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Dragomir and Gosa type inequalities on *b*-metric spaces

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Abstract

In this paper, we investigate Dragomir and Gosa type inequalities in the setting of *b*-metric spaces. As an application, we consider some inequalities in *b*-normed spaces. We prove that the inequalities admit geometrical interpretation.

Keywords: Dragomir and Gosa type inequalities; *b*-metric space; Inequality

1 Introduction and preliminaries

It is a natural trend in fixed point theory to refine a standard metric space structure with a weaker one. One of the interesting extensions of the notion of a metric space is the concept of a *b*-metric space which was introduced by Czerwik [8].

Definition 1.1 ([8]) Let X be a nonempty set and $s \geq 1$ a given real number. A mapping $d: X \times X \to [0,\infty)$ is said to be a *b*-metric if for all $x,y,z \in X$ the following conditions are satisfied:

(bM_1) $d(x,y) = 0$ if and only if $x = y$;

(bM_2) $d(x,y) = d(y,x)$ (symmetry);

(bM_3) $d(x,z) \leq s[d(x,y) + d(y,z)]$ (*b*-triangle inequality).

In this case, the pair (X,d) is called a *b*-metric space (with constant s).

Clearly, any metric space is a *b*-metric space (with constant $s = 1$).

Example 1.2 ([10]) Let $X = [0,1]$ and let $d : X \times X \longrightarrow [0,\infty)$ be defined by $d(x,y) = (x-y)^2$. Then, clearly, (X,d) is a *b*-metric space with $s = 2$.

The following is another constructive example of *b*-metric.

Example 1.3 ([1]) Let $X = \{x_i : 1 \leq i \leq M\}$ for some $M \in \mathbb{N}$ and $s \geq 2$. Define $d : X \times X \to \infty$ as

$$d(x_i,x_j) = \begin{cases} 0 & \text{if } i = j, \\ s & \text{if } (i,j) = (1,2) \text{ or } (i,j) = (2,1), \\ 1 & \text{otherwise.} \end{cases}$$

Consequently, we derive that

$$d(x_i,x_j) \leq \frac{s}{2}\big[d(x_i,x_k)+d(x_k,x_j)\big],$$

for all $i,j,k \in \{1,M\}$. Thus, (X,d) forms a b-metric for $s>2$ where the ordinary triangle inequality does not hold.

For more examples for b-metric, we may refer, e.g., to [1–7, 9, 12] and the corresponding references therein.

Example 1.4 (see, e.g., [6]) The space $L^p[0,1]$ (where $0<p<1$) of all real functions $x(t)$, $t\in[0,1]$ such that $\int_0^1 |x(t)|^p\,dt<\infty$, together with the functional

$$d(x,y) := \left(\int_0^1 \big|x(t)-y(t)\big|^p\,dt\right)^{1/p}, \quad \text{for each } x,y\in L^p[0,1],$$

is a b-metric space. Notice that $s=2^{1/p}$.

2 Main result

We start this section by recalling an interesting inequality that was proposed by Dragomir and Gosa in [11]. In what follows we investigate their inequality in the setting of a more general structure, namely that of b-metric spaces.

Theorem 2.1 *Let (X,d) be a b-metric space with constant $s\geq 1$, and $x_i\in X$, $p_i\geq 0$ ($i\in\{1,2,\dots,n\}$) with $\sum_{i=1}^n p_i=\frac{1}{s}$. Then we have*

$$\sum_{1\leq i<j\leq n} p_ip_jd(x_i,x_j) \leq \inf_{x\in X}\left[\sum_{i=1}^n p_id(x_i,x)\right]. \tag{1}$$

The inequality is sharp in the sense that the constant $c=1$ in front of the infimum cannot be replaced by a smaller constant.

Proof Using the b-triangle inequality, for any $x\in X$, $i,j\in\{1,2,\dots,n\}$ we have

$$d(x_i,x_j)\leq s\big[d(x_i,x)+d(x,x_j)\big]. \tag{2}$$

If we multiply (2) by p_i, p_j and sum over i and j from 1 to n, we get

$$\sum_{i,j=1}^n p_ip_jd(x_i,x_j)\leq s\left[\sum_{i,j=1}^n p_ip_j\big[d(x_i,x)+d(x,x_j)\big]\right].$$

Note that by symmetry we have

$$\sum_{i,j=1}^n p_ip_jd(x_i,x_j) = 2\sum_{1\leq i<j\leq n} p_ip_jd(x_i,x_j). \tag{3}$$

Now, using the condition $\sum_{i=1}^{n} p_i = \frac{1}{s}$, we can easily deduce that

$$\sum_{i,j=1}^{n} p_i p_j \bigl[d(x_i,x) + d(x,x_j)\bigr] = \frac{2}{s} \sum_{i=1}^{n} p_i d(x_i,x).$$

So, from (3) we have

$$\begin{aligned} \sum_{1\le i<j\le n} p_i p_j d(x_i,x_j) &= \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j d(x_i,x_j) \\ &\le \frac{s}{2} \left[\sum_{i,j=1}^{n} p_i p_j \bigl[d(x_i,x) + d(x,x_j)\bigr] \right] \\ &= \sum_{i=1}^{n} p_i d(x_i,x). \end{aligned}$$

Therefore,

$$\sum_{1\le i<j\le n} p_i p_j d(x_i,x_j) \le \sum_{i=1}^{n} p_i d(x_i,x)$$

for any $x \in X$. Using the fact that the infimum is the greatest lower bound, we deduce (1).

