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Inequalities for Katugampola conformable partial derivatives

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Abstract

In the paper, we introduce two concepts of Katugampola conformable partial derivatives and α-conformable integrals. As applications, we establish Opial type inequalities for Katugampola conformable partial derivatives and α-conformable integrals. The new inequalities in special cases yield some of the recent results on inequality of this type.

MSC: 26D15; 26A51

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1 Introduction

In 1960, Opial [1] established the following interesting and important inequality.

Theorem A *Suppose that* $f \in C^1[0,a]$ *satisfies* $f(0) = f(a) = 0$ *and* $f(x) > 0$ *for all* $x \in (0,a)$. *Then the inequality holds*

$$\int_0^a \left|f(x)f'(x)\right| dx \leq \frac{a}{4} \int_0^a \left(f'(x)\right)^2 dx, \tag{1.1}$$

where this constant $a/4$ *is best possible*.

Opial's inequality and its generalizations, extensions, and discretizations play a fundamental role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations as well as difference equations [2–6]. Inequality (1.1) has received considerable attention, and a large number of papers dealing with new proofs, extensions, generalizations, variants, and discrete analogues of Opial's inequality have appeared in the literature [7–18].

Recently, some new Opial's inequalities for the conformable fractional integrals have been established (see [19–22]). In the paper, we introduce two new concepts of Katugampola conformable partial derivatives and α-conformable integrals. As applications, we establish some Opial type inequalities for Katugampola conformable partial derivatives and α-conformable integrals.

2 Inequalities for Katugampola conformable partial derivatives

We recall the well-known Katugampola derivative formulation of conformable derivative of order for $\alpha \in (0,1]$ and $t \in [0,\infty)$, given by

$$D_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(te^{\varepsilon t^{-\alpha}}) - f(t)}{\varepsilon}, \tag{2.1}$$

and

$$D_\alpha(f)(0) = \lim_{t \to 0} D_\alpha(f)(t), \tag{2.2}$$

provided the limits exist. If f is fully differentiable at t, then

$$D_\alpha(f)(t) = t^{1-\alpha} \frac{df}{dt}(t).$$

A function f is α-differentiable at a point $t \geq 0$ if the limits in (2.1) and (2.2) exist and are finite. Inspired by this, we propose a new concept of α-conformable partial derivative. In the way of (2.1), we define α-conformable partial derivative.

Definition 2.1 (α-conformable partial derivative) Let $\alpha \in (0,1]$ and $s,t \in [0,\infty)$. Suppose that $f(s,t)$ is a continuous function and partially derivable, the α-conformable partial derivative at a point $s \geq 0$, denoted by $\frac{\partial}{\partial s}(f)_\alpha(s,t)$, is defined by

$$\frac{\partial}{\partial s}(f)_\alpha(s,t) = \lim_{\varepsilon \to 0} \frac{f(se^{\varepsilon s^{-\alpha}}, t) - f(s,t)}{\varepsilon}, \tag{2.3}$$

provided the limits exist, and is called α-conformable partially derivable.

To generalize Definition 2.1, we give the following definition.

Definition 2.2 (Katugampola conformable partial derivative) Let $\alpha \in (0,1]$ and $s,t \in [0,\infty)$. Suppose that $f(s,t)$ and $\frac{\partial}{\partial s}(f)_\alpha(s,t)$ are continuous functions and partially derivable, the Katugampola conformable partial derivative, denoted by $\frac{\partial^2}{\partial s \partial t}(f)_{\alpha^2}(s,t)$, is defined by

$$\frac{\partial^2}{\partial s \partial t}(f)_{\alpha^2}(s,t) = \lim_{\varepsilon \to 0} \frac{\frac{\partial}{\partial s}(f)_\alpha(s, te^{\varepsilon t^{-\alpha}}) - \frac{\partial}{\partial s}(f)_\alpha(s,t)}{\varepsilon}, \tag{2.4}$$

provided the limits exist, and is called Katugampola conformable partially derivable.

