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Monotonicity formulas for the first eigenvalue of the weighted p-Laplacian under the Ricci-harmonic flow

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Abstract

Let $\Delta_{p,\phi}$ be the weighted p-Laplacian defined on a smooth metric measure space. We study the evolution and monotonicity formulas for the first eigenvalue, $\lambda_1 = \lambda(\Delta_{p,\phi})$, of $\Delta_{p,\phi}$ under the Ricci-harmonic flow. We derive some monotonic quantities involving the first eigenvalue, and as a consequence, this shows that λ_1 is monotonically nondecreasing and almost everywhere differentiable along the flow existence.

MSC: Primary 53C21; 53C44; secondary 58C40

Keywords: Ricci harmonic flow; Laplace-Beltrami operator; Eigenvalue; Monotonicity; Ricci solitons

1 Introduction

In this paper we study evolution, monotonicity, and differentiability of the first weighted p-eigenvalue on an n-dimensional compact Riemannian manifold (M,g) equipped with measure $d\mu$, whose metric $g = g(t)$ evolves along the Ricci-harmonic flow (RHF)

$$\begin{cases} \frac{\partial}{\partial t} g(x,t) = -2Rc(x,t) + 2\alpha \nabla\phi(x,t) \otimes \nabla\phi(x,t), \\ \frac{\partial}{\partial t}\phi(x,t) = \Delta_g \phi(x,t). \end{cases} \tag{1.1}$$

Here $\phi(x,t) =: \phi : M \times [0,\infty) \to \mathbb{R}$ is a one-parameter family of smooth functions at least C^2 in x and C^1 in t, $\otimes$ is the tensor product, Rc is the Ricci curvature tensor of (M,g), ∇ is the gradient operator, α is a nonincreasing constant function of time bounded below by $\alpha_n > 0$ in time, and Δ is the Laplace–Beltrami operator on M. Sometimes, we shall refer to (1.1) simply as RHF. System (1.1) was first studied by List [18] with motivation coming from general relativity. It was generalized by Müller [20] to the situation where $\phi : (M,g) \to (N,h)$, $((N,h)$ is a compact Riemannian manifold endowed with static metric $h)$ and ϕ satisfies Eells and Sampson's heat flow [9] for harmonic map. Indeed, if ϕ is a constant function, the flow degenerates to the well-known Hamilton's Ricci flow [12].

In recent time, getting useful information about behaviors of eigenvalues of geometric operators on evolving manifolds has gained more popularity among researchers. This information usually turns to powerful tools in the study of geometry and topology of the underlying manifolds. Perelman [21] recorded a great success by proving that the first nonzero eigenvalue of $-4\Delta + R$ is nondecreasing along the Ricci flow via the monotonicity formula for his energy functional $\mathcal{F}$. Not quite long after Perelman's paper [21], Cao [7] extended Perelman's result to the first eigenvalue of $\Delta + \frac{R}{2}$ on the condition that the curvature operator is nonnegative. Later, Li [15] proved the same result without any curvature assumption. The following papers [8, 16] are some results along this idea. Recently, the first author studied the evolution and monotonicity of the first eigenvalue of p-Laplacian and weighted Laplacian in [1] and [2], respectively. He found some monotonic quantities under the respective flows. Azami [5] (see also [6]) extended these results to the setting of Ricci–Bourguignon flow. In [11] and [10], the authors studied the evolution of the first eigenvalue of $\Delta_\phi + \frac{R}{2}$ under the Yamabe flow and the Ricci flow, respectively, and they also obtained some monotonic quantities under these flows. For similar results, see [3, 4, 17, 25, 26] and the references therein. Motivated by the above works, in this paper we extend the results in [1] and [2] to the first eigenvalue of weighted p-Laplacian under the Ricci-harmonic flow.

The plan of the paper is as follows. In Sect. 2, we give background information in terms of basic notation and relevant definitions. In Sect. 3 we discuss regularization procedure for the nonlinear and degenerate operator and then present some evolution equations that will be useful in the proofs of the main results. In the last section, we state and prove the main results of the paper. Here, we discuss the time evolution and monotonicity of $\lambda_1(\Delta_{p,\phi})$ without differentiability assumption on its corresponding eigenfunction. In fact, the differentiability of the eigenvalue is a consequence of the monotonicity formula derived.

