Liu and Zhang *Journal of Inequalities and Applications* (2019) 2019:67
https://doi.org/10.1186/s13660-019-2023-9

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

A note on certain integrals along polynomial compound curves

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Abstract

In this paper we consider a singular integral operator and a parametric Marcinkiewicz integral operator with rough kernel. These operators have singularity along sets of the form curves $\{x = P(\varphi(|y|))y'\}$, where P is a real polynomial satisfying $P(0) = 0$ and φ satisfies certain smooth conditions. Under the conditions that $\Omega \in H^1(\mathbf{S}^{n-1})$ and $h \in \Delta_\gamma(\mathbf{R}_+)$ for some $\gamma > 1$, we prove that the above operators are bounded on the Lebesgue space $L^2(\mathbf{R}^n)$. Moreover, the L^2-bounds of the maximal functions related to the above integrals are also established. Particularly, the bounds are independent of the coefficients of the polynomial P. In addition, we also present certain Hardy type inequalities related to these operators.

MSC: Primary 42B20; secondary 42B15; 42B25; 42B99

Keywords: Singular integrals; Marcinkiewicz integrals; Maximal operators; Polynomial compound curves; Hardy spaces

1 Introduction

Let $\mathbf{R}^n$ $(n \geq 2)$ be the n-dimensional Euclidean space and $\mathbf{S}^{n-1}$ denote the unit sphere in $\mathbf{R}^n$ equipped with the induced Lebesgue measure $d\sigma$. Let $K(\cdot)$ be a kernel of Calderón–Zygmund type on $\mathbf{R}^n$ given by

$$K(y) = \frac{\Omega(y)h(|y|)}{|y|^n},$$

where h is a suitable function defined on $\mathbf{R}_+ := (0, \infty)$ and Ω is homogeneous of degree zero, with $\Omega \in L^1(\mathbf{S}^{n-1})$ and

$$\int_{\mathbf{S}^{n-1}} \Omega(u)\,d\sigma(u) = 0. \tag{1.1}$$

Suppose that P is a real polynomial on $\mathbf{R}$ of degree N and satisfies $P(0) = 0$. For a suitable function φ defined on $\mathbf{R}_+$, we consider that the singular integral operator $T_{h,\Omega,P,\varphi}$ along the "polynomial compound curve" $P(\varphi(|y|))y'$ on $\mathbf{R}^n$ is defined by

$$T_{h,\Omega,P,\varphi}f(x) := \text{p.v.} \int_{\mathbf{R}^n} f\big(x - P\big(\varphi\big(|y|\big)\big)y'\big)K(y)\,dy, \tag{1.2}$$

where $y' = \frac{y}{|y|}$ for any nonzero point $y \in \mathbf{R}^n$.

For the sake of simplicity, we denote $T_{h,\Omega,P,\varphi} = T_{h,\Omega}$ if $P(t) \equiv t$ and $\varphi(t) \equiv t$ and $T_{h,\Omega} = T_\Omega$ if $h(t) \equiv 1$. In their fundamental work on singular integrals, Calderón and Zygmund [1] proved that T_Ω is bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$, provided that $\Omega \in L\log L(\mathbf{S}^{n-1})$. The same conclusion under the less restrictive condition that $\Omega \in H^1(\mathbf{S}^{n-1})$ was later obtained independently by Coifman and Weiss [2] and Connett [3]. It should be pointed out that the condition that $\Omega \in H^1(\mathbf{S}^{n-1})$ turns out to be the most desirable size condition for the L^p boundedness of T_Ω. For the singular integral operator with radial kernel, Fefferman [4] in 1978 first introduced the singular integral operator $T_{h,\Omega}$ and established the L^p $(1 < p < \infty)$ boundedness of $T_{h,\Omega}$, provided that $\Omega \in \mathrm{Lip}_\alpha(\mathbf{S}^{n-1})$ for some $\alpha > 0$ and $h \in L^\infty(\mathbf{R}_+)$. Later on, Namazi [5] improved Fefferman's result by assuming $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q > 1$ instead of $\Omega \in \mathrm{Lip}_\alpha(\mathbf{S}^{n-1})$. In 1986, Duoandikoetxea and Rubio de Francia [6] used the Littlewood–Paley theory to improve the above results to the case $\Omega \in L^q(\mathbf{S}^{n-1})$ for any $q > 1$ and $h \in \Delta_2(\mathbf{R}_+)$. Here $\Delta_\gamma(\mathbf{R}_+)$ $(\gamma > 0)$ is the set of all measurable functions $h : \mathbf{R}_+ \to \mathbf{R}$ satisfying

$$\|h\|_{\Delta_\gamma(\mathbf{R}_+)} := \sup_{R>0}\left(R^{-1}\int_0^R \left|h(t)\right|^\gamma \, dt\right)^{1/\gamma} < \infty.$$

It is clear that

$$L^\infty(\mathbf{R}_+) = \Delta_\infty(\mathbf{R}_+) \subsetneq \Delta_{\gamma_2}(\mathbf{R}_+) \subsetneq \Delta_{\gamma_1}(\mathbf{R}_+), \quad \forall 1 \le \gamma_1 < \gamma_2 < \infty \tag{1.3}$$

and

$$\mathrm{Lip}_\alpha\bigl(\mathbf{S}^{n-1}\bigr) \subsetneq L^q\bigl(\mathbf{S}^{n-1}\bigr) \subsetneq L\log L\bigl(\mathbf{S}^{n-1}\bigr) \subsetneq H^1\bigl(\mathbf{S}^{n-1}\bigr) \subsetneq L^1\bigl(\mathbf{S}^{n-1}\bigr), \quad \forall \alpha > 0 \text{ and } q > 1.$$

In both T_Ω and $T_{h,\Omega}$, the singularity is along the diagonal $\{x = y\}$. However, many problems in analysis have led one to consider singular integral operators with singularity along more general sets. One of the principal motivations for the study of such operators is the requirements of several complex variables and large classes of "subelliptic" equations. We refer the readers to Stein's survey articles [7, 8] for more background information. During the last several years the L^p mapping properties for singular integral operators with singularity along various sets and with rough kernel in $H^1(\mathbf{S}^{n-1})$ have been actively studied by many authors. For example, see [9] for polynomial mappings, [10] for real-analytic submanifolds, [11, 12] for homogeneous mappings, [13] for polynomial curves. For further results on the singular integral operators with singularity along the above curves or surfaces, we refer the readers to consult [14–17], among others. Particularly, Fan and Pan [9] established the L^p boundedness of the singular integral operator along polynomial mappings for p with $|1/p - 1/2| < \max\{1/2, 1/\gamma'\}$ under the conditions that $\Omega \in H^1(\mathbf{S}^{n-1})$ and $h \in \Delta_\gamma(\mathbf{R}_+)$ for some $\gamma > 1$. In this paper we focus on the L^p-boundedness of the singular integral operator along the polynomial compound curves with rough kernels $\Omega \in H^1(\mathbf{S}^{n-1})$ and $h \in \Delta_\gamma(\mathbf{R}_+)$ for some $\gamma > 1$. Recently, Fan and Pan [13] proved the following result.

Theorem A ([13]) *Let $\varphi(t) \equiv t$. If $\Omega \in H^1(\mathbf{S}^{n-1})$ satisfies (1.1) and $h \in L^\infty(\mathbf{R}_+)$, then the operator $T_{h,\Omega,P,\varphi}$ is bounded on $L^2(\mathbf{R}^n)$. More precisely, we have*

$$\|T_{h,\Omega,P,\varphi}f\|_{L^2(\mathbf{R}^n)} \le C\|h\|_{L^\infty(\mathbf{R}_+)}\|\Omega\|_{H^1(\mathbf{S}^{n-1})}\|f\|_{L^2(\mathbf{R}^n)}.$$

Here $C > 0$ is independent of h, Ω, f and the coefficients of P.

Before stating our main results, let us introduce two classes of functions.

Definition 1.1 ($\mathcal{G}$ class) Let $\mathcal{G}$ be the set of all nonnegative (or non-positive) and monotonic $\mathcal{C}^1(\mathbf{R}_+)$ functions φ such that $\Upsilon_\varphi(t) := \frac{\varphi(t)}{t\varphi'(t)}$ with $|\Upsilon_\varphi(t)| \le C_\varphi$, where C_φ is a positive constant that depends only on φ.

Definition 1.2 ($\mathfrak{F}$ class) We denote by $\mathfrak{F}$ the set of all functions ϕ satisfying one of the following conditions:

(i) φ is a positive increasing $\mathcal{C}^1(\mathbf{R}_+)$ function such that $t\varphi'(t) \ge C_\varphi\varphi(t)$ and $\varphi(2t) \le c_\varphi\varphi(t)$ for all $t > 0$, where C_φ and c_φ are independent of t.

