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Inclusion relations for certain families of integral operators associated with conic regions

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Abstract

In this work, we introduce certain subclasses of analytic functions involving the integral operators that generalize the class of uniformly starlike, convex, and close-to-convex functions with respect to symmetric points. We then establish various inclusion relations for these newly defined classes.

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1 Introduction

Let $\mathcal{A}$ be the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

analytic in the open unit disc $\mathfrak{A} = \{z \in \mathbb{C} : |z| < 1\}$, and let $\mathcal{S}$ be the class of functions in $\mathcal{A}$ that are univalent in $\mathfrak{A}$. Also let $\mathcal{S}^*$, $\mathcal{C}$, $\mathcal{K}$, and $\mathcal{C}^*$ be the subclasses of $\mathcal{A}$ consisting of all functions that are starlike, convex, close-to-convex, and quasiconvex, respectively; for details, see [1].

Let f and g be analytic in $\mathfrak{A}$. We say that f is subordinate to g, written as $f(z) \prec g(z)$, if there exists a Schwarz function w that is analytic in $\mathfrak{A}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathfrak{A})$ and such that $f(z) = g(w(z))$. In particular, when g is univalent, then such a subordination is equivalent to $f(0) = g(0)$ and $f(\mathfrak{A}) \subseteq g(\mathfrak{A})$; see [1].

Two points A and A' are said to be symmetrical with respect to M if M is the midpoint of the line segment AA'. Sakaguchi [2] introduced and studied the class $\mathcal{S}_s^*$ of starlike functions with respect to symmetrical points z and $-z$ belonging to the open unit disc $\mathfrak{A}$. The class $\mathcal{S}_s^*$ includes the classes of convex and odd starlike functions with respect to the origin. It was shown [2] that a necessary and sufficient condition for $f(z) \in \mathcal{S}_s^*$ to be univalent and starlike with respect to symmetrical points in $\mathfrak{A}$ is that

$$\frac{2zf'(z)}{f(z) - f(-z)} \in \mathcal{P}, \quad z \in \mathfrak{A}.$$

Das and Singh [3] defined the classes $\mathcal{C}_s$ of convex functions with respect to symmetrical points and showed that a necessary and sufficient condition for $f(z) \in \mathcal{C}_s$ is that

$$\frac{2(zf'(z))'}{(f(z)-f(-z))'} \in \mathcal{P}, \quad z \in \mathfrak{A}.$$

It is also well known [3] that $f(z) \in \mathcal{C}_s$ if and only if $zf(z) \in \mathcal{S}_s^*$.

The classes $k - \mathcal{CV}$ and $k - \mathcal{ST}$ with $k \geq 0$ denote the famous classes of k-uniformly convex and k-starlike functions, respectively, introduced by Kanas and Wisniowska, respectively. For some details see [4–7].

Consider the domain

$$\Omega_k = \left\{u + iv; u > k\sqrt{(u-1)^2 + v^2}\right\}. \tag{1.2}$$

For fixed k, Ω_k represents the conic region bounded successively by the imaginary axis ($k = 0$), the right branch of a hyperbola ($0 < k < 1$), a parabola ($k = 1$), and an ellipse ($k > 1$). This domain was studied by Kanas [4–6]. The function p_k with $p_k(0) = 1$ and $p_k'(0) > 0$ plays the role of extremal and is given by

$$p_k(z) = \begin{cases} \frac{1+z}{1-z}, & k = 0, \\ 1 + \frac{2}{\pi^2}(\log \frac{1+\sqrt{z}}{1-\sqrt{z}})^2, & k = 1, \\ 1 + \frac{2}{1-k^2}\sinh^2[(\frac{2}{\pi}\arccos k)\operatorname{arc\,tanh}\sqrt{z}], & 0 < k < 1, \\ 1 + \frac{1}{k^2-1}\sin[\frac{\pi}{2R(t)}\int_0^{\frac{u(z)}{\sqrt{t}}} \frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}\,dx] + \frac{1}{k^2-1}, & k > 1, \end{cases} \tag{1.3}$$

