}.$$

Some basic properties of an orthogonal projection were introduced by Bauschke et al. in [12], and they are listed in the following proposition.

Proposition 2.2 ([12]) *Let S be a closed, convex, and nonempty subset of H, then, for any $x, y \in H$ and $z \in S$,*

(i) $\langle x - P_S x, z - P_S x\rangle \leq 0$;
(ii) $\|P_S x - P_S y\|^2 \leq \langle P_S x - P_S y, x - y\rangle$;
(iii) $\|P_S x - z\|^2 \leq \|x - z\|^2 - \|P_S x - x\|^2$.

Remark 2.3 By the Cauchy–Schwarz inequality, we can easily see that a firmly non-expansive mapping is non-expansive. From Proposition 2.2, it can be deduced that P_S is firmly non-expansive and non-expansive.

Let $G : H \to H_3$ be a bounded linear operator. The kernel of G is denoted $\ker G = \{y \in H : Gy = 0\}$, and the orthogonal complement of $\ker G$ is denoted $(\ker G)^{\perp} = \{x \in H : \langle y, x\rangle = 0, \forall y \in \ker G\}$. As is well known, both $\ker G$ and $(\ker G)^{\perp}$ are closed subspaces of H. Throughout this paper, we use Γ to denote the solution set of SEP (1.1), that is,

$$\Gamma := \{w \in S : Gw = 0\} = S \cap G^{-1}(0) = S \cap \ker G.$$

We assume that the SEP is consistent, thus, Γ is a closed, convex, and nonempty set.

Recall that a sequence $\{w_k\}$ in H is called linearly convergent to its limit w^* (with rate $\alpha \in [0, 1)$), if there exist $\beta > 0$ and a positive integer N such that

$$\|w_k - w^*\| \le \beta\alpha^k \quad \text{for all } k \ge N.$$

To investigate the linear convergence property of the projection algorithm for solving convex feasibility problems, Zhao et al. [13] presented the linear regularity for a family of closed convex subsets in a real Hilbert space, as defined below.

Definition 2.4 ([13]) Let $\{S_i\}_{i\in I}$ be a family of closed convex subsets of a real Hilbert space H and $S = \bigcap_{i\in I} S_i \neq \emptyset$. The family $\{S_i\}_{i\in I}$ is called bounded linearly regular if, for each $a > 0$, there exists a constant $\gamma_a > 0$ such that

$$d_S(w) \le \gamma_a \sup\bigl\{d_{S_i}(w) : i \in I\bigr\} \quad \text{for all } w \in a\mathbf{B}.$$

Bauschke [14] proved the following lemma for the case H is the Euclidean space. It provides sufficient conditions for the bounded linear regularity property for two closed convex subsets of H.

Lemma 2.5 ([14]) *Let E and F be closed convex subsets of H. Then E, F is bounded linearly regular provided that at least one of the following conditions holds*:

(a) $\operatorname{ri} E \cap F \neq \emptyset$ *and F is a polyhedron*;
(b) $\operatorname{ri} E \cap \operatorname{ri} F \neq \emptyset$ *and E is finite codimensional*;
(c) $\operatorname{ri} E \cap \operatorname{ri} F \neq \emptyset$ *and E is finite dimensional*.

Next, we will introduce the concept of bounded linear regularity for SEP (1.1).

Definition 2.6 ([10]) SEP (1.1) is said to have the bounded linear regularity property if for each $a > 0$, there exists a constant $\gamma_a > 0$ such that

$$\gamma_a d_\Gamma(w) \le \|Gw\| \quad \text{for all } w \in a\mathbf{B} \cap S. \tag{2.1}$$

Shi et al. [10] construct some moderate sufficient conditions to ensure the bounded linear regularity property for SEP (1.1). This is shown in the lemma below.

Lemma 2.7 ([10]) *SEP* (1.1) *satisfies the bounded linear regularity property if one of the following conditions holds*:

(a) *C and Q are polyhedrons, and G has closed range*;
(b) $\operatorname{ri} S \cap \ker G \neq \emptyset$ *and $\ker G$ is finite codimensional*;

(c) $\operatorname{ri} S \cap \ker G \neq \emptyset$ *and* $\ker G$ *is finite dimensional*;
(d) $\operatorname{ri} S \cap \ker G \neq \emptyset$, *$G$ has closed range and $S = C \times Q$ is finite codimensional*;
(e) $\operatorname{ri} S \cap \ker G \neq \emptyset$, *$G$ has closed range and $S = C \times Q$ is finite dimensional.*

Now, we will present the definition of sub-differential which is vital for constructing iterative algorithms later.