(ii) φ is a positive decreasing $\mathcal{C}^1(\mathbf{R}_+)$ function such that $t\varphi'(t) \le -C_\varphi\varphi(t)$ and $\varphi(t) \le c_\varphi\varphi(2t)$ for all $t > 0$, where C_φ and c_φ are independent of t.

Remark 1.1 It is clear that $\mathfrak{F} \subsetneq \mathcal{G}$. There are some model examples for the class $\mathfrak{F}$, such as t^α ($\alpha \neq 0$), $t^\alpha(\ln(1+t))^\beta$ ($\alpha, \beta > 0$), $t\ln\ln(\mathbf{e}+t)$, $t^{-1}\ln(1+t^{-1})$, real-valued polynomials P on $\mathbf{R}$ with positive coefficients and $P(0) = 0$, and so on. Note that the following facts are valid (see [18]): if $\varphi \in \mathcal{G}$, then

(a) $\lim_{t\to 0}\varphi(t) = 0$ and $\lim_{t\to\infty}|\varphi(t)| = \infty$ if φ is nonnegative and increasing, or non-positive and decreasing;

(b) $\lim_{t\to 0}|\varphi(t)| = \infty$ and $\lim_{t\to\infty}\varphi(t) = 0$ if φ is nonnegative and decreasing, or non-positive and increasing.

In this paper we shall establish the following.

Theorem 1.1 *Let $T_{h,\Omega,P,\varphi}$ be the singular integral operator defined by* (1.2) *and $\varphi \in \mathcal{G}$. If $\Omega \in H^1(\mathbf{S}^{n-1})$ satisfies* (1.1) *and $h \in \Delta_\gamma(\mathbf{R}_+)$ for some $\gamma \in (1,\infty]$, then the operator $T_{h,\Omega,P,\varphi}$ is bounded on $L^2(\mathbf{R}^n)$. More precisely, we have*

$$\|T_{h,\Omega,P,\varphi}f\|_{L^2(\mathbf{R}^n)} \le C\|h\|_{\Delta_\gamma(\mathbf{R}_+)}\|\Omega\|_{H^1(\mathbf{S}^{n-1})}\|f\|_{L^2(\mathbf{R}^n)},$$

where $C > 0$ is independent of h, γ, Ω, f and the coefficients of P, but depends on φ and $\deg(P)$.

Remark 1.2 By (1.3), Theorem 1.1 generalizes and improves Theorem A.

The second type of our operators we consider is the parametric Marcinkiewicz integral operator along polynomial compound curves. More precisely, let h, Ω, P, φ be given as in (1.2). For a complex number $\rho = \tau + i\vartheta$ ($\tau, \vartheta \in \mathbf{R}$ with $\tau > 0$), we consider the parametric Marcinkiewicz integral operator $\mathfrak{M}_{h,\Omega,P,\varphi,\rho}$ along "polynomial compound curve" $P(\varphi(|y|))y'$ on $\mathbf{R}^n$ by

$$\mathfrak{M}_{h,\Omega,P,\varphi,\rho}f(x) = \left(\int_0^\infty\left|\frac{1}{t^\rho}\int_{|y|\le t} f\big(x - P\big(\varphi\big(|y|\big)\big)y'\big)\frac{h(|y|)\Omega(y')}{|y|^{n-\rho}}\,dy\right|^2\frac{dt}{t}\right)^{1/2}. \tag{1.4}$$

If $\rho = 1$, $P(t) \equiv t$, and $\varphi(t) \equiv t$, $\mathfrak{M}_{h,\Omega,P,\varphi,\rho}$ is just the classical Marcinkiewicz integral operator, which is denoted by $\mathfrak{M}_{h,\Omega}$. The L^p mapping properties of $\mathfrak{M}_{h,\Omega}$ and other extensions have been studied by many authors extensively (see [19–28] for example). Particularly, Ding et al. [21] proved that the parametric Marcinkiewicz integral operator along polynomial mappings is of type (p,p) for $1 < p < \infty$ if $\Omega \in H^1(\mathbf{S}^{n-1})$ and $h \in L^\infty(\mathbf{R}_+)$. In this paper we focus on the parametric Marcinkiewicz integral operator along polynomial compound curves. More precisely, we shall establish the following result.

Theorem 1.2 *Let $\mathfrak{M}_{h,\Omega,P,\varphi,\rho}$ be the Marcinkiewicz integral operator defined by (1.4) and $\varphi \in \mathfrak{F}$. If $\Omega \in H^1(\mathbf{S}^{n-1})$ satisfies (1.1) and $h \in \Delta_\gamma(\mathbf{R}_+)$ for some $\gamma \in (1,\infty]$, then the operator $\mathfrak{M}_{h,\Omega,P,\varphi,\rho}$ is bounded on $L^2(\mathbf{R}^n)$. More precisely, we have*

$$\|\mathfrak{M}_{h,\Omega,P,\varphi,\rho}f\|_{L^2(\mathbf{R}^n)} \le C\frac{\gamma}{\gamma-1}\|h\|_{\Delta_\gamma(\mathbf{R}_+)}\|\Omega\|_{H^1(\mathbf{S}^{n-1})}\|f\|_{L^2(\mathbf{R}^n)},$$

where $C > 0$ is independent of h, γ, Ω, f and the coefficients of P, but depends on φ, ρ and $\deg(P)$.

Remark 1.3

(i) It was shown in [22] that the operator $\mathfrak{M}_{h,\Omega,P,\varphi,\rho}$ with $\rho > 0$ and $P(t) \equiv t$ is bounded on $L^p(\mathbf{R}^n)$ for p with $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$ under the same conditions of Theorem 1.2 (see also [27] for more generalization results).

(ii) Theorem 1.2 is new even in the special case $\rho = 1$, $h(t) \equiv 1$, and $\varphi(t) \equiv t$.

In 2009, Sato [29] introduced a class of functions $\mathcal{N}_\alpha(\mathbf{R}_+)$, which may be the most desirable size condition on the radial kernel of rough singular integrals so far. Here $\mathcal{N}_\alpha(\mathbf{R}_+)$ for $\alpha > 0$ is the set of all measurable functions $h : \mathbf{R}_+ \to \mathbf{R}$ satisfying

$$N_\alpha(h) = \sum_{m=1} m^\alpha 2^m \sup_{k\in\mathbb{Z}} 2^{-k}\bigl|E(k,m)\bigr| < \infty,$$

where $E(k,1) = \{t \in (2^k, 2^{k+1}]; |h(t)| \le 2\}$ and

$$E(k,m) = \bigl\{t \in (2^k, 2^{k+1}]; 2^{m-1} < \bigl|h(t)\bigr| \le 2^m\bigr\} \quad \text{for } m \ge 2.$$

It is easy to check that

$$\Delta_\gamma(\mathbf{R}_+) \subsetneq \mathcal{N}_\alpha(\mathbf{R}_+), \quad \forall \gamma > 1 \text{ and } \alpha > 0.$$

Applying Theorems 1.1–1.2 and the extrapolation arguments following from [29], we have the following result.

Corollary 1.1 *Let $T_{h,\Omega,P,\varphi}$ and $\mathfrak{M}_{h,\Omega,P,\varphi,\rho}$ be defined by (1.2) and (1.4), respectively. Let Ω satisfy (1.1) and $\Omega \in H^1(\mathbf{S}^{n-1})$. Then*

(i) *If $h \in \mathcal{N}_\alpha(\mathbf{R}_+)$ for some $\alpha > 0$ and $\varphi \in \mathcal{G}$, then the operator $T_{h,\Omega,P,\varphi}$ is bounded on $L^2(\mathbf{R}^n)$. More precisely, we have*

$$\|T_{h,\Omega,P,\varphi}f\|_{L^2(\mathbf{R}^n)} \le C\bigl(1 + N_\alpha(h)\bigr)\|\Omega\|_{H^1(\mathbf{S}^{n-1})}\|f\|_{L^2(\mathbf{R}^n)},$$

where $C > 0$ is independent of h, α, Ω, f and the coefficients of P, but depends on φ and $\deg(P)$.

(ii) *If $h \in \mathcal{N}_1(\mathbf{R}_+)$ and $\varphi \in \mathfrak{F}$, then the operator $\mathfrak{M}_{h,\Omega,P,\varphi,\rho}$ is bounded on $L^2(\mathbf{R}^n)$. More precisely, we have*

$$\|\mathfrak{M}_{h,\Omega,P,\varphi,\rho}f\|_{L^2(\mathbf{R}^n)} \leq C\big(1+N_1(h)\big)\|\Omega\|_{H^1(\mathbf{S}^{n-1})}\|f\|_{L^2(\mathbf{R}^n)},$$

where $C>0$ is independent of h, Ω, f and the coefficients of P, but depends on φ, ρ and $\deg(P)$.