with $u(z) = \frac{z-\sqrt{t}}{1-\sqrt{tz}}$, $t \in (0,1)$, $z \in E$, and t chosen such that $k = \cosh(\frac{\pi R'(t)}{4R(t)})$, where $R(t)$ is Legendre's complete elliptic integral of the first kind, and $R'(t)$ is the complementary integral of $R(t)$ (see [5, 6]). Let $\mathcal{P}_{p_k}$ denote the class of all functions $p(z)$ that are analytic in E with $p(0) = 1$ and $p(z) \prec p_k(z)$ for $z \in E$. Clearly, we can see that $\mathcal{P}_{p_k} \subset \mathcal{P}$, where $\mathcal{P}$ is the class of functions with positive real parts (see [1]). More precisely,

$$\mathcal{P}_{p_k} \subset \mathcal{P}\left(\frac{k}{1+k}\right) \subset \mathcal{P}.$$

For more detail regarding conic domains and related classes, see [4–6, 8–11].

Recently, Noor [12] defined the classes $k - \mathcal{ST}_s$, $k - \mathcal{UCV}_s$, and $k - \mathcal{UK}_s$ of k-uniformly starlike, convex, and close to convex functions with respect to symmetrical points and studied various interesting properties for these classes.

We consider the following one-parameter families of integral operators:

$$\mathcal{I}_\beta^\alpha f(z) = \frac{(\beta+1)^\alpha}{\Gamma(\alpha)z^\beta}\int_0^z t^{\beta-1}\left(\log\frac{z}{t}\right)^{\alpha-1} f(t)\,dt, \tag{1.4}$$

$$\mathfrak{L}_\beta^\alpha f(z) = \binom{\alpha+\beta}{\beta}\frac{\alpha}{z^\beta}\int_0^z t^{\beta-1}\left(1-\frac{t}{z}\right)^{\alpha-1} f(t)\,dt, \tag{1.5}$$

and

$$\mathfrak{I}_\beta f(z) = \frac{\beta+1}{z^\beta}\int_0^z t^{\beta-1} f(t)\,dt, \tag{1.6}$$

where $\alpha \geq 0$, $\beta > -1$, and Γ is the familiar gamma function. We note that $\mathfrak{J}_\beta : \mathcal{A} \to \mathcal{A}$ defined by (1.6) is the generalized Bernardi operator introduced in [13] for $\beta = 1, 2, 3, \ldots$, and for any real number $\beta > -1$, this operator was studied by Owa and Srivastava [14, 15]. For the operators $\mathfrak{L}_\beta^\alpha$ and $\mathcal{I}_\beta^\alpha$, we refer to [16, 17]. Also, for $\alpha = 1$, we see that

$$\mathfrak{J}_\beta f(z) = \mathfrak{L}_\beta^1 f(z) = \mathcal{I}_\beta^1 f(z).$$

We can represent these operators as follows:

$$\begin{aligned} \mathcal{I}_\beta^\alpha f(z) &= z + \sum_{n=2}^{\infty} \left(\frac{\beta+1}{\beta+n}\right)^\alpha a_n z^n \\ &= \left(z + \sum_{n=2}^{\infty} \left(\frac{\beta+1}{\beta+n}\right)^\alpha z^n\right) * f(z), \end{aligned} \tag{1.7}$$

$$\begin{aligned} \mathfrak{L}_\beta^\alpha f(z) &= z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+\beta+n)\Gamma(\beta+1)} a_n z^n \\ &= \binom{\alpha+\beta}{\beta} z {}_2F_1(1, \beta; \alpha+\beta; z) * f(z), \end{aligned} \tag{1.8}$$

and

$$\mathfrak{J}_\beta f(z) = z + \sum_{n=2}^{\infty} \left(\frac{\beta+1}{\beta+n}\right) a_n z^n, \tag{1.9}$$

where ${}_2F_1$ denotes the Gaussian hypergeometric function, and the symbol $*$ stands for the convolution (Hadamard product).