Definition 2.8 ([15]) Let $f : H \to R$ be a convex function. The sub-differential of f at x is defined as

$$\partial f(x) := \{\xi \in H : f(y) \geq f(x) + \langle \xi, y - x\rangle \text{ for all } y \in H\}.$$

Lemma 2.9 ([15]) *Let $f : H \to R$ be a convex function, $x_0 \in H$, and f be sub-differentiable at x_0. Suppose that $D = \{x \in H : f(x) \leq 0\}$ is nonempty for any $g(x_0) \in \partial f(x_0)$, define $\tilde{D}$ by*

$$\tilde{D} := \{x \in H : f(x_0) + \langle g(x_0), x - x_0\rangle \leq 0\}.$$

Then:
(i) $D \subseteq \tilde{D}$. *If $g(x_0) \neq 0$, then $\tilde{D}$ is a half-space*; *otherwise*, $\tilde{D} = H$;
(ii) $P_{\tilde{D}}(x_0) = x_0 - \frac{\max\{f(x_0),0\}}{\|g(x_0)\|^2} g(x_0)$;
(iii) $d_{\tilde{D}}(x_0) = \frac{\max\{f(x_0),0\}}{\|g(x_0)\|}$.

Finally, in order to complete the convergence analysis, the following equality and concept of Fejér monotone sequence are essential.

Lemma 2.10 ([12]) *Let $\{x_i\}_{i\in I}$ be a finite family in H, and $\{\lambda_i\}_{i\in I}$ be a finite family in R with $\sum_{i\in I} \lambda_i = 1$, then the following equality holds*:

$$\left\| \sum_{i\in I} \lambda_i x_i \right\|^2 = \sum_{i\in I} \lambda_i \|x_i\|^2 - \frac{1}{2} \sum_{i\in I} \sum_{j\in I} \lambda_i \lambda_j \|x_i - x_j\|^2, \quad i \geq 2.$$

Definition 2.11 ([12]) Let C be a nonempty subset of H, and $\{x_i\}$ be a sequence in H. $\{x_i\}$ is called Fejér monotone with respect to C, if

$$\|x_{i+1} - z\| \leq \|x_i - z\|, \quad \forall z \in C.$$

Obviously, $\lim_{i\to\infty} \|x_i - z\|$ exists.

3 Main result

In this section, we mainly use the bounded linear regularity property for SEP (1.1) to prove the linear convergence of the relaxed gradient projection algorithm when using different types of step sizes.

We start by reviewing the relaxed gradient projection algorithm in detail. Note that Moudafi [5] presented the relaxed simultaneous iterative algorithm (RSSEA) for solving the approximate SEP and established its weak convergence:

$$\text{(RSSEA)} \quad \begin{cases} x_{k+1} = P_{C_k}(x_k - \gamma_k A^*(Ax_k - By_k)), \\ y_{k+1} = P_{Q_k}(y_k + \gamma_k B^*(Ax_k - By_k)), \end{cases} \tag{3.1}$$

where C_k and Q_k are two sequences of closed convex sets, defined by

$$C_k = \{x \in H_1 : c(x_k) + \langle \xi_k, x - x_k \rangle \le 0\}, \quad \text{where } \xi_k \in \partial c(x_k),$$

and

$$Q_k = \{y \in H_2 : q(y_k) + \langle \eta_k, y - y_k \rangle \le 0\}, \quad \text{where } \eta_k \in \partial q(y_k),$$

where $c : H_1 \to R$ and $q : H_2 \to R$ are convex, sub-differentiable functions, and where the sub-differentials are bounded on bounded sets. Applying the definition of sub-differential, one finds that $C \subseteq C_k$ and $Q \subseteq Q_k$, where C and Q are two nonempty closed convex level sets:

$$C = \{x \in H_1 : c(x) \le 0\},$$

and

$$Q = \{y \in H_2 : q(y) \le 0\}.$$

For convenience, we define $h : H_1 \times H_2$ to be

$$h(w) = h(x, y) = c(x) + q(y),$$

then

$$C \times Q \subseteq S, \quad \text{where } S = \{w \in H_1 \times H_2 : h(w) \le 0\}.$$