The third type of our operators is the maximal functions related to the singular integrals and Marcinkiewicz integrals along polynomial compound curves. More precisely, let $T_{h,\Omega,P,\varphi}$ and $\mathfrak{M}_{h,\Omega,P,\varphi,\varrho}$ be defined as in (1.2) and (1.4), respectively. We define the maximal operators $\mathcal{S}_{\Omega,P,\varphi}$ and $\mathcal{M}_{\Omega,P,\varphi,\rho}$ along the "polynomial compound curve" $P(\varphi(|y|))y'$ on $\mathbf{R}^n$ by

$$\mathcal{S}_{\Omega,P,\varphi}f(x) = \sup_{h\in\mathcal{K}_2}\big|T_{h,\Omega,P,\varphi}f(x)\big|, \tag{1.5}$$

$$\mathcal{M}_{\Omega,P,\varphi,\rho}f(x) = \sup_{h\in\mathcal{K}_2}\big|\mathfrak{M}_{h,\Omega,P,\varphi,\rho}f(x)\big|, \tag{1.6}$$

where $\mathcal{K}_2$ is the set of all measurable functions $h:\mathbf{R}_+\to\mathbf{R}$ with $\|h\|_{L^2(\mathbf{R}_+,r^{-1}\,dr)}\leq 1$. Here $L^\gamma(\mathbf{R}_+,r^{-1}\,dr)$ ($\gamma>0$) is the set of all measurable functions $h:\mathbf{R}_+\to\mathbf{R}$ that satisfies

$$\|h\|_{L^\gamma(\mathbf{R}_+,r^{-1}\,dr)} := \left(\int_0^\infty \big|h(r)\big|^\gamma r^{-1}\,dr\right)^{1/\gamma} < \infty.$$

Clearly, $L^\gamma(\mathbf{R}_+,r^{-1}\,dr) \subsetneq \Delta_\gamma(\mathbf{R}_+)$ for $\gamma>0$.

The rest of the main results can be formulated as follows.

Theorem 1.3 *Let $\mathcal{S}_{\Omega,P,\varphi}$ and $\mathcal{M}_{\Omega,P,\varphi,\rho}$ be the maximal operators defined by* (1.5) *and* (1.6), *respectively. If $\Omega\in H^1(\mathbf{S}^{n-1})$ satisfies* (1.1) *and $\varphi\in\mathcal{G}$, then both the operators $\mathcal{S}_{\Omega,P,\varphi}$ and $\mathcal{M}_{\Omega,P,\varphi,\rho}$ are bounded on $L^2(\mathbf{R}^n)$. More precisely, we have*

$$\|\mathcal{S}_{\Omega,P,\varphi}f\|_{L^2(\mathbf{R}^n)} \leq C\|\Omega\|_{H^1(\mathbf{S}^{n-1})}\|f\|_{L^2(\mathbf{R}^n)},$$
$$\|\mathcal{M}_{\Omega,P,\varphi,\rho}f\|_{L^2(\mathbf{R}^n)} \leq C\|\Omega\|_{H^1(\mathbf{S}^{n-1})}\|f\|_{L^2(\mathbf{R}^n)},$$

where $C>0$ is independent of Ω, f and the coefficients of P, but depends on φ and $\deg(P)$.

Remark 1.4 The maximal operator related to singular integrals, which is denoted by $\mathcal{S}_\Omega$ and corresponds to the special case of $\mathcal{S}_{\Omega,P,\varphi}$ with $P(t)=\varphi(t)=t$, was first introduced by Chen and Lin [30]. Chen and Lin proved that if $\Omega\in\mathcal{C}(\mathbf{S}^{n-1})$, then $\mathcal{S}_\Omega$ is of type (p,p) for any $p>2n/(2n-1)$ and the range of p is best possible. Subsequently, the L^p mapping properties of $\mathcal{S}_\Omega$ have been discussed extensively by many authors. Particularly, Xu et al. [31] established the $L^p(\mathbf{R}^n)$ bounds for $\mathcal{S}_\Omega$ with $2\leq p<\infty$, provided that $\Omega\in H^1(\mathbf{S}^{n-1})$ satisfying (1.1). It should be pointed out that Theorem 1.3 is new even in the special case $\varphi(t)\equiv t$.

The rest of this paper is organized as follows. After recalling some preliminary notations and lemmas in Sect. 2, we prove the main results in Sect. 3. Finally, we present certain

Hardy type inequalities related to the parametric Marcinkiewicz integral operators and maximal operators related to singular integrals along polynomial compound curves in Sect. 4. We would like to remark that our main results and proofs are inspired by the work in [13], but our main results and proofs are more delicate and complex than those of [13]. Some ideas in our proofs are taken from [18, 29, 32, 33]. Throughout the paper, we denote by p' the conjugate index of p, which satisfies $1/p + 1/p' = 1$. The letter C or c, sometimes with certain parameters, will stand for positive constants that are not necessarily the same ones at each occurrence, but are independent of the essential variables. In what follows, we denote $\mathbf{e} = \lim_{x\to\infty}(1+\frac{1}{x})^x$.

2 Preliminary definitions and lemmas

Recall that the Hardy space $H^1(\mathbf{S}^{n-1})$ is the set of all functions $\Omega \in L^1(\mathbf{S}^{n-1})$ satisfying the condition

$$\|\Omega\|_{H^1(\mathbf{S}^{n-1})} := \left\| \sup_{0\le r<1} \left| \int_{\mathbf{S}^{n-1}} \Omega(\theta) P_{r(\cdot)}(\theta)\,d\sigma(\theta) \right| \right\|_{L^1(\mathbf{S}^{n-1})} < \infty,$$

where $P_{rw}(\theta) = \frac{1-r^2}{|rw-\theta|^n}$ for $0 \le r < 1$ and $\theta, w \in \mathbf{S}^{n-1}$ denotes the Poisson kernel on $\mathbf{S}^{n-1}$.

Definition 2.1 ($H^1(\mathbf{S}^{n-1})$ atoms) A function $a : \mathbf{S}^{n-1} \to \mathbf{C}$ is a $(1,\infty)$ atom if there exist $\vartheta \in \mathbf{S}^{n-1}$ and $\varrho \in (0,2]$ such that

$$\operatorname{supp}(a) \subset \mathbf{S}^{n-1} \cap B(\vartheta,\varrho), \quad \text{where } B(\vartheta,\varrho) = \bigl\{y \in \mathbf{R}^n : |y-\vartheta| < \varrho\bigr\}; \tag{2.1}$$

$$\|a\|_{L^\infty(\mathbf{S}^{n-1})} \le \varrho^{-n+1}; \tag{2.2}$$

$$\int_{\mathbf{S}^{n-1}} a(y)\,d\sigma(y) = 0. \tag{2.3}$$

An important property of $H^1(\mathbf{S}^{n-1})$ is the atomic decomposition, which is listed as follows:

Lemma 2.1 ([34, 35]) *If $\Omega \in H^1(\mathbf{S}^{n-1})$ satisfies the cancelation condition (1.1), then there exist a sequence of complex numbers $\{c_j\}$ and a sequence of $(1,\infty)$ atoms $\{\Omega_j\}$ such that $\Omega = \sum_j c_j \Omega_j$ and $\|\Omega\|_{H^1(\mathbf{S}^{n-1})} \approx \sum_j |c_j|$.*

The following results are known (see [36]).

Lemma 2.2 ([36]) *Suppose that $n \ge 3$ and $b(\cdot)$ is a $(1,\infty)$ atom on $\mathbf{S}^{n-1}$ supported in $\mathbf{S}^{n-1} \cap B(\zeta,\varrho)$, where $\zeta \in \mathbf{S}^{n-1}$ and $\varrho \in (0,2]$. Let*

$$F_b(s) = \bigl(1-s^2\bigr)^{(n-3)/2} \chi_{(-1,1)}(s) \int_{\mathbf{S}^{n-2}} b\bigl(s, \bigl(1-s^2\bigr)^{1/2}\tilde{y}\bigr)\,d\sigma(\tilde{y}).$$

Then there exists a constant C, independent of b, such that

$$\operatorname{supp}(F_b) \subset \bigl(\zeta_1 - 2r(\zeta), \zeta_1 + 2r(\zeta)\bigr); \tag{2.4}$$

$$\|F_b\|_{L^\infty(\mathbf{R})} \le \frac{C}{r(\zeta)}; \tag{2.5}$$

$$\int_{\mathbf{R}} F_b(s)\,ds = 0, \tag{2.6}$$

where $\zeta = (\zeta_1, \ldots, \zeta_n)$, $r(\zeta) = |L_\varrho(\zeta)|$, *and* $L_\varrho(\zeta) = (\varrho^2\zeta_1, \varrho\zeta_2, \ldots, \varrho\zeta_n)$.