By (1.7) and (1.8) we can easily derive the identities

$$z\big(\mathcal{I}_\beta^\alpha f(z)\big)' = (\beta+1)\mathcal{I}_\beta^{\alpha-1} f(z) - \beta \mathcal{I}_\beta^\alpha f(z) \tag{1.10}$$

and

$$z\big(\mathfrak{L}_\beta^\alpha f(z)\big)' = (\alpha+\beta)\mathfrak{L}_\beta^{\alpha-1} f(z) - (\alpha+\beta-1)\mathfrak{L}_\beta^\alpha f(z), \tag{1.11}$$

where $\alpha \geq 1$ and $\beta > -1$. From (1.10) we have

$$\left[\frac{1}{1+\beta} p(z) + \frac{\beta}{1+\beta}\right] = \frac{\mathcal{I}_\beta^{\alpha-1} f(z)}{\mathcal{I}_\beta^\alpha f(z)}$$

with

$$p(z) = \frac{z(\mathcal{I}_\beta^\alpha f(z))'}{\mathcal{I}_\beta^\alpha f(z)}.$$

With the help of these integral operators, we now define the following classes.

Definition 1.1 Let $f(z) \in \mathcal{A}$. Then $f(z) \in k - \mathcal{ST}_s(\alpha, \beta)$, $\alpha \geq 0$, $\beta > -1$, if $\mathcal{I}_\beta^\alpha f(z) \in k - \mathcal{ST}_s$ in $\mathfrak{A}$.

Definition 1.2 Let $f(z) \in \mathcal{A}$. Then $f(z) \in k-\mathcal{ST}_s^*(\alpha,\beta)$, $\alpha \geq 0$, $\beta > -1$, if $\mathfrak{L}_\beta^\alpha f(z) \in k-\mathcal{ST}_s$ in $\mathfrak{A}$.

Definition 1.3 Let $f(z) \in \mathcal{A}$. Then $f(z) \in k-\mathcal{UK}_s(\alpha,\beta)$, $\alpha \geq 0$, $\beta > -1$, if $\mathcal{I}_\beta^\alpha f(z) \in k-\mathcal{UK}_s$ in $\mathfrak{A}$.

Definition 1.4 Let $f(z) \in \mathcal{A}$. Then $f(z) \in k-\mathcal{UK}_s^*(\alpha,\beta)$, $\alpha \geq 0$, $\beta > -1$, if $\mathfrak{L}_\beta^\alpha f(z) \in k-\mathcal{UK}_s$ in $\mathfrak{A}$.

2 A set of lemmas

In this section, we give the following lemmas, which will be used in our investigation.

Lemma 2.1 ([4]) *Let $k \geq 0$, and let $\beta_1,\gamma \in \mathbb{C}$ be such that $\beta_1 \neq 0$ and $\mathfrak{Re}\{\frac{\beta_1 k}{k+1}+\gamma\} > 0$. Suppose that $p(z)$ is analytic in $\mathfrak{A}$ with $p(0)=1$ and satisfies*

$$\left(p(z)+\frac{zp'(z)}{\beta_1 p(z)+\gamma}\right) \prec p_k(z) \tag{2.1}$$

and that $q(z)$ is an analytic function satisfying

$$q(z)+\frac{zq'(z)}{\beta_1 q(z)+\gamma} = p_k(z). \tag{2.2}$$

Then $q(z)$ is univalent, $p(z) \prec q(z) \prec p_k(z)$, and $q(z)$ is the best dominant of (2.1) *given as*

$$q(z) = \left[\beta_1 \int_0^1 \left(t^{\beta_1+\gamma-1} \exp \int_z^{tz} \frac{p_k(u)-1}{u}\,du\right)dt\right]^{-1} - \frac{\gamma}{\beta_1}. \tag{2.3}$$

Lemma 2.2 ([18]) *Let $\lambda,\rho \in \mathbb{C}$ be such that $\lambda \neq 0$, and let $\phi(z) \in \mathcal{A}$ be convex and univalent in $\mathbb{U}$ with $\mathfrak{Re}\{\lambda\phi(z)+\rho\} > 0$ $(z \in \mathbb{U})$. Also, let $q(z) \in \mathcal{A}$ and $q(z) \prec \phi(z)$. If $p(z)$ is analytic in $\mathbb{U}$ with $p(0)=1$ and satisfies*

$$\left(p(z)+\frac{zp'(z)}{\lambda q(z)+\rho}\right) \prec \phi(z), \tag{2.4}$$

then $p(z) \prec \phi(z)$.