Definition 2.3 (α-conformable integral) Let $\alpha \in (0,1]$, $0 \leq a < b$, and $0 \leq c < d$. A function $f(x,y) : [a,b] \times [c,d] \to \mathbb{R}$ is α-conformable integrable if the integral

$$\int_a^b \int_c^d f(x,y)\, d_\alpha x\, d_\alpha y := \int_a^b \int_c^d (xy)^{\alpha-1} f(x,y)\, dx\, dy \tag{2.5}$$

exists and is finite.

Lemma 2.1 *Let $\alpha \in (0,1]$, $s,t \in [0,\infty)$, and $f(s,t)$, $g(s,t)$ be Katugampola conformable partially differentiable, then*

$$\frac{\partial^2}{\partial s\partial t}(f\circ g)_{\alpha^2}(s,t) = f'\bigl(g(s,t)\bigr)\cdot\frac{\partial^2}{\partial s\partial t}(g)_{\alpha^2}(s,t) + \frac{\partial}{\partial t}(g)_\alpha(s,t)\cdot\frac{\partial}{\partial t}\bigl(f'\bigl(g(s,t)\bigr)\bigr)_\alpha(s,t), \quad (2.6)$$

where f has derivative at $g(s,t)$.

Proof From Definitions 2.1 and 2.2, we obtain

$$\begin{aligned}\frac{\partial}{\partial s}(f\circ g)_\alpha(s,t) &= \frac{\partial}{\partial s}\bigl(f\bigl(g(s,t)\bigr)\bigr)_\alpha(s,t)\\ &= s^{1-\alpha}\frac{\partial}{\partial s}\bigl(f\bigl(g(s,t)\bigr)\bigr)\\ &= s^{1-\alpha}f'\bigl(g(s,t)\bigr)\frac{\partial}{\partial s}\bigl(g(s,t)\bigr)\\ &= f'\bigl(g(s,t)\bigr)\frac{\partial}{\partial s}(g)_\alpha(s,t).\end{aligned}$$

Hence

$$\begin{aligned}\frac{\partial^2}{\partial s\partial t}(f\circ g)_{\alpha^2}(s,t) &= \frac{\partial}{\partial t}\left(\frac{\partial}{\partial s}(f\circ g)_\alpha(s,t)\right)_\alpha(s,t)\\ &= \frac{\partial}{\partial t}\left(f'\bigl(g(s,t)\bigr)\frac{\partial}{\partial s}(g)_\alpha(s,t)\right)_\alpha(s,t)\\ &= t^{1-\alpha}\frac{\partial}{\partial t}\left(f'\bigl(g(s,t)\bigr)\cdot\frac{\partial}{\partial s}(g)_\alpha(s,t)\right)\\ &= t^{1-\alpha}\frac{\partial}{\partial t}\bigl(f'\bigl(g(s,t)\bigr)\bigr)\cdot\frac{\partial}{\partial t}(g)_\alpha(s,t) + t^{1-\alpha}f'\bigl(g(s,t)\bigr)\cdot\frac{\partial}{\partial t}\left(\frac{\partial}{\partial s}(g)_\alpha(s,t)\right)\\ &= \frac{\partial}{\partial t}(g)_\alpha(s,t)\cdot\frac{\partial}{\partial t}\bigl(f'\bigl(g(s,t)\bigr)\bigr)_\alpha(s,t) + f'\bigl(g(s,t)\bigr)\cdot\frac{\partial^2}{\partial s\partial t}(g)_{\alpha^2}(s,t).\end{aligned}$$

This completes the proof. □

This similar chain rule theorem is important, but it is also understood. In order for the reader to better understand this theorem, we give another proof below.