Lemma 2.3 ([36]) *Suppose that* $n = 2$ *and* $b(\cdot)$ *is a* $(1,\infty)$ *atom on* $\mathbf{S}^1$ *supported in* $\mathbf{S}^1 \cap B(\zeta, \varrho)$, *where* $\zeta \in \mathbf{S}^1$ *and* $\varrho \in (0,2]$. *Let*

$$F_b(s) = \left(1-s^2\right)^{-1/2}\chi_{(-1,1)}(s)\left(b\left(s,\left(1-s^2\right)^{1/2}\right) + b\left(s,-\left(1-s^2\right)^{1/2}\right)\right).$$

Then $F_b(\cdot)$ *satisfies* (2.4), (2.6) *and*

$$\|F_b\|_{L^q(\mathbf{R})} \le C\left|L_\varrho(\zeta)\right|^{-1+1/q}$$

for some $q \in (1,2)$, *where* $\zeta = (\zeta_1, \zeta_2)$, $r(\zeta) = |L_\varrho(\zeta)|$, *and* $L_\varrho(\zeta) = (\varrho^2\zeta_1, \varrho\zeta_2)$.

The following lemmas will play key roles in the proof of Theorem 1.1.

Lemma 2.4 ([18]) *Let* $\varphi \in \mathcal{G}$ *and* $h \in \Delta_\gamma(\mathbf{R}_+)$ *for some* $\gamma > 1$, *then*

$$\left\|h\left(\varphi^{-1}\right)\Upsilon_\varphi\left(\varphi^{-1}\right)\right\|_{\Delta_\gamma(\mathbf{R}_+)} \le C\|h\|_{\Delta_\gamma(\mathbf{R}_+)},$$

where the constant $C > 0$ *depends only on* φ.

Lemma 2.5 *Let* $\varphi \in \mathcal{G}$ *and* $\Upsilon_\varphi(t) = \frac{\varphi(t)}{t\varphi'(t)}$. *Then*

(i) *if* φ *is nonnegative and increasing,* $T_{h,\Omega,P,\varphi}f = T_{h(\varphi^{-1})\Upsilon_\varphi(\varphi^{-1}),\Omega,P}f$;

(ii) *if* φ *is nonnegative and decreasing,* $T_{h,\Omega,P,\varphi}f = -T_{h(\varphi^{-1})\Upsilon_\varphi(\varphi^{-1}),\Omega,P}f$;

(iii) *if* φ *is non-positive and decreasing,* $T_{h,\Omega,P,\varphi}f = T_{h(\varphi^{-1})\Upsilon_\varphi(\varphi^{-1}),\tilde{\Omega},P}f$;

(iv) *if* φ *is non-positive and increasing,* $T_{h,\Omega,P,\varphi}f = -T_{h(\varphi^{-1})\Upsilon_\varphi(\varphi^{-1}),\tilde{\Omega},P}f$,

where $\tilde{\Omega}(y) = \Omega(-y)$.

Lemma 2.5 can be proved by similar arguments as in the proof of [18, Lemma 2.3], we omit the details.

3 Proofs of the main results

In this section we shall prove Theorems 1.1–1.3. In what follows, we let $\deg(P) = N$ and $P(t) = \sum_{k=1}^N a_k t^k$ and assume that $\Omega \in H^1(\mathbf{S}^{n-1})$ satisfies (1.1).

Proof of Theorem 1.1 By Lemmas 2.4 and 2.5, to prove Theorem 1.1, it suffices to show that

$$\|T_{h,\Omega,P}f\|_{L^2(\mathbf{R}^n)} \le C\|h\|_{\Delta_\gamma(\mathbf{R}_+)}\|\Omega\|_{H^1(\mathbf{S}^{n-1})}\|f\|_{L^2(\mathbf{R}^n)}, \tag{3.1}$$

where $C > 0$ is independent of h, Ω, f and the coefficients of P, but depends on $\deg(P)$. It is clear that $T_{h,\Omega,P}f(x) = K_{h,\Omega,P} * f(x)$, where the function $K_{h,\Omega,P}$ is defined by

$$\widehat{K_{h,\Omega,P}}(\xi) = \int_{\mathbf{R}^n} \mathbf{e}^{-2\pi i P(|y|)y'\cdot\xi}\,\frac{h(|y|)\Omega(y')}{|y|^n}\,dy.$$

By Plancherel's theorem, (3.1) reduces to the following:

$$\|\widehat{K_{h,\Omega,P}}\|_{L^\infty(\mathbf{R}^n)} \le C\|h\|_{\Delta_\gamma(\mathbf{R}_+)}\|\Omega\|_{H^1(\mathbf{S}^{n-1})}, \tag{3.2}$$

where $C > 0$ is independent of h, Ω and the coefficients of P. Invoking Lemma 2.1, we can write $\Omega = \sum_j \lambda_j a_j$ satisfying $\sum_j |\lambda_j| \le C\|\Omega\|_{H^1(\mathbf{S}^{n-1})}$, where each a_j is a $(1,\infty)$ atom and each λ_j is a complex number. Therefore, to prove (3.2), it is enough to prove that

$$\|\widehat{K_{h,\Omega,P}}\|_{L^\infty(\mathbf{R}^n)} \le C\|h\|_{\Delta_\gamma(\mathbf{R}_+)}, \tag{3.3}$$

where Ω is a $(1,\infty)$ atom and $C > 0$ is independent of h, Ω and the coefficients of P.

We now prove (3.3). Let Ω be a $(1,\infty)$ atom satisfying (2.1)–(2.3). Without loss of generality, we may assume that $\vartheta = (1,0,\dots,0)$. We only prove the case $n \ge 3$, since the proof for $n = 2$ is essentially the same (using Lemma 2.3 instead of Lemma 2.2). Fix $\xi \ne (0,0,\dots,0)$ and write $\xi' = \xi/|\xi| = (\xi_1',\dots,\xi_n')$. We can choose a rotation $\mathcal{O}$ such that $\mathcal{O}(\xi') = \vartheta$. By the change of variables, we write

$$\widehat{K_{h,\Omega,P}}(\xi) = \int_0^\infty h(t)\int_{\mathbf{S}^{n-1}} \mathbf{e}^{-2\pi i P(t)|\xi|y'\cdot\iota}\,\Omega\big(\mathcal{O}^{-1}y'\big)\,d\sigma\big(y'\big)\frac{dt}{t}.$$

Let $b(y') = \Omega(\mathcal{O}^{-1}y')$. It is easy to see that b is a regular $(1,\infty)$ atom satisfying (2.2)–(2.3) and $\operatorname{supp}(b) \subset B(\xi',\varrho)\cap\mathbf{S}^{n-1}$. By the change of variables, we have

$$\widehat{K_{h,\Omega,P}}(\xi) = \int_0^\infty h(t)\int_{\mathbf{R}} F_b(s)\mathbf{e}^{-2\pi i P(t)|\xi|s}\,ds\frac{dt}{t},$$

where F_b is the function defined in Lemma 2.2. We know by Lemma 2.2 that $\operatorname{supp}(F_b) \subset (\xi_1' - 2r(\xi'), \xi_1' + 2r(\xi'))$ and F_b satisfies (2.5)–(2.6), where $r(\xi') = |(\varrho^2\xi_1', \varrho\xi_2', \dots, \varrho\xi_n')|$. We set $N_b(s) = r(\xi')F_b(r(\xi')s + \xi_1')$. One can easily check that $\operatorname{supp}(N_b) \subset (-2,2)$ and $\|N_b\|_{L^\infty(\mathbf{R})} \le C$ and $\int_{\mathbf{R}} N_b(s)\,ds = 0$. By the change of variables again,

$$\widehat{K_{h,\Omega,P}}(\xi) = \int_0^\infty h(t)\int_{\mathbf{R}} N_b(s)\mathbf{e}^{-2\pi i P(t)|\xi|r(\xi')s}\,ds\frac{dt}{t}.$$

For convenience, we set

$$b_k = 2\pi a_k|\xi|r\big(\xi'\big), \qquad |\beta_\kappa|^{1/\kappa} = \max_{1\le k\le N}|b_k|^{1/k}, \quad \text{and} \quad \beta = |\beta_\kappa|^{-1/\kappa}.$$

We can write

$$\begin{aligned}\widehat{K_{h,\Omega,P}}(\xi) &= \int_0^\beta h(t)\int_{\mathbf{R}} N_b(s)\mathbf{e}^{-2\pi i P(t)|\xi|r(\xi')s}\,ds\frac{dt}{t}\\ &\quad + \int_\beta^\infty h(t)\int_{\mathbf{R}} N_b(s)\mathbf{e}^{-2\pi i P(t)|\xi|r(\xi')s}\,ds\frac{dt}{t}\\ &=: I_1 + I_2.\end{aligned} \tag{3.4}$$