Now, suppose that there exists $c > 0$ such that

$$\sum_{1\le i<j\le n} p_i p_j d(x_i,x_j) \le c \inf_{x\in X} \left[\sum_{i=1}^{n} p_i d(x_i,x) \right];$$

and choose $n = 2$, $p_1 = p$ and $p_2 = 1 - p$ where $p \in (0,1)$. Then,

$$p(1-p)d(x_1,x_2) \le c\bigl[pd(x_1,x) + (1-p)d(x,x_2)\bigr]. \tag{4}$$

If we let $x = x_1$ in (4), we get

$$p(1-p)d(x_1,x_2) \le c(1-p)d(x_1,x_2).$$

As $d(x_1,x_2) > 0$ and $1 - p > 0$, so $p \le c$ for any $p \in (0,1)$. Using the fact that the supremum is the least upper bound, we deduce that $c \ge 1$. □

The following corollary is a generalization of Corollary 1 in [11] to the case of a b-metric space.

Corollary 2.2 *Let (X,d) be a b-metric space with constant $s \ge 1$, and $x_i \in X$, $i \in \{1,2,\ldots,n\}$, then*

$$\sum_{1\le i<j\le n} d(x_i,x_j) \le \frac{n}{s} \inf_{x\in X} \left[\sum_{i=1}^{n} d(x_i,x) \right].$$

The proof follows directly by taking $p_i = \frac{1}{ns}$, $i \in \{1, 2, \dots, n\}$ in the previous theorem.

The above corollary can be interpreted geometrically as follows: The sum of all edges and diagonals of a polygon with n vertices in a b-metric space is less than or equal to $\frac{n}{s}$-times the sum of the distances from any arbitrary point in the space to its vertices.

The next corollary is a generalization of Corollary 2 in [11] in the framework of b-metric spaces.

Corollary 2.3 *Let (X, d) be a b-metric space with constant s and $x_i \in X$, $i \in \{1, 2, \dots, n\}$. If there exist $z \in X$ and $r > 0$ such that the closed ball $\overline{B}(z, r) = \{y \in X : d(z, y) \le r\}$ contains all the points x_i, then for any $p_i \ge 0$ with $\sum_{i=1}^{n} p_i = \frac{1}{s}$ we have*

$$\sum_{1 \le i < j \le n} p_i p_j d(x_i, x_j) \le \frac{r}{s}.$$

Proof Using (1) we have

$$\begin{aligned}\sum_{1 \le i < j \le n} p_i p_j d(x_i, x_j) &\le \inf_{x \in X} \left[\sum_{i=1}^{n} p_i d(x_i, x) \right] \\ &\le \sum_{i=1}^{n} p_i d(x_i, z) \\ &\le \frac{r}{s}.\end{aligned}$$

□

3 Applications

In this section we define a new notion of a b-normed space and study some of its properties.

Definition 3.1 Let X be a vector space over a field K and let $s \ge 1$ be a constant. A function $\| \cdot \|_b : X \longrightarrow [0, \infty)$ is said to be a b-norm if the following conditions hold for every $x, y \in X$, $c \in K$:

(Nb1) $\|x\|_b \ge 0$;

(Nb2) $\|x\|_b = 0 \Longleftrightarrow x = 0$;

(Nb3) $\|cx\|_b = |c|^{\log_2 s + 1} \|x\|_b$ (b-homogeneity);

(Nb4) $\|x + y\|_b \le s[\|x\|_b + \|y\|_b]$ (b-norm triangle inequality).

In this case $(X, \| \cdot \|_b)$ is called a b-normed space with constant s.

Here we give an example of a b-normed space.

Example 3.2 Let $X = \mathbb{R}$ and define $\| \cdot \|_b : X \longrightarrow [0, \infty)$ by $\|x\|_b = |x|^p$ where $p \in (1, \infty)$, then, using the relation $(x + y)^p \le 2^{p-1}(x + y)$, we can easily deduce that $(X, \| \cdot \|_b)$ is a b-normed space with constant $s = 2^{p-1}$.

Remark 3.3 Let $(X, \| \cdot \|_b)$ be a b-normed space with constant $s \ge 1$, $x_i \in X$, $i \in \{1, \dots, n\}$. Then it is easy to prove the following generalized b-triangle inequality:

$$\left\| \sum_{i=1}^{n} x_i \right\| \le \sum_{i=1}^{n} s^i \|x_i\|.$$

Remark 3.4 Any b-norm with $s \geq 1$ defines a b-metric as follows:

$$d(x,y) = \|x-y\|_b.$$

The question now is the following: Is any b-metric induced from a b-norm? The following remark can answer this question.