Second proof Let

$$\delta = g\bigl(se^{\varepsilon s^{-\alpha}},t\bigr) - g(s,t).$$

Obviously, if $\varepsilon\to 0$, then $\delta\to 0$. From the hypotheses, we obtain

$$\begin{aligned}\frac{\partial^2}{\partial s\partial t}(f\circ g)_{\alpha^2}(s,t) &= \frac{\partial}{\partial t}\left(\frac{\partial}{\partial s}\bigl(f\bigl(g(s,t)\bigr)\bigr)_\alpha(s,t)\right)_\alpha(s,t)\\ &= \frac{\partial}{\partial t}\left(\lim_{\varepsilon\to 0}\frac{f(g(se^{\varepsilon s^{-\alpha}},t)) - f(g(s,t))}{\varepsilon}\right)_\alpha(s,t)\\ &= \frac{\partial}{\partial t}\left(\lim_{\delta\to 0}\frac{f(g(s,t)+\delta) - f(g(s,t))}{\delta}\cdot\lim_{\varepsilon\to 0}\frac{\delta}{\varepsilon}\right)_\alpha(s,t)\end{aligned}$$

$$= \frac{\partial}{\partial t}\left(f'\big(g(s,t)\big)\frac{\partial}{\partial s}(g)_\alpha(s,t)\right)_\alpha(s,t)$$
$$= f'\big(g(s,t)\big)\cdot\frac{\partial^2}{\partial s\partial t}(g)_{\alpha^2}(s,t) + \frac{\partial}{\partial t}(g)_\alpha(s,t)\cdot\frac{\partial}{\partial t}\big(f'\big(g(s,t)\big)\big)_\alpha(s,t).$$

This completes the proof. □

Theorem 2.1 *Let $p(s,t), u(s,t) : [a,b] \times [c,d] \to \mathbb{R}$ with $a, c \geq 0$ be Katugampola conformable partially derivable such that $\frac{\partial^2}{\partial s\partial t}(p)_{\alpha^2}(s,t) > 0$, $\alpha \in (0,1]$ and $p(a,c) = p(a,d) = p(b,c) = p(b,d) = 0$, and F be derivable on $[0,\infty)$ and F' be increasing. Let φ be a convex and increasing function on $[0,\infty)$, and define*

$$z(s,t) = \int_a^s \int_c^t \frac{\partial^2}{\partial\sigma\partial\tau}(p)_{\alpha^2}(\sigma,\tau)\cdot\varphi\left(\frac{|\frac{\partial^2}{\partial\sigma\partial\tau}(u)_{\alpha^2}(\sigma,\tau)|}{\frac{\partial^2}{\partial\sigma\partial\tau}(p)_{\alpha^2}(\sigma,\tau)}\right) d_\alpha\sigma\, d_\alpha\tau. \tag{2.7}$$

Then

$$\int_a^b \int_c^d \left\{\frac{\partial^2}{\partial s\partial t}(z)_{\alpha^2}(s,t)\cdot F'\left(p(s,t)\cdot\varphi\left(\frac{|u(s,t)|}{p(s,t)}\right)\right) + \frac{\partial}{\partial t}(z)_\alpha(s,t)\cdot\frac{\partial}{\partial t}\big(F'\big(z(s,t)\big)\big)_\alpha(s,t)\right\} d_\alpha s\, d_\alpha t$$
$$\leq F\left(\int_a^b \int_c^d \frac{\partial^2}{\partial s\partial t}(p)_{\alpha^2}(s,t)\cdot\varphi\left(\frac{|\frac{\partial^2}{\partial s\partial t}(u)_{\alpha^2}(s,t)|}{\frac{\partial^2}{\partial s\partial t}(p)_{\alpha^2}(s,t)}\right) d_\alpha s\, d_\alpha t\right), \tag{2.8}$$

where

$$\frac{\partial}{\partial t}\big(F'\big(z(s,t)\big)\big)_\alpha(s,t) = t^{1-\alpha}\frac{\partial}{\partial t}F'\big(z(s,t)\big).$$

Proof Let

$$y(s,t) = \int_a^s \int_c^t \left|\frac{\partial^2}{\partial s\partial t}(u)_{\alpha^2}(\sigma,\tau)\right| d_\alpha\sigma\, d_\alpha\tau$$

such that

$$\frac{\partial^2}{\partial s\partial t}(y)_{\alpha^2}(s,t) = \left|\frac{\partial^2}{\partial s\partial t}(u)_{\alpha^2}(s,t)\right|$$

and $y(s,t) \geq |u(s,t)|$. Since φ is convex and increasing, by using Jensen's inequality, we get