For I_1, let us choose an integer K_0 such that $2^{K_0} \le \beta < 2^{K_0+1}$. By the cancelation condition of N_b and Hölder's inequality, we have

$$\begin{aligned}
|I_1| &= \left|\int_0^\beta h(t)\int_{\mathbf{R}} N_b(s)\big(\mathbf{e}^{-2\pi iP(t)|\xi|r(\xi')s}-1\big)\,ds\frac{dt}{t}\right| \\
&\le \sum_{j=-\infty}^{K_0}\int_{2^j}^{2^{j+1}}\big|h(t)\big|\int_{\mathbf{R}}\big|N_b(s)\big(\mathbf{e}^{-2\pi iP(t)|\xi|r(\xi')s}-1\big)\big|\,ds\frac{dt}{t} \\
&\le C\|h\|_{\Delta_\gamma(\mathbf{R}_+)}\sum_{j=-\infty}^{K_0}\left(\int_{2^j}^{2^{j+1}}\left(\sum_{k=1}^{N}|b_k|t^k\right)^{\gamma'}\frac{dt}{t}\right)^{1/\gamma'} \\
&\le C\|h\|_{\Delta_\gamma(\mathbf{R}_+)}\sum_{j=-\infty}^{K_0}\sum_{k=1}^{N}|b_k|\left(\int_{2^j}^{2^{j+1}}t^{k\gamma'-1}\,dt\right)^{1/\gamma'} \\
&= C\|h\|_{\Delta_\gamma(\mathbf{R}_+)}\sum_{j=-\infty}^{K_0}\sum_{k=1}^{N}|b_k|\big(k\gamma'\big)^{-1/\gamma'}\big(2^{k\gamma'}-1\big)^{1/\gamma'}2^{kj} \\
&\le C\|h\|_{\Delta_\gamma(\mathbf{R}_+)},
\end{aligned} \tag{3.5}$$

where in the last inequality of (3.5) we have used the fact that $\alpha^\alpha \le 1$ for all $\alpha \in (0,1]$ and $\max_{1\le k\le N}|b_k|2^{K_0k} \le 1$. Here $C>0$ is independent of f, ζ and the coefficients of P.

For I_2, let $\Phi \in \mathcal{C}_0^\infty(\mathbf{R})$ such that $\Phi(t)\equiv 1$ if $|t|\le 1$ and $\Phi(t)\equiv 0$ if $|t|\ge 2$. For any $j\in\mathbf{Z}$, we set $R_j=[2^j,2^{j+1})$ and define the operator T_j by

$$T_jf(t)=\chi_{R_j}(t)\int_{\mathbf{R}}\Phi(s)f(s)\mathbf{e}^{-2\pi iP(t)|\xi|r(\xi')s}\,ds.$$

From the estimate on page 60 in [32], there exists a large integer $\Lambda>0$ independent of j such that

$$\|T_jf\|_{L^2(\mathbf{R})}\le C2^{j/2}|\beta_k|^{-1/(2\Lambda)}2^{-j\kappa/(2\Lambda)}\|f\|_{L^2(\mathbf{R})}, \tag{3.6}$$

where $C>0$ is independent of f, ξ and the coefficients of P. By (3.6) and Hölder's inequality,

$$\begin{aligned}
|I_2| &\le \sum_{j=K_0}^{\infty}\int_{2^j}^{2^{j+1}}\big|h(t)\big|\left|\int_{\mathbf{R}}N_b(s)\mathbf{e}^{-2\pi iP(t)|\xi|r(\xi')s}\,ds\right|\frac{dt}{t} \\
&\le 2\|h\|_{\Delta_\gamma(\mathbf{R}_+)}\sum_{j=K_0}^{\infty}\left(\int_{2^j}^{2^{j+1}}\left|\int_{\mathbf{R}}N_b(s)\mathbf{e}^{-2\pi iP(t)|\xi|r(\xi')s}\,ds\right|^{\gamma'}\frac{dt}{t}\right)^{1/\gamma'} \\
&\le C\|h\|_{\Delta_\gamma(\mathbf{R}_+)}\sum_{j=K_0}^{\infty}\left(\int_{2^j}^{2^{j+1}}\left|\int_{\mathbf{R}}N_b(s)\mathbf{e}^{-2\pi iP(t)|\xi|r(\xi')s}\,ds\right|^{2}\frac{dt}{t}\right)^{1/\tilde{\gamma}} \\
&\le C\|h\|_{\Delta_\gamma(\mathbf{R}_+)}\sum_{j=K_0}^{\infty}2^{-j/\tilde{\gamma}}\left(\int_{\mathbf{R}}\left|\chi_{R_j}(t)\int_{\mathbf{R}}N_b(s)\mathbf{e}^{-2\pi iP(t)|\xi|r(\xi')s}\,ds\right|^{2}dt\right)^{1/\tilde{\gamma}} \\
&\le C\|h\|_{\Delta_\gamma(\mathbf{R}_+)}\sum_{j=K_0}^{\infty}2^{-j/\tilde{\gamma}}\big(2^j|\beta_\kappa|^{-1/\Lambda}2^{-j\kappa/\Lambda}\|N_b\|_{L^2(\mathbf{R})}^2\big)^{1/\tilde{\gamma}} \\
&\le C\|h\|_{\Delta_\gamma(\mathbf{R}_+)},
\end{aligned} \tag{3.7}$$

where in the last inequality of (3.7) we have used the fact that $2^{K_0\kappa} \geq 2^{-\kappa}\beta_\kappa^{-1}$. Here $\tilde{\gamma} = \max\{2, \gamma'\}$ and the constant $C > 0$ is independent of h, b, ξ and the coefficients of P. We get from (3.4)–(3.5) and (3.7) that

$$\big|\widehat{K_{h,\Omega,P}}(\xi)\big| \leq C\|h\|_{\Delta_\gamma(\mathbf{R}_+)},$$

where $C > 0$ is independent of h, Ω, ξ and the coefficients of P. This yields (3.3) and completes the proof of Theorem 1.1. □

Proof of Theorem 1.2 Let h, Ω, φ be given as in Theorem 1.2. We only prove Theorem 1.2 for the case $\varphi \in \mathfrak{F}$ satisfying condition (a), and another case is discussed similarly. For $t > 0$, we define the measure $\sigma_{h,\Omega,P,\varphi,t}$ by

$$\widehat{\sigma_{h,\Omega,P,\varphi,t}}(x) = \frac{1}{t^\rho}\int_{t/2<|y|\leq t} \mathbf{e}^{-2\pi iP(\varphi(|y|))x\cdot y'}\frac{h(|y|)\Omega(y')}{|y|^{n-\rho}}\,dy. \tag{3.8}$$

By Minkowski's inequality and the change of variables, one can easily verify that

$$\mathfrak{M}_{h,\Omega,P,\varphi,\rho}f(x) \leq \frac{1}{1-2^{-\tau}}\left(\int_0^\infty \big|\sigma_{h,\Omega,P,\varphi,t} * f(x)\big|^2\frac{dt}{t}\right)^{1/2}. \tag{3.9}$$

By Plancherel's theorem and Fubini's theorem, to prove Theorem 1.2, we only need to show that

$$\left\|\left(\int_0^\infty \big|\widehat{\sigma_{h,\Omega,P,\varphi,t}}(\cdot)\big|^2\frac{dt}{t}\right)^{1/2}\right\|_{L^\infty(\mathbf{R}^n)} \leq C\|h\|_{\Delta_\gamma(\mathbf{R}_+)}\|\Omega\|_{H^1(\mathbf{S}^{n-1})}, \tag{3.10}$$

where $C > 0$ is independent of h, Ω and the coefficients of P, but depends on φ and N.

Invoking Lemma 2.1, (3.10) reduces to the following:

$$\left\|\left(\int_0^\infty \big|\widehat{\sigma_{h,\Omega,P,\varphi,t}}(\cdot)\big|^2\frac{dt}{t}\right)^{1/2}\right\|_{L^\infty(\mathbf{R}^n)} \leq C\|h\|_{\Delta_\gamma(\mathbf{R}_+)}, \tag{3.11}$$

where Ω is a $(1,\infty)$ atom and $C > 0$ is independent of h, Ω and the coefficients of P, but depends on φ and N.

Given $\xi \neq (0,0,\dots,0)$. To prove (3.11), we want to show that

$$\int_0^\infty \big|\widehat{\sigma_{h,\Omega,P,\varphi,t}}(\xi)\big|^2\frac{dt}{t} \leq C\|h\|^2_{\Delta_\gamma(\mathbf{R}_+)}, \tag{3.12}$$

where Ω is a $(1,\infty)$ atom and $C > 0$ is independent of h, Ω, ξ and the coefficients of P.