Remark 3.5 Let X be a vector space over a field K. Any b-metric $d : X \times X \longrightarrow [0,\infty)$ with constant $s \geq 1$ induced from a b-norm must satisfy the following properties for each $x,y,z \in X$, $c \in K$:

(i) $d(x+z,y+z) = d(x,y)$ (translation invariance);

(ii) $d(cx,cy) = |c|^{\log_2 s+1} d(x,y)$ (b-homogeneity).

Proposition 3.6 *A b-homogeneous translation invariant b-metric* $d : X \times X \longrightarrow [0,\infty)$ *with constant* $s \geq 1$ *can define a b-norm* $\|\cdot\|_b : X \longrightarrow [0,\infty)$ *as follows*:

$$\|x\|_b = d(x,0) \quad \forall x \in X.$$

Proof Clearly, (Nb1) and (Nb2) are satisfied.

As d is homogeneous, $\|cx\| = d(cx,0) = |c|^{\log_2 s+1} d(x,0) = |c|^{\log_2 s+1}\|x\|_b$.

As d is translation invariant,

$$\begin{aligned}\|x+y\|_b = d(x+y,0) &\leq s\big[d(x+y,x) + d(x,0)\big] \\ &= s\big[d(y,0) + d(x,0)\big] \\ &= s\big[\|x\|_b + \|y\|_b\big],\end{aligned}$$

which prove (Nb3) and (Nb4), respectively. □

Now, we rewrite inequality (1) in the sense of b-normed spaces and obtain some corollaries.

If $(X,\|\cdot\|_b)$ is a b-normed space with constant $s \geq 1$, $x_i \in X$, and $p_i \geq 0$, $i \in \{1,\dots,n\}$ with $\sum_{i=1}^n p_i = \frac{1}{s}$, then by (1) we have

$$\sum_{1\leq i<j\leq n} p_i p_j \|x_i - x_j\| \leq \inf_{x\in X}\left[\sum_{i=1}^n p_i \|x_i - x_j\|\right]. \tag{5}$$

The following proposition is a generalization of Proposition 2 in [11] to the case of a b-normed space.

Proposition 3.7 *Let* $(X,\|\cdot\|_b)$ *be a b-normed space with constant* $s \geq 1$, $x_i \in X$ *and* $p_i \geq 0$, $i \in \{1,\dots,n\}$ *with* $\sum_{i=1}^n p_i = \frac{1}{s}$. *Let* $x_p = \sum_{i=1}^n p_i x_i$, *then*

$$\frac{1}{2}\sum_{i=1}^n p_i \|x_i - x_p\| \leq s^n \sum_{1\leq i<j\leq n} p_i p_j \|x_i - x_j\| \leq s^n \sum_{i=1}^n p_i \|x_i - x_p\|. \tag{6}$$

Proof As the infimum is a lower bound, the second part of inequality (6) is trivial. For the first part, we use a generalized b-norm inequality as follows:

$$\begin{aligned}
\frac{1}{2}\sum_{i=1}^{n} p_i\|x_i - x_p\| &= \frac{1}{2}\sum_{i=1}^{n} p_i\left\|x_i - \sum_{j=1}^{n} p_j x_j\right\| \\
&= \frac{1}{2}\sum_{i=1}^{n} p_i\left\|\sum_{j=1}^{n}(x_i - p_j x_j)\right\| \\
&\leq \frac{1}{2}\sum_{i,j=1}^{n} p_i s^j\|x_i - p_j x_j\| \\
&\leq \frac{s^n}{2}\sum_{i,j=1}^{n} p_i p_j\|x_i - x_j\| \\
&= s^n \sum_{1\leq i<j\leq n} p_i p_j\|x_i - x_j\|,
\end{aligned}$$

which completes the proof. □

We have the following corollary, which has a nice geometric interpretation.

Corollary 3.8 *Let* $(X, \|\cdot\|_b)$ *be a b-normed space with constant* $s \geq 1$ *and* $x_i \in X$, $i \in \{1, \ldots, n\}$. *If* $\overline{x} = \frac{x_1+\cdots+x_n}{n}$ *is the gravity center of the vectors* $\{x_1, \ldots, x_n\}$, *then we have*

$$\frac{n}{2}\sum_{i=1}^{n}\|x_i - \overline{x}\| \leq s^n \sum_{1\leq i<j\leq n}\|x_i - x_j\| \leq ns^n \sum_{i=1}^{n}\|x_i - \overline{x}\|.$$

Geometrically, the last corollary means that the sum of the edges and diagonals of a polygon with n vertices in a b-normed space is less than or equal to n-times the sum of the distances from the gravity center to its vertices and greater than or equal to $\frac{n}{2s^n}$-times this quantity.

4 Conclusion

Similarly, we can generalize more inequalities on metric and normed spaces.

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