$$\varphi\left(\frac{|u(s,t)|}{p(s,t)}\right) \leq \varphi\left(\frac{y(s,t)}{p(s,t)}\right)$$
$$= \varphi\left(\frac{\int_a^s\int_c^t \frac{\partial^2}{\partial\sigma\partial\tau}(p)_{\alpha^2}(\sigma,\tau)\frac{|\frac{\partial^2}{\partial\sigma\partial\tau}(u)_{\alpha^2}(\sigma,\tau)|}{\frac{\partial^2}{\partial\sigma\partial\tau}(p)_{\alpha^2}(\sigma,\tau)}\, d_\alpha\sigma\, d_\alpha\tau}{\int_a^s\int_c^t \frac{\partial^2}{\partial\sigma\partial\tau}(p)_{\alpha^2}(\sigma,\tau)\, d_\alpha\sigma\, d_\alpha\tau}\right)$$

$$\le \frac{1}{p(s,t)}\int_a^s\int_c^t \frac{\partial^2}{\partial\sigma\partial\tau}(p)_{\alpha^2}(\sigma,\tau)\cdot\varphi\left(\frac{|\frac{\partial^2}{\partial\sigma\partial\tau}(u)_{\alpha^2}(\sigma,\tau)|}{\frac{\partial^2}{\partial\sigma\partial\tau}(p)_{\alpha^2}(\sigma,\tau)}\right)d_\alpha\sigma\, d_\alpha\tau$$
$$= \frac{1}{p(s,t)}\int_a^s\int_c^t \frac{\partial^2}{\partial\sigma\partial\tau}(p)_{\alpha^2}(\sigma,\tau)\cdot\varphi\left(\frac{\frac{\partial^2}{\partial\sigma\partial\tau}(y)_{\alpha^2}(\sigma,\tau)}{\frac{\partial^2}{\partial\sigma\partial\tau}(p)_{\alpha^2}(\sigma,\tau)}\right)d_\alpha\sigma\, d_\alpha\tau. \tag{2.9}$$

From (2.9) and noting that F' is increasing, and Lemma 2.1, (2.7) and in view of that F is derivable on $[0,\infty)$, we obtain

$$\begin{aligned}
&\int_a^b\int_c^d\left\{\frac{\partial^2}{\partial s\partial t}(z)_{\alpha^2}(s,t)\cdot F'\left(p(s,t)\cdot\varphi\left(\frac{|u(s,t)|}{p(s,t)}\right)\right)\right.\\
&\qquad\left.+\frac{\partial}{\partial t}(z)_\alpha(s,t)\cdot\frac{\partial}{\partial t}\big(F'(z(s,t))\big)_\alpha(s,t)\right\}d_\alpha s\, d_\alpha t\\
&\le\int_a^b\int_c^d\left\{\frac{\partial^2}{\partial s\partial t}(z)_{\alpha^2}(s,t)\cdot F'\big(z(s,t)\big)\right.\\
&\qquad\left.+\frac{\partial}{\partial t}(z)_\alpha(s,t)\cdot\frac{\partial}{\partial t}\big(F'(z(s,t))\big)_\alpha(s,t)\right\}d_\alpha s\, d_\alpha t\\
&=\int_a^b\int_c^d\frac{\partial^2}{\partial s\partial t}(F\circ z)_{\alpha^2}(s,t)\,d_\alpha s\, d_\alpha t\\
&=\int_a^b\int_c^d\frac{\partial^2}{\partial s\partial t}\left(F\left(\int_a^s\int_b^t\frac{\partial^2}{\partial\sigma\partial\tau}(p)_{\alpha^2}(\sigma,\tau)\right.\right.\\
&\qquad\left.\left.\cdot\varphi\left(\frac{\frac{\partial^2}{\partial\sigma\partial\tau}(y)_{\alpha^2}(\sigma,\tau)}{\frac{\partial^2}{\partial\sigma\partial\tau}(p)_{\alpha^2}(\sigma,\tau)}\right)d_\alpha\sigma\, d_\alpha\tau\right)\right)_{\alpha^2}(s,t)\,d_\alpha s\, d_\alpha t\\
&=F\left(\int_a^b\int_c^d\frac{\partial^2}{\partial\sigma\partial\tau}(p)_{\alpha^2}(\sigma,\tau)\cdot\varphi\left(\frac{\frac{\partial^2}{\partial\sigma\partial\tau}(y)_{\alpha^2}(\sigma,\tau)}{\frac{\partial^2}{\partial\sigma\partial\tau}(p)_{\alpha^2}(\sigma,\tau)}\right)d_\alpha\sigma\, d_\alpha\tau\right)\\
&=F\left(\int_a^b\int_c^d\frac{\partial^2}{\partial s\partial t}(p)_{\alpha^2}(s,t)\cdot\varphi\left(\frac{|\frac{\partial^2}{\partial s\partial t}(u)_{\alpha^2}(s,t)|}{\frac{\partial^2}{\partial s\partial t}(p)_{\alpha^2}(s,t)}\right)d_\alpha s\, d_\alpha t\right).
\end{aligned}$$