We now prove (3.12). Without loss of generality we may assume that Ω is a $(1,\infty)$ atom satisfying (2.1)–(2.3) with $\vartheta = (1,0,\dots,0)$ and only prove the case $n \geq 3$. Let b and N_b be given as in the proof of Theorem 1.1. By some change of variables, we write

$$\widehat{\sigma_{h,\Omega,P,\varphi,t}}(\xi) = \frac{1}{t^\rho}\int_{t/2}^t h(r)\int_{\mathbf{R}} N_b(s)\mathbf{e}^{-2\pi iP(\varphi(r))|\xi|r(\xi')s}\,ds\frac{dr}{r^{1-\rho}}. \tag{3.13}$$

By (3.13) and Hölder's inequality, we obtain that

$$\left|\widehat{\sigma_{h,\Omega,P,\varphi,t}}(\xi)\right| \le C\|h\|_{\Delta_\gamma(\mathbf{R}_+)}\left(\int_{t/2}^{t}\left|\int_{\mathbf{R}} N_b(s)\mathbf{e}^{-2\pi i P(\varphi(r))|\xi|r(\xi')s}\,ds\right|^{\gamma'}\frac{dr}{r}\right)^{1/\gamma'}.$$

Hence, to prove (3.12), it suffices to show that

$$\int_0^\infty\left(\int_{t/2}^{t}\left|\int_{\mathbf{R}} N_b(s)\mathbf{e}^{-2\pi i P(\varphi(r))|\xi|r(\xi')s}\,ds\right|^{\gamma'}\frac{dr}{r}\right)^{2/\gamma'}\frac{dt}{t} \le C, \tag{3.14}$$

where $C > 0$ is independent of h, Ω, ξ and the coefficients of P.

Next we shall prove (3.14). By a change of variable and the properties of φ,

$$\begin{aligned}
&\int_0^\infty\left(\int_{t/2}^{t}\left|\int_{\mathbf{R}} N_b(s)\mathbf{e}^{-2\pi i P(\varphi(r))|\xi|r(\xi')s}\,ds\right|^{\gamma'}\frac{dr}{r}\right)^{2/\gamma'}\frac{dt}{t}\\
&\quad\le C_\varphi\int_0^\infty\left(\int_{\varphi(t/2)}^{\varphi(t)}\left|\int_{\mathbf{R}} N_b(s)\mathbf{e}^{-2\pi i P(r)|\xi|r(\xi')s}\,ds\right|^{\gamma'}\frac{dr}{r}\right)^{2/\gamma'}\frac{dt}{t}.
\end{aligned}\tag{3.15}$$

Let β be given as in the proof of Theorem 1.1 and $\delta = \varphi^{-1}(\beta)$. We write

$$\begin{aligned}
&\int_0^\infty\left(\int_{\varphi(t/2)}^{\varphi(t)}\left|\int_{\mathbf{R}} N_b(s)\mathbf{e}^{-2\pi i P(r)|\xi|r(\xi')s}\,ds\right|^{\gamma'}\frac{dr}{r}\right)^{2/\gamma'}\frac{dt}{t}\\
&\quad=\int_0^\delta\left(\int_{\varphi(t/2)}^{\varphi(t)}\left|\int_{\mathbf{R}} N_b(s)\mathbf{e}^{-2\pi i P(r)|\xi|r(\xi')s}\,ds\right|^{\gamma'}\frac{dr}{r}\right)^{2/\gamma'}\frac{dt}{t}\\
&\qquad+\int_\delta^\infty\left(\int_{\varphi(t/2)}^{\varphi(t)}\left|\int_{\mathbf{R}} N_b(s)\mathbf{e}^{-2\pi i P(r)|\xi|r(\xi')s}\,ds\right|^{\gamma'}\frac{dr}{r}\right)^{2/\gamma'}\frac{dt}{t}\\
&\quad=: J_1 + J_2.
\end{aligned}\tag{3.16}$$

For J_1, by the fact that $\int_{\mathbf{R}} N_b(s)\,ds = 0$ and the change of variables, we have

$$\begin{aligned}
J_1 &= \int_0^\delta\left(\int_{\varphi(t/2)}^{\varphi(t)}\left|\int_{\mathbf{R}} N_b(s)\left(\mathbf{e}^{-2\pi i P(r)|\xi|r(\xi')s}-1\right)ds\right|^{\gamma'}\frac{dr}{r}\right)^{2/\gamma'}\frac{dt}{t}\\
&\le C\sum_{k=1}^N |b_k|^2\int_0^\delta\left(\int_{\varphi(t/2)}^{\varphi(t)} r^{k\gamma'-1}\,dr\right)^{2/\gamma'}\frac{dt}{t}\\
&\le C\sum_{k=1}^N |b_k|^2\int_0^\delta\left(\varphi(t)\right)^{2k}\frac{dt}{t}\\
&\le C\sum_{k=1}^N |b_k|^2\int_0^{\varphi(\delta)} t^{2k-1}\,dt\\
&\le C\sum_{k=1}^N |b_k|^2\beta^{2k} \le C,
\end{aligned}\tag{3.17}$$

where $C > 0$ is independent of h, Ω, ξ and the coefficients of P, but depends on φ and N.

For J_2. Fix $t \geq \delta$, there exists an integer j_0 such that $2^{j_0} \leq \varphi(t/2) < 2^{j_0+1}$. By the fact that $\varphi(2t) \leq c_\varphi \varphi(t)$, there exists an integer k_0 such that $2^{k_0} \leq c_\varphi < 2^{k_0+1}$ and then $\varphi(t) \leq 2^{k_0+j_0+2}$. From (3.6) we have

$$\begin{aligned}
&\int_{\varphi(t/2)}^{\varphi(t)} \left| \int_{\mathbf{R}} N_b(s) \mathbf{e}^{-2\pi i P(r)|\xi| r(\xi')s} \, ds \right|^2 \frac{dr}{r} \\
&\quad \leq \sum_{j=j_0}^{k_0+j_0+1} \int_{2^j}^{2^{j+1}} \left| \int_{\mathbf{R}} N_b(s) \mathbf{e}^{-2\pi i P(r)|\xi| r(\xi')s} \, ds \right|^2 \frac{dr}{r} \\
&\quad \leq \sum_{j=j_0}^{k_0+j_0+1} 2^{-j} 2^j |\beta_\kappa|^{-1/\Lambda} 2^{-j\kappa/\Lambda} \|N_b\|_{L^2(\mathbf{R})}^2 \\
&\quad \leq C \sum_{j=j_0}^{k_0+j_0+1} |\beta_\kappa|^{-1/\Lambda} 2^{-j\kappa/\Lambda} \\
&\quad \leq C |\beta_\kappa|^{-1/\Lambda} \varphi(t)^{-\kappa/\Lambda},
\end{aligned} \tag{3.18}$$

where $C > 0$ is independent of Ω, ξ and the coefficients of P, but depends on φ. (3.18) together with Hölder's inequality and the change of variable shows that

$$\begin{aligned}
J_2 &\leq C(\varphi) \int_\delta^\infty \left(\int_{\varphi(t/2)}^{\varphi(t)} \left| \int_{\mathbf{R}} N_b(s) \mathbf{e}^{-2\pi i P(r)|\xi| r(\xi')s} \, ds \right|^2 \frac{dr}{r} \right)^{2/\tilde{\gamma}} \frac{dt}{t} \\
&\leq C \int_\delta^\infty \left(|\beta_\kappa|^{-1/\Lambda} \varphi(t)^{-\kappa/\Lambda} \right)^{2/\tilde{\gamma}} \frac{dt}{t} \\
&\leq C |\beta_\kappa|^{-2/(\Lambda\tilde{\gamma})} \int_\delta^\infty \varphi(t)^{-2\kappa/(\Lambda\tilde{\gamma})} \frac{dt}{t} \\
&\leq C |\beta_\kappa|^{-2/(\Lambda\tilde{\gamma})} \int_\beta^\infty t^{-2\kappa/(\Lambda\tilde{\gamma})-1} \, dt \\
&\leq C \frac{\Lambda\tilde{\gamma}}{2\kappa}.
\end{aligned} \tag{3.19}$$

Here $C > 0$ is independent of h, Ω, ξ and the coefficients of P, but depends on φ. Then (3.14) follows from (3.15)–(3.17) and (3.19). This proves Theorem 1.2. □

Proof of Theorem 1.3 Let Ω, φ be given as in Theorem 1.3. By arguments similar to those used in deriving (3.14) and (3.16) in [37], one can easily get that

$$\mathcal{M}_{\Omega,P,\varphi,\rho} f(x) \leq C(\rho) \mathcal{S}_{\Omega,P,\varphi} f(x).$$

Thus, we only prove Theorem 1.3 for the operator $\mathcal{S}_{\Omega,P,\varphi}$. Define the measure $\sigma_{\Omega,P,\varphi,t}$ by

$$\widehat{\sigma_{\Omega,P,\varphi,t}}(\xi) = \int_{\mathbf{S}^{n-1}} \mathbf{e}^{-2\pi i P(\varphi(t)) y' \cdot \xi} \Omega(y') \, d\sigma(y'). \tag{3.20}$$

By duality we can write

$$\mathcal{S}_{\Omega,P,\varphi} f(x) = \left(\int_0^\infty \left| \sigma_{\Omega,P,\varphi,t} * f(x) \right|^2 \frac{dt}{t} \right)^{1/2}. \tag{3.21}$$

By (3.21) and the same arguments as in the proof of Theorem 1.2, to prove Theorem 1.3 for the operator $\mathcal{S}_{\Omega,P,\varphi}$, it suffices to show that

$$\left\| \left(\int_0^\infty \left| \widehat{\sigma_{\Omega,P,\varphi,t}}(\cdot) \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^\infty(\mathbf{R}^n)} \le C\|\Omega\|_{H^1(\mathbf{S}^{n-1})}, \tag{3.22}$$

where $C > 0$ is independent of Ω and the coefficients of P. By Lemma 2.1 and Minkowski's inequality, (3.22) reduces to the following:

$$\left\| \left(\int_0^\infty \left| \widehat{\sigma_{\Omega,P,\varphi,t}}(\cdot) \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^\infty(\mathbf{R}^n)} \le C, \tag{3.23}$$

where Ω is a $(1,\infty)$ atom and $C > 0$ is independent of Ω and the coefficients of P.