3 The main results and their consequences

Our first main result is stated as the following:

Theorem 3.1 *Let $f(z) \in k-\mathcal{ST}_s(\alpha,\beta)$. Then the odd function*

$$\psi(z) = \frac{1}{2}\big[f(z)-f(-z)\big] \in k-\mathcal{ST}(\alpha,\beta).$$

Proof Note that

$$\mathcal{I}_\beta^\alpha \psi(z) = \frac{1}{2}\big[\mathcal{I}_\beta^\alpha f(z) - \mathcal{I}_\beta^\alpha f(-z)\big].$$

We want to show that $\mathcal{I}_\beta^\alpha \psi(z) \in k - \mathcal{ST}$. Now, for $f(z) \in k - \mathcal{ST}_s(\alpha,\beta)$, this implies that $\mathcal{I}_\beta^\alpha f(z) \in k - \mathcal{ST}_s$. Then, for $z \in \mathfrak{A}$,

$$\begin{aligned}\frac{z(\mathcal{I}_\beta^\alpha \psi(z))'}{\mathcal{I}_\beta^\alpha \psi(z)} &= \frac{1}{2}\left[\frac{2z(\mathcal{I}_\beta^\alpha f(z))'}{\mathcal{I}_\beta^\alpha f(z) - \mathcal{I}_\beta^\alpha f(-z)} + \frac{2(-z)(\mathcal{I}_\beta^\alpha f(-z))'}{\mathcal{I}_\beta^\alpha f(-z) - \mathcal{I}_\beta^\alpha f(z)}\right] \\ &= \frac{1}{2}\left[h_1(z) + h_2(z)\right] \\ &= h(z).\end{aligned}$$

and $h_i(z) \prec p_k(z)$, $i = 1,2$. This implies that $h(z) \prec p_k(z)$ in $\mathfrak{A}$, and therefore $\mathcal{I}_\beta^\alpha \psi(z) \in k - \mathcal{ST}$. Consequently, $\psi(z) \in k - \mathcal{ST}(\alpha,\beta)$ in $\mathfrak{A}$. □

Similarly, we can prove that if $f(z) \in k - \mathcal{ST}_s^*(\alpha,\beta)$, then

$$\phi(z) = \frac{1}{2}\left[f(z) - f(-z)\right] \in k - \mathcal{ST}^*(\alpha,\beta).$$

Taking $\alpha = 0$, we obtain the following result proved by Noor [12].

Corollary 3.2 *Let $f(z) \in k - \mathcal{ST}_s$. Then the odd function*

$$\psi(z) = \frac{1}{2}\left[f(z) - f(-z)\right] \in k - \mathcal{ST}.$$

Note that, for $k = \alpha = 0$, the function $\psi(z) = \frac{1}{2}[f(z) - f(-z)]$ is a starlike function in $\mathfrak{A}$; see [2].

Theorem 3.3 *Let $\alpha \geq 2$ and $\beta > -1$. Then $k - \mathcal{ST}(\alpha - 1,\beta) \subset k - \mathcal{ST}(\alpha,\beta)$.*

Proof Let $f(z) \in k - \mathcal{ST}(\alpha - 1,\beta)$ and set

$$p(z) = \frac{z(\mathcal{I}_\beta^\alpha f(z))'}{\mathcal{I}_\beta^\alpha f(z)}. \tag{3.1}$$

Note that $p(z)$ is analytic in $\mathfrak{A}$ with $p(0) = 1$.