We define

$$S_k = \{w \in H_1 \times H_2 : h(w_k) + \langle \theta_k, w - w_k \rangle \le 0\}, \quad \text{where } \theta_k \in \partial h(w_k),$$

then

$$S \subseteq S_k, \qquad C_k \times Q_k \subseteq S_k.$$

Moreover, let $S = C \times Q \subseteq H = H_1 \times H_2$. $G = [A, -B] : H \to H_3$. The adjoint operator of G is denoted by G^*. Then G and G^*G have the following matrix form:

$$G = [A, -B], \qquad G^*G = \begin{bmatrix} A^*A & -A^*B \\ -B^*B & B^*B \end{bmatrix}.$$

On that basis, the original problem (1.1) can be modified as

$$\text{to find } w = (x, y) \in S \text{ such that } Gw = 0. \tag{3.2}$$

And then the algorithm (3.1) reduces to the following relaxed gradient projection algorithm (in short, RGPA):

$$\text{(RGPA)} \quad w_{k+1} = P_{S_k}\bigl(w_k - \gamma_k G^* G w_k\bigr). \tag{3.3}$$

The lemma below will be a powerful tool in our proof later.

Lemma 3.1 *Assume that a vector x^k in S_k minimizes the function $f(t) = \frac{1}{2}\|Gt\|^2$ over all t in S_k. Then $x^k = P_{S_k}(x^k - \gamma_k \nabla f(x^k))$ with $\gamma_k \in (0, +\infty)$.*

Proof Since a vector x^k in S_k minimizes the function $f(t) = \frac{1}{2}\|Gt\|^2$ over all t in S_k we have $\langle \nabla f(x^k), t - x^k\rangle \geq 0$, where $\nabla f(x^k) = G^* G x^k$. This is equivalent to $\langle x^k - (x^k - \gamma_k \nabla f(x^k)), t - x^k\rangle \geq 0$ from which we infer that $x^k = P_{S_k}(x^k - \gamma_k \nabla f(x^k))$. The proof is complete. □

Now we give the main theorem and proof of this paper.

Theorem 3.2 *Assume that SEP (3.2) satisfies the bounded linear regularity property. Then the sequence $\{w_k\}$ generated by RGPA (3.3) with $\gamma_k \in (0, +\infty)$ converges to a solution w^* of SEP (1.1) such that*

$$\bigl\| w_k - w^* \bigr\| \leq \sigma p^{\sum_{i=1}^{k} \gamma_i}, \tag{3.4}$$

for $\sigma \geq 1$ and $0 < p < 1$, provided that one of the following conditions is assumed:

$$\begin{cases} \text{(a)} \quad 0 < \lim_{k\to\infty} \inf \gamma_k \leq \lim_{k\to\infty} \sup \gamma_k < \frac{2}{\|G\|^2}; \\ \text{(b)} \quad \gamma_k = \begin{cases} 0, & w_k \in \Gamma; \\ \frac{\rho_k \|G w_k\|^2}{\|G^* G w_k\|^2} \quad \textit{and} \quad 0 \leq \lim_{k\to\infty} \inf \rho_k \leq \lim_{k\to\infty} \sup \rho_k < 2, & \textit{otherwise}; \end{cases} \\ \text{(c)} \quad \lim_{k\to\infty} \gamma_k = 0 \quad \textit{and} \quad \sum_{k=1}^{\infty} \gamma_k = \infty. \end{cases} \tag{3.5}$$

Consequently, $\{w_k\}$ converges to w^* linearly in the case when (a) *or* (b) *is supposed.*

Proof Without loss of generality, we assume that w_k is not in Γ for all $k \geq 1$. Otherwise, RGPA (3.3) terminates in finite number of iterates, and then the conclusions follow clearly.

Firstly, we will show that the sequence $\{w_k\}$ is Fejér monotone with respect to Γ and the sequence $\|G w_k\|^2$ converges to zero.