This completes the proof. □

Remark 2.1 Putting $\varphi(x)=x$ in (2.7), we have

$$\begin{aligned}
&\int_a^b\int_c^d\left\{\left|\frac{\partial^2}{\partial s\partial t}(u)_{\alpha^2}(s,t)\right|\cdot F'\big(|u(s,t)|\big)\right.\\
&\qquad\left.+\frac{\partial}{\partial t}(y)_\alpha(s,t)\cdot\frac{\partial}{\partial t}\big(F'(y(s,t))\big)_\alpha(s,t)\right\}d_\alpha s\, d_\alpha t\\
&\le F\left(\int_a^b\int_c^d\left|\frac{\partial^2}{\partial s\partial t}(u)_{\alpha^2}(s,t)\right|d_\alpha s\, d_\alpha t\right),
\end{aligned}\tag{2.10}$$

where

$$y(s,t)=\int_a^s\int_c^t\left|\frac{\partial^2}{\partial s\partial t}(u)_{\alpha^2}(\sigma,\tau)\right|d_\alpha\sigma\, d_\alpha\tau.$$

This inequality (2.10) is just a two-dimensional generalization of the following inequality which was established in [20] and [21]:

$$\int_a^b \left|D_\alpha u(t)\right| \cdot F'\left(\left|u(t)\right|\right) d_\alpha t \le F\left(\int_a^b \left|D_\alpha u(t)\right| d_\alpha t\right).$$

Theorem 2.2 *Let $\alpha, p(s,t), u(s,t), z(s,t), \varphi, F$ be as in Theorem 2.1 and replace $[a,b]\times[c,d]$ by $[0,a]\times[0,b]$. Let h be a concave and increasing function on $[0,\infty)$, and ϕ be a continuous and positive function on $[0,\infty)$ and such that*

$$\frac{\partial^2}{\partial s\partial t}(F\circ z)_{\alpha^2}(s,t)\cdot\phi\left(\frac{1}{\frac{\partial^2}{\partial s\partial t}(z)_{\alpha^2}(s,t)}\right) \le \frac{F(z(a,b))}{z(a,b)}\cdot\phi'\left(\frac{t}{z(a,b)}\right). \tag{2.11}$$

Then

$$\begin{aligned}&\int_0^a\int_0^b\left\{\psi\left(\frac{\partial^2}{\partial s\partial t}(p)_{\alpha^2}(s,t)\cdot\varphi\left(\frac{|\frac{\partial^2}{\partial s\partial t}(u)_{\alpha^2}(s,t)|}{\frac{\partial^2}{\partial s\partial t}(p)_{\alpha^2}(s,t)}\right)\right)\cdot F'\left(p(s,t)\cdot\varphi\left(\frac{|u(s,t)|}{p(s,t)}\right)\right)\right.\\&\qquad\left.+\psi\left(\frac{\partial^2}{\partial s\partial t}(z)_{\alpha^2}(s,t)\right)\cdot\frac{\partial}{\partial t}\left(F'\left(z(s,t)\right)\right)_\alpha(s,t)\cdot\frac{\frac{\partial}{\partial t}(z(s,t))_\alpha(s,t)}{\frac{\partial^2}{\partial s\partial t}(z)_{\alpha^2}(s,t)}\right\}d_\alpha s\,d_\alpha t\\&\quad\le\Phi\left(\int_0^a\int_0^b\frac{\partial^2}{\partial s\partial t}(p)_{\alpha^2}(s,t)\cdot\varphi\left(\frac{|\frac{\partial^2}{\partial s\partial t}(u)_{\alpha^2}(s,t)|}{\frac{\partial^2}{\partial s\partial t}(p)_{\alpha^2}(s,t)}\right)d_\alpha s\,d_\alpha t\right),\end{aligned} \tag{2.12}$$