From (3.1) and identity (1.10) we have

$$\frac{\mathcal{I}_\beta^{\alpha-1} f(z)}{\mathcal{I}_\beta^\alpha f(z)} = (1-\gamma)p(z) + \gamma \tag{3.2}$$

with

$$\gamma = \frac{\beta}{\beta+1}. \tag{3.3}$$

Logarithmic differentiation of (3.2) yields

$$\frac{z(\mathcal{I}_\beta^{\alpha-1} f(z))'}{\mathcal{I}_\beta^{\alpha-1} f(z)} = \left\{p(z) + \frac{(1-\gamma)zp'(z)}{(1-\gamma)zp(z)+\gamma}\right\},$$

and thus it follows that

$$\left(p(z) + \frac{zp'(z)}{zp(z) + \beta}\right) \prec p_k(z).$$

Using Lemma 2.1, we have

$$p(z) \prec q(z) \prec p_k(z)$$

with

$$q(z) = \left[\int_0^1 \left(t^\beta \exp \int_z^{tz} \frac{p_k(u) - 1}{u}\, du\right) dt\right]^{-1} - \beta.$$

This proves that $f(z) \in k - \mathcal{ST}(\alpha, \beta)$ in $\mathfrak{A}$, and the proof is complete. □

Theorem 3.4 *Let* $\alpha \geq 2$ *and* $\beta > -1$. *Then* $k - \mathcal{ST}^*(\alpha - 1, \beta) \subset k - \mathcal{ST}^*(\alpha, \beta)$.

Proof Let

$$\frac{z(\mathfrak{L}_\beta^\alpha f(z))'}{\mathfrak{L}_\beta^\alpha f(z)} = h(z), \tag{3.4}$$

where $h(z)$ is analytic in $\mathfrak{A}$ with $h(0) = 1$.

From (3.4) and identity (1.11) we get

$$\frac{1}{\alpha + \beta} \frac{z(\mathfrak{L}_\beta^\alpha f(z))'}{\mathfrak{L}_\beta^\alpha f(z)} + \left(1 - \frac{1}{\alpha + \beta}\right) = \frac{\mathfrak{L}_\beta^{\alpha-1} f(z)}{\mathfrak{L}_\beta^\alpha f(z)}. \tag{3.5}$$

Logarithmic differentiation of (3.5), together with (3.4), gives us

$$\begin{aligned}\frac{z(\mathfrak{L}_\beta^{\alpha-1} f(z))'}{\mathfrak{L}_\beta^{\alpha-1} f(z)} &= h(z) + \frac{\frac{1}{\alpha+\beta} zh'(z)}{\frac{1}{\alpha+\beta} h(z) + \frac{\alpha+\beta-1}{\alpha+\beta}} \\ &= h(z) + \frac{zh'(z)}{h(z) + \alpha + \beta - 1}.\end{aligned}$$

Since $f(z) \in k - \mathcal{ST}^*(\alpha - 1, \beta)$, it follows that

$$h(z) + \frac{zh'(z)}{h(z) + \alpha + \beta - 1} \prec p_k(z).$$

Applying Lemma 2.1, we have

$$h(z) \prec p_k(z).$$

This proves our result. □

Theorem 3.5 *Let* $\alpha \geq 2$ *and* $\beta > -1$. *Then* $k - \mathcal{ST}_s(\alpha - 1, \beta) \subset k - \mathcal{ST}_s(\alpha, \beta)$.

Proof Let $f(z) \in k - \mathcal{ST}_s(\alpha - 1, \beta)$. Then, using Theorems 3.1 and 3.3, we have

$$\psi(z) = \frac{f(z) - f(-z)}{2} \in k - \mathcal{ST}(\alpha - 1, \beta) \subset k - \mathcal{ST}(\alpha, \beta).$$

From this it easily follows that $f(z) \in k - \mathcal{ST}_s(\alpha, \beta)$, and this completes the proof. □

A similar result for the class $k - \mathcal{ST}_s^*(\alpha, \beta)$ can be easily proved.

Theorem 3.6 *Let* $\alpha \geq 1$ *and* $\beta > 0$. *Then* $k - \mathcal{UK}_s(\alpha - 1, \beta) \subset k - \mathcal{UK}_s(\alpha, \beta)$.