Let $z \in \Gamma$, then $Gz = 0$, that is, z minimizes $f(t) = \frac{1}{2}\|Gt\|^2$ over $t \in S_k$, for all k. From Lemma 3.1,

$$z = P_{S_k} z = P_{S_k}\bigl(z - \gamma_k G^* G z\bigr), \tag{3.6}$$

for all k. Since we have (3.6) and P_{S_k} is non-expansive, we obtain

$$\begin{aligned} &\|z - w_{k+1}\|^2 \\ &\quad = \bigl\| P_{S_k}\bigl(z - \gamma_k G^* G z\bigr) - P_{S_k}\bigl(w_k - \gamma_k G^* G w_k\bigr) \bigr\|^2 \end{aligned}$$

$$
\begin{aligned}
&\le \left\| z - \gamma_k G^* G z - w_k + \gamma_k G^* G w_k \right\|^2 \\
&= \|z - w_k\|^2 - 2\gamma_k \langle z - w_k, G^* G z - G^* G w_k \rangle + \gamma_k^2 \left\| G^* G z - G^* G w_k \right\|^2,
\end{aligned}
$$

which is equivalent to

$$
\|z - w_k\|^2 - \|z - w_{k+1}\|^2 \ge \left(2\gamma_k - \gamma_k^2 \frac{\|G^* G w_k\|^2}{\|G w_k\|^2} \right) \|G w_k\|^2. \tag{3.7}
$$

Further, from the condition in (3.5), we get the following assertions:

(i) If (a) or (c) holds, then there exist $\eta > 0$ and $M \in \mathbf{N}$ such that

$$
\gamma_k \le \eta < \frac{2}{\|G\|^2} \quad \text{for any } k \ge M.
$$

(ii) If (a) or (b) holds, then

$$
\liminf_{k \to \infty} \gamma_k > 0.
$$

Using the above assertions, we deduce that there exists $M \in \mathbf{N}$ such that

$$
\gamma_k^2 \left\| G^* G w_k \right\| \le 2 \|G w_k\|, \tag{3.8}
$$

for any $k \ge M$, if (a), (b), (c) are assumed. Substituting (3.8) in (3.7), we see that $\|w_k - z\|_{k \ge M}$ is monotone decreasing. From Definition 2.11, we infer that the sequence $\{w_k\}$ is Fejér monotone with respect to Γ. Hence $\lim_{k \to \infty} \|w_k - z\|$ exists and the sequence $\|G w_k\|^2$ converges to zero.

Then we show that the sequence $\{\|w_k - w_{k+1}\|\}$ converges to zero.

In view of the property of the orthogonal projection, we infer

$$
\langle w_{k+1} - (w_k - \gamma_k G^* G w_k), z - w_{k+1} \rangle \ge 0,
$$

that is,

$$
\langle w_k - w_{k+1}, z - w_{k+1} \rangle \le \gamma_k \langle G^* G w_k, z - w_{k+1} \rangle \le \gamma_k \left\| G^* G w_k \right\| \|z - w_{k+1}\|. \tag{3.9}
$$

Combining (3.9) and

$$
\|w_k - w_{k+1}\|^2 = \|z - w_k\|^2 - \|z - w_{k+1}\|^2 + 2\langle w_{k+1} - w_k, w_{k+1} - z \rangle,
$$

we obtain

$$
\|w_k - w_{k+1}\|^2 \le \|z - w_k\|^2 - \|z - w_{k+1}\|^2 + 2\gamma_k \left\| G^* G w_k \right\| \|z - w_{k+1}\|.
$$

Since the sequence $\{\|w_k - z\|\}$ is bounded, the right hand side converges to zero. Therefore, the sequence $\{\|w_k - w_{k+1}\|\}$ converges to zero.

Next, we show that $\{w_k\}$ converges to a solution w^* of SEP and (3.4) holds. Because $w_{k+1} \in S_k$, we get

$$
h(w_k) + \langle \theta_k, w_{k+1} - w_k \rangle \le 0, \quad \text{where } \theta_k \in \partial h(w_k),
$$

which implies that

$$h(w_k) \le -\langle \theta_k, w_{k+1} - w_k \rangle \le \theta \|w_{k+1} - w_k\|, \quad \text{where } \|\theta_k\| \le \theta \text{ for all } k.$$

Then there exists $L \in \mathbf{N}$, when $k \ge L$, and by virtue of the sequence $\{\|w_k - w_{k+1}\|\}$ converging to zero, it follows that $h(w_k) \le 0$. Consequently, $w_k \in S$ for any $k \ge L$.