where

$$\psi(r) = rh\left(\phi\left(\frac{1}{r}\right)\right), \tag{2.13}$$

and

$$\Phi(r) = F(r)\cdot h\left(\frac{1}{r}\int_0^a\int_0^b\phi'\left(\frac{t}{r}\right)d_\alpha s\,d_\alpha t\right). \tag{2.14}$$

Proof From (2.9), we have

$$\varphi\left(\frac{|u(s,t)|}{p(s,t)}\right) \le \frac{z(s,t)}{p(s,t)}. \tag{2.15}$$

From (2.7), (2.15), (2.13) (2 times), Lemma 2.1, and noting that h is a concave, increasing and using reverse Jensen's inequality, and (2.11) and (2.14), we obtain

$$\begin{aligned}&\int_0^a\int_0^b\left\{\psi\left(\frac{\partial^2}{\partial s\partial t}(p)_{\alpha^2}(s,t)\cdot\varphi\left(\frac{|\frac{\partial^2}{\partial s\partial t}(u)_{\alpha^2}(s,t)|}{\frac{\partial^2}{\partial s\partial t}(p)_{\alpha^2}(s,t)}\right)\right)\cdot F'\left(p(s,t)\cdot\varphi\left(\frac{|u(s,t)|}{p(s,t)}\right)\right)\right.\\&\qquad\left.+\psi\left(\frac{\partial^2}{\partial s\partial t}(z)_{\alpha^2}(s,t)\right)\cdot\frac{\partial}{\partial t}\left(F'\left(z(s,t)\right)\right)_\alpha(s,t)\cdot\frac{\frac{\partial}{\partial t}(z)_\alpha(s,t)}{\frac{\partial^2}{\partial s\partial t}(z)_{\alpha^2}(s,t)}\right\}d_\alpha s\,d_\alpha t\\&\quad\le\int_0^a\int_0^b\left\{\psi\left(\frac{\partial^2}{\partial s\partial t}(z)_{\alpha^2}(s,t)\right)\cdot F'\left(z(s,t)\right)\right.\\&\qquad\left.+h\left(\phi\left(\frac{1}{\frac{\partial^2}{\partial s\partial t}(z)_{\alpha^2}(s,t)}\right)\right)\frac{\partial}{\partial t}(z)_\alpha(s,t)\frac{\partial}{\partial t}\left(F'\left(z(s,t)\right)\right)_\alpha(s,t)\right\}d_\alpha s\,d_\alpha t\end{aligned}$$