Proof Let $f(z) \in k - \mathcal{UK}_s(\alpha - 1, \beta)$. Then there exists $g(z) \in k - \mathcal{ST}_s(\alpha - 1, \beta)$ such that

$$\frac{2z(\mathcal{I}_\beta^{\alpha-1} f(z))'}{\mathcal{I}_\beta^{\alpha-1} g(z) - \mathcal{I}_\beta^{\alpha-1} g(-z)} = \frac{z(\mathcal{I}_\beta^{\alpha-1} f(z))'}{\mathcal{I}_\beta^{\alpha-1} \psi(z)} \in P,$$

where $\psi(z) = \frac{\mathcal{I}_\beta^{\alpha-1} g(z) - \mathcal{I}_\beta^{\alpha-1} g(-z)}{2} \in k - \mathcal{ST}(\alpha - 1, \beta) \subset k - \mathcal{ST}(\alpha, \beta)$ in $\mathfrak{A}$.

Let us set

$$\frac{z(\mathcal{I}_\beta^{\alpha} f(z))'}{\mathcal{I}_\beta^{\alpha} \psi(z)} = p(z), \tag{3.6}$$

where $p(z)$ is analytic in $\mathfrak{A}$ with $p(0) = 1$. Then by (3.6) and identity (1.10) we get

$$\frac{\mathcal{I}_\beta^{\alpha-1} \psi(z)}{\mathcal{I}_\beta^{\alpha} \psi(z)} = (1 - \gamma) p_0(z) + \gamma,$$

where $p_0(z) = \frac{z(\mathcal{I}_\beta^{\alpha} \psi(z))'}{\mathcal{I}_\beta^{\alpha} \psi(z)}$, and γ is given by (3.3). Now by simple computations we obtain

$$\begin{aligned}
\frac{z(\mathcal{I}_\beta^{\alpha-1} f(z))'}{z\mathcal{I}_\beta^{\alpha-1} \psi(z)} &= \frac{z(\mathcal{I}_\beta^{\alpha-1} f(z))'}{\mathcal{I}_\beta^{\alpha} \psi(z)[(1-\gamma)p_0(z) + \gamma]} \\
&= \frac{z[(z(\mathcal{I}_\beta^{\alpha} f(z))')'] + \beta z(\mathcal{I}_\beta^{\alpha} f(z))'}{(\beta + 1)\mathcal{I}_\beta^{\alpha} \psi(z)[(1-\gamma)p_0(z) + \gamma]} \\
&= \frac{\beta p(z) + p(z)p_0(z) + zp'(z)}{(\beta + 1)[(1 - \frac{\beta}{1+\beta})p_0(z) + \frac{\beta}{1+\beta}]} \\
&= \frac{\beta p(z) + p(z)p_0(z) + zp'(z)}{p_0(z) + \beta} \\
&= p(z) + \frac{zp'(z)}{p_0(z) + \beta}.
\end{aligned}$$

Since $f(z) \in k - \mathcal{UK}_s(\alpha - 1, \beta)$, it follows that

$$p(z) + \frac{zp'(z)}{p_0(z) + \beta} \in \mathcal{P} \quad \text{in } \mathfrak{A}.$$

Applying Lemma.2.2, we have $p(z) \in \mathcal{P}$ in $\mathfrak{A}$. This proves $f(z) \in k - \mathcal{UK}_s(\alpha, \beta)$ in $\mathfrak{A}$. □

By a similar argument we can easily prove the following inclusion result.

Theorem 3.7 *Let $\alpha \geq 1$ and $\beta > 0$. Then $k-\mathcal{UK}^*(\alpha-1,\beta) \subset k-\mathcal{UK}^*(\alpha,\beta)$.*

Theorem 3.8 *Let $f(z) \in k-\mathcal{ST}_s(\alpha,\beta)$ in $\mathfrak{A}$. Then*

$$\mathfrak{Re}\left\{\frac{z(\mathcal{I}_\beta^{\alpha-1}f(z))'}{\mathcal{I}_\beta^{\alpha-1}\varphi(z)}\right\} > 0$$

for $|z| < R(\beta,\gamma_0)$, where

$$R(\beta,\gamma_0) = \frac{(1+\beta)}{(2-\gamma_0)+\sqrt{(2-\gamma_0)^2+(1+\beta)(\beta+2\gamma_0-1)}}$$