Since the SEP satisfies the bounded linear regularity property and $w_k \in S$ for all $k \ge L$, there exists $\beta > 0$ such that

$$\beta d_\Gamma(w_k) \le \|Gw_k\|, \tag{3.10}$$

for all $k \ge L$. Combining (3.10) with (3.7), we obtain

$$\|w_{k+1} - z\|^2 \le \|w_k - z\|^2 - \beta^2 \gamma_k \left(2 - \gamma_k \frac{\|G^* G w_k\|^2}{\|Gw_k\|^2}\right) d_\Gamma^2(w_k),$$

for each $z \in \Gamma$, which equals

$$d_\Gamma(w_{k+1})^2 \le \left(1 - \beta^2 \gamma_k \left(2 - \gamma_k \frac{\|G^* G w_k\|^2}{\|Gw_k\|^2}\right)\right) d_\Gamma^2(w_k). \tag{3.11}$$

Note that if (a), (b) and (c) hold, then

$$\lim_{k\to\infty} \inf \left(2 - \gamma_k \frac{\|G^* G w_k\|^2}{\|Gw_k\|^2}\right) > 0.$$

Hence, there exists T such that

$$\alpha = \inf_{k \ge T} \beta^2 \left(2 - \gamma_k \frac{\|G^* G w_k\|^2}{\|Gw_k\|^2}\right) > 0. \tag{3.12}$$

Using (3.12) and (3.11), we infer that

$$d_\Gamma^2(w_{k+1}) \le (1 - \alpha\gamma_k) d_\Gamma^2(w_k), \quad \text{for all } k \ge N = \max\{M, L, T\}.$$

By induction,

$$d_\Gamma^2(w_{k+1}) \le d_\Gamma^2(w_N) \prod_{i=N+1}^{k} (1 - \alpha\gamma_i), \tag{3.13}$$

for all $k \ge N = \max\{M, L, T\}$. Observe that, for each $z \in \Gamma$, $\|w_{k+1} - z\|$ is monotone decreasing for k, hence

$$\begin{aligned} \|w_m - w_{k+1}\| &\le \|w_m - P_\Gamma(w_{k+1})\| + \|w_{k+1} - P_\Gamma(w_{k+1})\| \\ &\le 2\|w_{k+1} - P_\Gamma(w_{k+1})\| = 2d_\Gamma(w_{k+1}), \end{aligned} \tag{3.14}$$

for all $m > k > N$. Substituting (3.13) in (3.14), one can easily show that

$$\|w_m - w_{k+1}\| \le 2d_\Gamma(w_N) \prod_{i=N+1}^{k} \sqrt{1 - \alpha\gamma_i}, \tag{3.15}$$

for all $m \geq k+1$. Let $p := e^{-\frac{\alpha}{2}} \in (0,1)$, then

$$\prod_{i=N+1}^{k} \sqrt{1-\alpha\gamma_i} = \exp\left\{\frac{1}{2}\sum_{i=N+1}^{k} \ln(1-\alpha\gamma_i)\right\} \leq p^{\sum_{i=N+1}^{k}\gamma_i}. \tag{3.16}$$

From (3.15) and (3.16), we get

$$\|w_m - w_{k+1}\| \leq 2d_\Gamma(w_N)p^{\sum_{i=N+1}^{k}\gamma_i}, \quad \text{for all } m \geq k+1.$$

By virtue of $\sum_{k=1}^{\infty}\gamma_k = \infty$, $\{w_k\}$ is a Cauchy sequence and converges to a solution w^* of SEP (1.1) satisfying

$$\left\|w_{k+1} - w^*\right\| \leq 2d_\Gamma(w_N)p^{\sum_{i=N+1}^{k}\gamma_i}, \quad \text{for all } k \geq N.$$

For convenience, let

$$\sigma = \max\left\{2d_\Gamma(w_N)p^{-\sum_{i=1}^{N}\gamma_i}, \max\left\{\left\|w_i - w^*\right\|p^{-\sum_{j=1}^{i}\gamma_j}, i = 1,2,\dots,N\right\}\right\}.$$

Then

$$\left\|w_k - w^*\right\| \leq \sigma p^{\sum_{i=1}^{k}\gamma_i}.$$

Moreover, if (a) or (b) is assumed, then $\lim_{k\to\infty}\inf\gamma_k > 0$. One can derive that $\{w_k\}$ converges to w^* linearly. This completes the proof. □

As a direct consequence of Lemma 2.7 and Theorem 3.2, we propose the following corollary.