We now prove (3.23). We only assume that Ω is a $(1,\infty)$ atom satisfying (2.1)–(2.3) with $\vartheta = (1,0,\ldots,0)$ and consider the case $n \ge 3$. Let ξ, b, and N_b be given as in the proof of Theorem 1.1. By the change of variables, we have

$$\int_0^\infty \left| \widehat{\sigma_{\Omega,P,\varphi,t}}(\xi) \right|^2 \frac{dt}{t} = \int_0^\infty \left| \int_{\mathbf{R}} N_b(s) \mathbf{e}^{-2\pi i P(\varphi(t))|\xi| r(\xi')s} \, ds \right|^2 \frac{dt}{t}.$$

By a change of variable and the properties of φ, we obtain that

$$\int_0^\infty \left| \widehat{\sigma_{\Omega,P,\varphi,t}}(\xi) \right|^2 \frac{dt}{t} \le C(\varphi) \int_0^\infty \left| \int_{\mathbf{R}} N_b(s) e^{-2\pi i P(t)|\xi| r(\xi')s} \, ds \right|^2 \frac{dt}{t}. \tag{3.24}$$

Let b_k, β be given as in the proof Theorem 1.1. We write

$$\begin{aligned}
&\int_0^\infty \left| \int_{\mathbf{R}} N_b(s) \mathbf{e}^{-2\pi i P(t)|\xi| r(\xi')s} \, ds \right|^2 \frac{dt}{t} \\
&\quad = \int_0^\beta \left| \int_{\mathbf{R}} N_b(s) \mathbf{e}^{-2\pi i P(t)|\xi| r(\xi')s} \, ds \right|^2 \frac{dt}{t} + \int_\beta^\infty \left| \int_{\mathbf{R}} N_b(s) \mathbf{e}^{-2\pi i P(t)|\xi| r(\xi')s} \, ds \right|^2 \frac{dt}{t} \\
&\quad =: L_1 + L_2.
\end{aligned} \tag{3.25}$$

For L_1, by the cancelation condition of N_b, we have

$$L_1 = \int_0^\beta \left| \int_{\mathbf{R}} N_b(s) \left(\mathbf{e}^{-2\pi i P(t)|\xi| r(\xi')s} - 1 \right) ds \right|^2 \frac{dt}{t} \le C \sum_{k=1}^N |b_k|^2 \int_0^\beta t^{-1+2k} \, dt \le C, \tag{3.26}$$

where $C > 0$ is independent of b, ξ and the coefficients of P.

For L_2, let K_0 be given as in the proof of Theorem 1.1. Applying (3.6), we get

$$\begin{aligned}
L_2 &\le \sum_{j=K_0}^\infty \int_{2^j}^{2^{j+1}} \left| \int_{\mathbf{R}} N_b(s) \mathbf{e}^{-2\pi i P(t)|\xi| r(\xi')s} \, ds \right|^2 \frac{dt}{t} \\
&\le \sum_{j=K_0}^\infty 2^{-j} \int_R \left| \chi_{R_j}(t) \int_{\mathbf{R}} N_b(s) \mathbf{e}^{-2\pi i P(t)|\xi| r(\xi')s} \, ds \right|^2 dt \\
&\le C \sum_{j=K_0}^\infty 2^{-j} 2^j |\beta_\kappa|^{-1/\Lambda} 2^{-j\kappa/\Lambda} \|N_b\|_{L^2(\mathbf{R})}^2 \le C,
\end{aligned} \tag{3.27}$$

where in the last inequality of (3.27) we have used the fact that $2^{K_0\kappa} \geq 2^{-\kappa}\beta_\kappa^{-1}$ and $C > 0$ is independent of b, ξ and the coefficients of P. We get from (3.24)–(3.27) that

$$\int_0^\infty \left|\widehat{\sigma_{\Omega,P,\varphi,t}}(\xi)\right|^2 \frac{dt}{t} \leq C,$$

where $C > 0$ is independent of Ω, ξ and the coefficients of P. This yields (3.23) and completes the proof of Theorem 1.3. □

4 Hardy-type inequalities

In this section we shall establish the following Hardy-type inequalities.

Theorem 4.1 *Let P be a real polynomial on $\mathbf{R}$ satisfying $P(0) = 0$ and $\Omega \in H^1(\mathbf{S}^{n-1})$ satisfying (1.1). Then we have*

(i) *If $\varphi \in \mathfrak{F}$, then*

$$\left(\int_{\mathbf{R}^n} \left|\frac{1}{|x|^\rho}\int_{|x|/2\leq|y|<|x|} \mathbf{e}^{-2\pi i P(\varphi(|y|))x'\cdot y'} \frac{\Omega(y)}{|y|^{n-\rho}}\,dy\right|^2 \frac{dx}{|x|^n}\right)^{1/2} \leq C\|\Omega\|_{H^1(\mathbf{S}^{n-1})}, \tag{4.1}$$

where C is a positive constant independent of Ω and the coefficients of P, but depends on ρ, φ, and deg(P).

(ii) *If $\varphi \in \mathcal{G}$, then*

$$\left(\int_{\mathbf{R}^n} \left|\int_{\mathbf{S}^{n-1}} \mathbf{e}^{-2\pi i P(\varphi(|x|))x'\cdot y'} \Omega(y')\,d\sigma(y')\right|^2 \frac{dx}{|x|^n}\right)^{1/2} \leq C\|\Omega\|_{H^1(\mathbf{S}^{n-1})}, \tag{4.2}$$

where C is a positive constant independent of Ω and the coefficients of P, but depends on φ and deg(P).

Proof of Theorem 4.1 We first prove (i). Using Lemma 2.1, (4.1) reduces to the following:

$$\left(\int_{\mathbf{R}^n} \left|\frac{1}{|x|^\rho}\int_{|x|/2\leq|y|<|x|} \mathbf{e}^{-2\pi i P(\varphi(|y|))x'\cdot y'} \frac{\Omega(y)}{|y|^{n-\rho}}\,dy\right|^2 \frac{dx}{|x|^n}\right)^{1/2} \leq C, \tag{4.3}$$

where Ω is a $(1,\infty)$ atom and C is a positive constant independent of Ω and the coefficients of P. By the polar coordinates,

$$\begin{aligned}
&\int_{\mathbf{R}^n} \left|\frac{1}{|x|^\rho}\int_{|x|/2\leq|y|<|x|} \mathbf{e}^{-2\pi i P(\varphi(|y|))x'\cdot y'} \frac{\Omega(y)}{|y|^{n-\rho}}\,dy\right|^2 \frac{dx}{|x|^n} \\
&\quad = \int_{\mathbf{S}^{n-1}}\int_0^\infty \left|\frac{1}{t^\rho}\int_{t/2}^t\int_{\mathbf{S}^{n-1}} \mathbf{e}^{-2\pi i P(\varphi(r))x'\cdot y'}\Omega(y')\,d\sigma(y')\frac{dr}{r^{1-\rho}}\right|^2 \frac{dt}{t}\,d\sigma(x') \\
&\quad = \int_{\mathbf{S}^{n-1}}\int_0^\infty \left|\widehat{\sigma_{b,\Omega,P,\varphi,t}}(x')\right|^2 \frac{dt}{t}\,d\sigma(x') \\
&\quad \leq \omega_{n-1}\left\|\left(\int_0^\infty \left|\widehat{\sigma_{b,\Omega,P,\varphi,t}}(\cdot)\right|^2\frac{dt}{t}\right)^{1/2}\right\|^2_{L^\infty(\mathbf{S}^{n-1})}.
\end{aligned} \tag{4.4}$$

Here ω_{n-1} is the surface area of the unit $\mathbf{S}^{n-1}$ and $\sigma_{b,\Omega,P,\varphi,t}$ is defined as in (3.8) with $b(\cdot) \equiv 1$. Then (4.3) follows from (4.4) and (3.11).