with

$$\gamma_0 = \frac{k}{k+1}. \tag{3.7}$$

Proof Let $f(z) \in k-\mathcal{ST}_s(\alpha,\beta)$. Then

$$\varphi(z) = \frac{f(z)-f(-z)}{2} \in k-\mathcal{ST}(\alpha,\beta),$$

and hence

$$\frac{z(\mathcal{I}_\beta^{\alpha}f(z))'}{\mathcal{I}_\beta^{\alpha}\varphi(z)} \in \mathcal{P}(p_k) \subset \mathcal{P}(\gamma_0),$$

where γ_0 is given by (3.7). Let

$$\begin{aligned}\frac{z(\mathcal{I}_\beta^{\alpha}f(z))'}{\mathcal{I}_\beta^{\alpha}\varphi(z)} &= h(z), \quad h(z) \in \mathcal{P}(\gamma_0),\\ &= (1-\gamma_0)h_0(z)+\gamma_0, \quad h_0(z) \in \mathcal{P}.\end{aligned} \tag{3.8}$$

Then, proceeding as in Theorem 3.5, we have

$$\frac{z(\mathcal{I}_\beta^{\alpha-1}f(z))'}{\mathcal{I}_\beta^{\alpha-1}\varphi(z)} = h(z) + \frac{zh'(z)}{p(z)+\beta}, \tag{3.9}$$

where $p(z) = \frac{z(\mathcal{I}_\beta^{\alpha}\varphi(z))'}{\mathcal{I}_\beta^{\alpha}\varphi(z)} \in \mathcal{P}(\gamma)$. Using (3.8) and $p(z) = (1-\gamma_0)p_0(z)+\gamma_0$ in (3.9), we have

$$\frac{z(\mathcal{I}_\beta^{\alpha-1}f(z))'}{\mathcal{I}_\beta^{\alpha-1}\varphi(z)} = (1-\gamma_0)h_0(z)+\gamma_0+\frac{(1-\gamma_0)zh_0'(z)}{(1-\gamma_0)p_0(z)+\gamma_0+\beta}$$

with $h_0(z) \in \mathcal{P}$, $p_0(z) \in \mathcal{P}$, that is,

$$\frac{1}{1-\gamma_0}\left[\frac{z(\mathcal{I}_\beta^{\alpha-1}f(z))'}{\mathcal{I}_\beta^{\alpha-1}\varphi(z)} - \gamma_0\right] = h_0(z) + \frac{zh_0'(z)}{(1-\gamma_0)p_0(z)+\gamma_0+\beta}.$$

Using the distortion result for the class $\mathcal{P}$, we obtain

$$\begin{aligned}
&\mathfrak{Re}\left[\frac{1}{1-\gamma_0}\left\{\frac{z(\mathcal{I}_\beta^{\alpha-1}f(z))'}{\mathcal{I}_\beta^{\alpha-1}\varphi(z)}-\gamma_0\right\}\right] \\
&\geq \mathfrak{Re}h_0(z)\left\{1-\frac{\frac{2r}{1-r^2}}{(1-\gamma_0)\frac{1-r}{1+r}+(\gamma_0+\beta)}\right\} \\
&= \mathfrak{Re}h_0(z)\left\{1-\frac{2r}{(1-\gamma_0)(1+r)^2+(1-r^2)(\gamma_0+\beta)}\right\}.
\end{aligned} \tag{3.10}$$

Right-hand side of (3.10) is greater than or equal to zero for $|z|<R(\beta,\gamma_0)$, where $R(\beta,\gamma_0)$ is the least positive root of the equation

$$T(r) := (1-\beta-2\gamma_0)r^2-2(2-\gamma_0)r+(1+\beta)=0,$$

that is,

$$\begin{aligned}
R(\beta,\gamma_0) &= \frac{2(2-\gamma_0)-\sqrt{4(2-\gamma_0)^2+4(1+\beta)(\beta+2\gamma_0-1)}}{2(1-\beta-2\gamma_0)} \\
&= \frac{(1+\beta)}{(2-\gamma_0)+\sqrt{(2-\gamma_0)^2+(1+\beta)(\beta+2\gamma_0-1)}}.
\end{aligned}$$