Corollary 3.3 *Assume that one of statements* (a)–(e) *of Lemma* 2.7 *holds. Then the sequence* $\{w_k\}$ *generated by RGPA* (3.3) *with* $\gamma_k \in (0,+\infty)$ *converges to a solution* w^* *of SEP* (3.2) *satisfying* (3.4), *provided that one of the conditions in* (3.5) *is assumed. In particular,* $\{w_k\}$ *converges to* w^* *linearly in the case when* (a) *or* (b) *in* (3.5) *is assumed.*

4 Numerical experiments

In this section, we give an example to verify the validity of our results. All codes were written in Wolfram Mathematica (version 10.3). All the numerical procedures were performed on a personal Asus computer with AMD A9-9420 RADEON R5, 5 COMPUTE. CORES 2C+3G 3.00 GHz and RAM 8.00 GB.

Let $H_1 = R$, $H_2 = R^2$ and $H_3 = R^3$. We have the SEP with $C = C_k = \{x \in H_2 : \|x\| \leq 20\}$, $Q = Q_k = \{x \in H_1 : \|X\| \leq 10\}$, and $A : H_2 \to H_3$, $B : H_1 \to H_3$ are defined by

$$A(x,y) = (x,y,0) \quad \text{and} \quad B(z) = (0,z,0), \quad \text{for all } x,y,z \in R,$$

respectively. Let $S = C \times Q \subseteq R^3$. Define an operator $G = [A,-B] : S \to H_3$ by

$$G(x,y,z) = (x,y-z,0), \text{for all } (x,y,z) \in S.$$

Then $\ker G \cap \operatorname{ri} S = \{(0,z,z), z \in Q\} \neq \emptyset$, S is finite codimensional, G has closed range, and the solution set of the SEP is $\Gamma = (C \times Q) \cap \ker G = \{(0,z,z) : z \in Q\}$. By Lemma 2.7 it is easy to show that the SEP satisfies the bounded linear regularity property.

For $w = (x,y,z) \in S$, we have

$$d_\Gamma^2(w) = x^2 + \frac{(y-z)^2}{2}.$$

Let $w_0 = (x_0, y_0, z_0) \in C \times Q$. In view of RGPA (3.3), we infer

$$\begin{cases} x_{n+1} = x_n - \gamma_n x_n, \\ y_{n+1} = (1-\gamma_n)y_n + \gamma_n z_n, \\ z_{n+1} = (1-\gamma_n)z_n + \gamma_n y_n. \end{cases}$$

In algorithm (3.3), we take $\gamma_n = \frac{1}{2}, \frac{n}{n+1}$, respectively. Moreover, we select the error value to be 10^{-10}, 10^{-20}, and initial value $w_0 = (3,2,8)$. Then we get the numerical results displayed in Figs. 1 and 2.

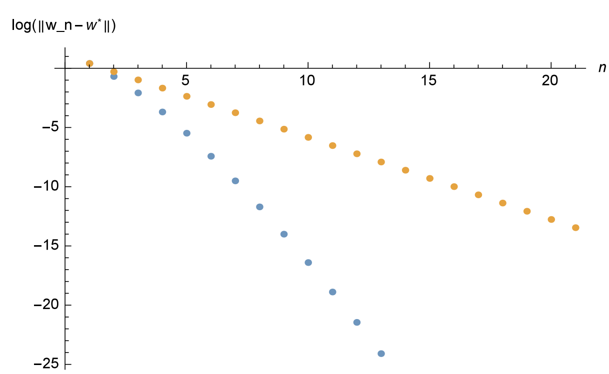

Figure 1 The x-coordinate indicates the number of iterative steps and the y-coordinate indicates the logarithm of the error. Initial conditions: $x_1 = 3, y_1 = 2, z_1 = 8$. $w^* = (0,5,5)$, error $= 10^{-10}$. ●: $\gamma_n = \frac{1}{2}$, ●: $\gamma_n = \frac{n}{n+1}$