$$
\begin{aligned}
&= \int_0^a \int_0^b h\left(\phi\left(\frac{1}{\frac{\partial^2}{\partial s \partial t}(z)_{\alpha^2}(s,t)}\right)\right) \cdot \left(\frac{\partial^2}{\partial s \partial t}(z)_{\alpha^2}(s,t) \cdot F'\bigl(z(s,t)\bigr) \right. \\
&\quad \left. + \frac{\partial}{\partial t}(z)_\alpha(s,t) \cdot \frac{\partial}{\partial t}\bigl(F'\bigl(z(s,t)\bigr)\bigr)_\alpha(s,t)\right) d_\alpha s\, d_\alpha t \\
&= \frac{\int_0^a \int_0^b \frac{\partial^2}{\partial s \partial t}(F \circ z)_{\alpha^2}(s,t) \cdot h(\phi(\frac{1}{\frac{\partial^2}{\partial s \partial t}(z)_{\alpha^2}(s,t)}))\, d_\alpha s\, d_\alpha t}{\int_0^a \int_0^b \frac{\partial^2}{\partial s \partial t}(F \circ z)_{\alpha^2}(s,t)\, d_\alpha s\, d_\alpha t} \int_0^a \int_0^b \frac{\partial^2}{\partial s \partial t}(F \circ z)_{\alpha^2}(s,t)\, d_\alpha s\, d_\alpha t \\
&\le h\left(\frac{\int_0^a \int_0^b \frac{\partial^2}{\partial s \partial t}(F \circ z)_{\alpha^2}(s,t) \cdot \phi(\frac{1}{\frac{\partial^2}{\partial s \partial t}(z)_{\alpha^2}(s,t)})\, d_\alpha s\, d_\alpha t}{\int_0^a \int_0^b \frac{\partial^2}{\partial s \partial t}(F \circ z)_{\alpha^2}(s,t)\, d_\alpha s\, d_\alpha t}\right) F\bigl(z(a,b)\bigr) \\
&\le h\left(\frac{\int_0^a \int_0^b \frac{F(z(a,b))}{z(a,b)} \phi'(\frac{t}{z(a,b)})\, d_\alpha s\, d_\alpha t}{F(z(a,b))}\right) F\bigl(z(a,b)\bigr) \\
&= \Phi\bigl(z(a,b)\bigr) \\
&= \Phi\left(\int_0^a \int_0^b \frac{\partial^2}{\partial s \partial t}(p)_{\alpha^2}(s,t) \cdot \varphi\left(\frac{|\frac{\partial^2}{\partial s \partial t}(u)_{\alpha^2}(s,t)|}{\frac{\partial^2}{\partial s \partial t}(p)_{\alpha^2}(s,t)}\right) d_\alpha s\, d_\alpha t\right).
\end{aligned}
$$

This completes the proof. □

Remark 2.2 Putting $\varphi(x) = x$ in (2.12), we have

$$
\begin{aligned}
&\int_0^b \psi\left(\left|\frac{\partial^2}{\partial s \partial t}(u)_{\alpha^2}(s,t)\right|\right) \cdot F'\bigl(\bigl|u(s,t)\bigr|\bigr)\, d_\alpha s\, d_\alpha t \\
&\quad \le \Phi\left(\int_0^a \int_0^b \left|\frac{\partial^2}{\partial s \partial t}(u)_{\alpha^2}(s,t)\right| d_\alpha s\, d_\alpha t\right) - N_\alpha(a,b),
\end{aligned}
\tag{2.16}
$$

where

$$
N_\alpha(a,b) = \int_0^a \int_0^b \psi\left(\frac{\partial^2}{\partial s \partial t}(z)_{\alpha^2}(s,t)\right) \cdot \frac{\partial}{\partial t}\bigl(F'\bigl(z(s,t)\bigr)\bigr)_\alpha(s,t) \cdot \frac{\frac{\partial}{\partial t}(z)_\alpha(s,t)}{\frac{\partial^2}{\partial s \partial t}(z)_{\alpha^2}(s,t)}\, d_\alpha s\, d_\alpha t.
$$

This inequality (2.16) is just a two-dimensional generalization of the following inequality which was established in [21]:

$$
\begin{aligned}
&\int_0^b \psi\left(D_\alpha p(t) \cdot \varphi\left(\frac{|D_\alpha u(t)|}{D_\alpha p(t)}\right)\right) \cdot F'\left(p(t) \cdot \varphi\left(\frac{|u(t)|}{p(t)}\right)\right) d_\alpha t \\
&\quad \le \Phi\left(\int_0^b D_\alpha p(t) \cdot \varphi\left(\frac{|D_\alpha u(t)|}{D_\alpha p(t)}\right) d_\alpha t\right),
\end{aligned}
$$

where $D_\alpha p(t) = D_\alpha(p)(t)$, $\psi(r) = rh(\phi(\frac{1}{r}))$ and $\Phi(r) = F(r)h(\phi(\frac{b}{r}))$, and h is a concave and increasing function on $[0,\infty)$.

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Authors' contributions
C-JZ and W-SC jointly contributed to the main results. All authors read and approved the final manuscript.

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