The proof is completed. □

Particular Cases

(i) For $\beta=0$ and $\gamma_0=\frac{k}{k+1}=0$ (i.e., $k=0$), we have $f(z)\in\mathcal{S}_s^*(\alpha,0)$ ($\psi\in\mathcal{S}^*(\alpha,0)$) and

$$R(0,0)=\frac{1}{2+\sqrt{3}}.$$

(ii) For $k=1$ and $\beta=0$,

$$R\left(0,\frac{1}{2}\right)=\frac{1}{3}.$$

(iii) For $k=1$ and $\beta=1$,

$$R\left(1,\frac{1}{2}\right)=\frac{4}{4+\sqrt{17}}.$$

Theorem 3.9 *Let $\mathfrak{L}_\beta^\alpha f(z)\in k-\mathcal{ST}$. Then*

$$\mathfrak{L}_\beta^{\alpha-1}f(z)\in\mathcal{S}^*(\gamma_0),\qquad \gamma_0=\frac{k}{k+1}$$

for $|z|<R_1$, where

$$R_1(\alpha,\beta,\gamma_0)=\frac{\alpha+\beta}{2-\gamma_0+\sqrt{(2-\gamma_0)^2+(\alpha+\beta)(2\gamma_0+\alpha+\beta-2)}}.$$

Proof Since $\mathfrak{L}_\beta^\alpha f(z) \in k - \mathcal{ST}$, we have

$$\frac{z(\mathfrak{L}_\beta^\alpha f(z))'}{\mathfrak{L}_\beta^\alpha f(z)} = h(z), \quad h(z) \prec p_k(z)$$

in $\mathfrak{A}$. With a similar argument as in Theorem 3.5, we have

$$\frac{z(\mathfrak{L}_\beta^{\alpha-1} f(z))'}{\mathfrak{L}_\beta^{\alpha-1} f(z)} = h(z) + \frac{zh'(z)}{h(z) + \alpha + \beta - 1},$$

that is,

$$\begin{aligned} &\Re\mathfrak{e}\left[\frac{1}{1-\gamma_0}\left\{\frac{z(\mathfrak{L}_\beta^{\alpha-1} f(z))'}{\mathfrak{L}_\beta^{\alpha-1} f(z)} - \gamma_0\right\}\right] \\ &\quad = \Re\mathfrak{e}\left[h_0(z) + \frac{zh_0'(z)}{(1-\gamma_0)h_0(z) + (\gamma_0 + \alpha + \beta - 1)}\right] \\ &\quad \geq \Re\mathfrak{e} h_0(z)\left[1 - \frac{\frac{2r}{1-r^2}}{(1-\gamma_0)\frac{1-r}{1+r} + (\gamma_0 + \alpha + \beta - 1)}\right], \end{aligned} \tag{3.11}$$

where

$$h(z) = (1-\gamma_0)h_0(z) + \gamma_0, \quad h_0 \in \mathcal{P}, \gamma_0 = \frac{k}{k+1}.$$

The right-hand side of (3.11) is greater than or equal to zero for $|z| < R_1$, where R_1 is the least positive root of the equation

$$T(r) := (2 - 2\gamma_0 - \alpha - \beta)r^2 - 2(2-\gamma_0)r + \alpha + \beta = 0,$$

that is,

$$\begin{aligned} R_1(\alpha, \beta, \gamma_0) &= \frac{2 - \gamma_0 - \sqrt{(2-\gamma_0)^2 + (\alpha+\beta)(2\gamma_0 + \alpha + \beta - 2)}}{2(2 - \alpha - \beta - 2\gamma_0)} \\ &= \frac{\alpha + \beta}{2 - \gamma_0 + \sqrt{(2-\gamma_0)^2 + (\alpha+\beta)(2\gamma_0 + \alpha + \beta - 2)}}. \end{aligned}$$

This completes the proof. □

4 Conclusion

In this paper, we have defined some new classes of analytic functions involving integral operators. We have shown that these classes generalize the well-known classes, and already existing results can be obtained as a particular cases of our results. Inclusion relations of these classes are also a significant part of our work. We believe that the work presented in this paper will give researchers a new direction and will motivate them to explore more interesting facts on similar lines.

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