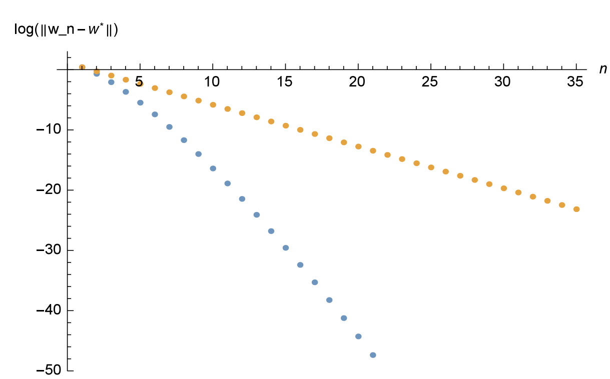

Figure 2 The x-coordinate indicates the number of iterative steps and the y-coordinate indicates the logarithm of the error. Initial conditions: $x_1 = 3, y_1 = 2, z_1 = 8$. $w^* = (0,5,5)$, error $= 10^{-20}$. ●: $\gamma_n = \frac{1}{2}$, ●: $\gamma_n = \frac{n}{n+1}$

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The authors declare that they have no competing interests.

Authors' contributions
The main idea of this paper was proposed by TT, LS and RC prepared the manuscript initially and performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

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References
1. Moudafi, A.: Alternating CQ-algorithm for convex feasibility and split fixed-point problems. J. Nonlinear Convex Anal. **15**(4), 809–818 (2013)
2. Censor, Y., Bortfeld, T., Martin, B.: A unified approach for inversion problems in intensity-modulated radiation therapy. Phys. Med. Biol. **51**, 2353–2365 (2006)
3. Censor, Y., Elfving, T., Kopf, N., Bortfled, T.: The multi-sets split feasibility problem and it applications to inverse problems. Inverse Probl. **21**, 2071–2084 (2005)
4. Moudafi, A.: A relaxed alternating CQ-algorithms for convex feasibility problems. Nonlinear Anal., Theory Methods Appl. **79**, 117–121 (2004)
5. Byrne, C., Moudafi, A.: Extensions of the CQ algorithm for split feasibility and split equality problems. hal-00776640-version 1 (2013)
6. Shi, L.Y., Chen, R.D., Wu, Y.J.: Strong convergence of iterative algorithms for split equality problem. J. Inequal. Appl. **2014**, Article ID 478 (2014)
7. Zhao, Y., Shi, L.Y.: Strong convergence of an extragradient-type algorithm for the multiple-sets split equality problem. J. Inequal. Appl. (2017). https://doi.org/10.1186/s13660-017-1326-y
8. Dong, Q.L., He, S.N., Zhao, J.: Solving the split equality problem without prior knowledge of operator norms. Optimization **64**(9), 1887–1906 (2015)
9. Dong, Q.L., He, S.N.: Modified projection algorithms for solving the split equality problems. Sci. World J. **2014**, Article ID 328787 (2014)
10. Shi, L.Y., Ansari, Q.H., Wen, C.F.: Linear convergence of gradient projection algorithm for split equality problems. Optimization, **67**, 2347–2358 (2018)
11. Wang, J.H., Hu, Y.H., Li, C., Yao, J.C.: Linear convergence of CQ algorithms and applications in gene regulatory network inference. Inverse Probl. **33**, 055017 (2017)
12. Bauschke, H.H., Combettes, P.L.: Convex Analysis and Monotone Operator Theory in Hilbert Spaces. Springer, London (2011)
13. Zhao, X.P., Kung, F.N., Li, C.: Linear regularity and linear convergence of projection-based methods for solving convex feasibility problems. Appl. Math. Optim. (2017). https://doi.org/10.1007/s00245-017-9417-1
14. Bauschke, H.H.: Projection algorithms and monotone operators. Ph.D. thesis, Department of Math-ematics, Simon Fraser University, Burnaby, BC (1996) http://oldweb.cecm.sfu.ca/preprints/1996pp.html
15. Bauschke, H.H., Borwein, J.M.: On projection algorithms for solving convex feasibility problems. SIAM Rev. **38**, 367–426 (1996)