It remains to prove (ii). To prove (4.2), by Lemma 2.1 it suffices to show that

$$\left(\int_{\mathbf{R}^n}\left|\int_{\mathbf{S}^{n-1}}\mathbf{e}^{-2\pi iP(\varphi(|x|))x'\cdot y'}\Omega\left(y'\right)d\sigma\left(y'\right)\right|^2\frac{dx}{|x|^n}\right)^{1/2}\leq C, \tag{4.5}$$

where Ω is a $(1,\infty)$ atom and C is a positive constant independent of Ω and the coefficients of P. Using the polar coordinates, we can obtain

$$\begin{aligned}
&\int_{\mathbf{R}^n}\left|\int_{\mathbf{S}^{n-1}}\mathbf{e}^{-2\pi iP(\varphi(|x|))x'\cdot y'}\Omega\left(y'\right)d\sigma\left(y'\right)\right|^2\frac{dx}{|x|^n}\\
&\quad=\int_{\mathbf{S}^{n-1}}\int_0^\infty\left|\int_{\mathbf{S}^{n-1}}\mathbf{e}^{-2\pi iP(\varphi(t))x'\cdot y'}\Omega\left(y'\right)d\sigma\left(y'\right)\right|^2\frac{dt}{t}\,d\sigma\left(x'\right)\\
&\quad=\int_{\mathbf{S}^{n-1}}\int_0^\infty\left|\widehat{\sigma_{\Omega,P,\varphi,t}}\left(x'\right)\right|^2\frac{dt}{t}\,d\sigma\left(x'\right)\\
&\quad\leq\omega_{n-1}\left\|\left(\int_0^\infty\left|\widehat{\sigma_{\Omega,P,\varphi,t}}\left(x'\right)\right|^2\frac{dt}{t}\right)^{1/2}\right\|_{L^\infty(\mathbf{S}^{n-1})}^2,
\end{aligned} \tag{4.6}$$

where $\widehat{\sigma_{\Omega,P,\varphi,t}}$ is defined as in (3.20). (4.6) together with (3.22) yields (4.5). Theorem 4.1 is proved. □

Acknowledgements
The authors would like to express their sincere thanks to the referees for their careful reading and invaluable comments.

Funding
The first author was supported partially by the NNSF of China (No. 11701333) and Support Program for Outstanding Young Scientific and Technological Top-notch Talents of College of Mathematics and Systems Science (No. Sxy2016K01). The second author was supported partially by the NNSF of China (No. 11571160).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors worked jointly in drafting and approving the final manuscript.

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Received: 20 November 2018 Accepted: 13 March 2019 Published online: 19 March 2019

References

1. Calderón, A.P., Zygmund, A.: On singular integral. Am. J. Math. **78**, 289–309 (1956)
2. Coifman, R.R., Weiss, G.: Extensions of Hardy spaces and their use in analysis. Bull. Am. Math. Soc. **83**(6), 569–645 (1977)
3. Connett, W.C.: Singular integrals near L^1. In: Harmonic Analysis in Euclidean Spaces (Proc. Sympos. Pure Math., Williams Coll., Willamstown, 1978), pp. 163–165. Am. Math. Soc., Providence (1979)
4. Fefferman, R.: A note on singular integrals. Proc. Am. Math. Soc. **74**(2), 266–270 (1979)
5. Namazi, J.: A singular integral. Proc. Am. Math. Soc. **96**, 201–219 (1986)
6. Duoandikoetxea, J., Rubio de Francia, J.L.: Maximal and singular integral operators via Fourier transform estimates. Invent. Math. **84**(3), 541–561 (1986)
7. Stein, E.M.: Problems in harmonic analysis related to curvature and oscillatory integrals. In: Proc. Internat. Congr. Math., Berkeley, 1986, vol. 1, pp. 196–221. Am. Math. Soc., Providence (1987)
8. Stein, E.M.: Some geometrical concepts arising in harmonic analysis. Geom. Funct. Anal. **Special volume, Part 1**, 434–453 (2000)
9. Fan, D., Pan, Y.: Singular integral operators with rough kernels supported by subvarieties. Am. J. Math. **119**(4), 799–839 (1997)

10. Fan, D., Guo, K., Pan, Y.: Singular integrals with rough kernels along real-analytic submanifolds in $\mathbf{R}^3$. Am. J. Math. **119**(4), 799–839 (1997)
11. Cheng, L.C.: Singular integrals related to homogeneous mappings. Mich. Math. J. **47**, 407–416 (2000)
12. Liu, F., Mao, S., Wu, H.: On rough singular integrals related to homogeneous mappings. Collect. Math. **67**(1), 113–132 (2016)
13. Fan, D., Pan, Y.: L^2-boundedness of a singular integral operator. Publ. Mat. **41**, 317–333 (1997)
14. Liu, F.: Rough singular integrals associated to surfaces of revolution on Triebel–Lizorkin spaces. Rocky Mt. J. Math. **47**(5), 1617–1653 (2017)
15. Liu, F., Wu, H.: Singular integrals related to homogeneous mappings in Triebel–Lizorkin spaces. J. Math. Inequal. **11**(4), 1075–1097 (2017)
16. Liu, F.: Boundedness and continuity of maximal operators associated to polynomial compound curves on Triebel–Lizorkin spaces. Math. Inequal. Appl. **22**(1), 25–44 (2019)
17. Liu, F., Fu, Z., Jhang, S.: Boundedness and continuity of Marcinkiewicz integrals associated to homogeneous mappings on Triebel–Lizorkin spaces. Front. Math. China **14**(1), 95–122 (2019)
18. Ding, Y., Xue, Q., Yabuta, K.: On singular integral operators with rough kernel along surfaces. Integral Equ. Oper. Theory **68**, 151–161 (2010)
19. Ding, Y., Fan, D., Pan, Y.: L^p-boundedness of Marcinkiewicz integrals with Hardy space function kernel. Acta Math. Appl. Sin. Engl. Ser. **16**, 593–600 (2000)
20. Liu, F.: A note of Littlewood–Paley functions on Triebel–Lizorkin spaces. Bull. Korean Math. Soc. **55**(2), 659–672 (2018)
21. Ding, Y., Fan, D., Pan, Y.: On the L^p boundedness of Marcinkiewicz integrals. Mich. Math. J. **50**, 17–26 (2002)
22. Ding, Y., Xue, Q., Yabuta, K.: Boundedness of the Marcinkiewicz integrals with rough kernel associated to surfaces. Tohoku Math. J. **62**, 233–262 (2010)
23. Liu, F.: A note on Marcinkiewicz integrals associated to surfaces of revolution. J. Aust. Math. Soc. **104**, 380–402 (2018)
24. Liu, F.: Integral operators of Marcinkiewicz type on Triebel–Lizorkin spaces. Math. Nachr. **290**(1), 75–96 (2017)
25. Bai, Z., Chen, Y., Lian, H., Sun, S.: On the existence of blow up solutions for a class of fractional differential equations. Fract. Calc. Appl. Anal. **17**(4), 1175–1187 (2014)
26. Bai, Z., Zhang, Y.: Solvability of fractional three-point boundary value problems with nonlinear growth. Appl. Math. Comput. **218**(5), 1719–1725 (2011)
27. Liu, F., Wu, H.: On Marcinkiewicz integrals associated to compound mappings with rough kernels. Acta Math. Sin. Engl. Ser. **30**(7), 1210–1230 (2014)
28. Li, Y., Cheng, H., Wang, J., Wang, Y.: Dynamic analysis of unilateral diffusion Gompertz model with impulsive control strategy. Adv. Differ. Equ. **2018**(32), 1 (2018)
29. Sato, S.: Estimates for singular integrals and extrapolation. Stud. Math. **192**, 219–233 (2009)
30. Chen, L., Lin, H.: A maximal operator related to a class of singular integrals. Ill. J. Math. **34**, 120–126 (1990)
31. Xu, H., Fan, D., Wang, M.: Some maximal operators related to families of singular integral operators. Acta Math. Sin. Engl. Ser. **20**(3), 441–452 (2004)
32. Pan, Y.: Hardy spaces and oscillatory singular integrals. Rev. Mat. Iberoam. **7**(1), 55–64 (1991)
33. Miao, A., Zhang, T., Zhang, J., Wang, C.: Dynamics of a stochastic SIR model with both horizontal and vertical transmission. J. Appl. Anal. Comput. **8**(4), 1108–1121 (2018)
34. Colzani, L.: Hardy spaces on sphere. PhD Thesis, Washington University, St Louis (1982)
35. Colzani, L., Taibleson, M., Weiss, G.: Maximal estimates for Cesàro and Riesz means on sphere. Indiana Univ. Math. J. **33**(6), 873–889 (1984)
36. Fan, D., Pan, Y.: A singular integral operator with rough kernel. Proc. Am. Math. Soc. **125**, 3695–3703 (1997)
37. Al-Qassem, H.M., Cheng, L.C., Pan, Y.: On certain rough integral operators and extrapolation. J. Inequal. Pure Appl. Math. **10**, 1–29 (2009)