2 Notation and preliminaries

By standard notations in the theory of Ricci-harmonic flow, we denote a symmetric 2-tensor by $Sc := Rc - \alpha\nabla\phi \otimes \nabla\phi$, its components by $S_{ij} := R_{ij} - \alpha\phi_i\phi_j$, and its trace by $S := R - \alpha|\nabla\phi|^2$, where R_{ij} are the Ricci tensor's components, R is the scalar curvature of (M,g), and $\phi_i = \nabla_i\phi = \frac{\partial}{\partial x^i}\phi$.

In most cases, our calculations will be performed in a local coordinate system $\{x^i\}_1^n$, where repeated indices are summed. The Riemannian metric $g(x)$ at any point $x \in M$ is a bilinear symmetric positive definite matrix written in local coordinates as

$$g(x) = g_{ij}\,dx^i\,dx^j.$$

We denote the Laplace–Beltrami operator, p-Laplacian, weighted Laplacian, and weighted p-Laplacian on (M,g) by Δ, Δ_p, Δ_ϕ, and $\Delta_{p,\phi}$, respectively. For instance, in a local coordinate system

$$\Delta = g^{ij}\left(\frac{\partial^2}{\partial x^i\,\partial x^j} - \Gamma_{ij}^k\frac{\partial}{\partial x^k}\right)$$

with respect to the Christoffel symbols Γ_{ij}^k, where $g^{ij} = (g_{ij})^{-1}$ is the inverse matrix. We denote dv as the Riemannian volume measure on (M,g) and $d\mu := e^{-\varphi(x)}\,dv$, the weighted volume measure, where $\varphi \in C^\infty(M)$. Throughout, M will be assumed to be closed (compact without boundary), except if otherwise stated.

Let $f : M \to \mathbb{R}$ be a smooth function.

1. For $p \in (1, +\infty)$, the p-Laplacian of f is defined as

$$\begin{aligned}\Delta_p f &= \operatorname{div}\bigl(|\nabla f|^{p-2}\nabla f\bigr)\\ &= |\nabla f|^{p-2}\Delta f + (p-2)|\nabla f|^{p-4}\operatorname{Hess} f(\nabla f, \nabla f),\end{aligned}$$

   where div is the divergence operator, the adjoint of gradient for the L^2-norm induced by the metric on the space of differential forms. When $p = 2$, Δ_p is the usual Laplace–Beltrami operator.

2. For the weighted volume measure $d\mu = e^{-\phi}\,dv$, the weighted-Laplacian is defined by

$$\Delta_\phi f := e^\phi \operatorname{div}\bigl(e^{-\phi}\nabla f\bigr) = \Delta f - \langle \nabla\phi, \nabla f\rangle,$$

   which is a symmetric diffusion operator on $L^2(M, g, d\mu)$ and self-adjoint with respect to the measure in the sense of integration by parts formula

$$\int_M \Delta_\phi u v\,d\mu = -\int_M \langle \nabla u, \nabla v\rangle\,d\mu = \int_M u\Delta_\phi v\,d\mu$$

   for any $u, v \in C^\infty(M)$ (M is closed). When ϕ is constant, the weighted Laplacian is just the Laplace–Beltrami operator.

3. The weighted p-Laplacian generalizes the p-Laplacian and the weighted Laplacian. It is defined by

$$\Delta_{p,\phi} := e^\phi \operatorname{div}\bigl(e^{-\phi}|\nabla f|^{p-2}\nabla f\bigr) = \Delta_p f - |\nabla f|^{p-2}\langle \nabla\phi, \nabla f\rangle.$$

   When $p = 2$, this is just the weighted Laplacian, and when ϕ is a constant, it is just the p-Laplacian.

The mini-max principle also holds for the weighted p-Laplacian where its first nonzero eigenvalue is characterized as follows:

$$\lambda_1(t) = \inf_f \left\{\int_M |\nabla f|^p\,d\mu : \int_M |f|^p\,d\mu = 1, f \neq 0, f \in W^{1,p}(M, g, d\mu)\right\} \tag{2.1}$$

satisfying the constraints $\int_M |f|^{p-2} f\,d\mu = 0$, where $W^{1,p}(M, g, d\mu)$ is the completion of $C^\infty(M, g, d\mu)$ with respect to the norm

$$\|f\|_{W^{1,p}} = \left(\int_M |f|^p\,d\mu + \int_M |\nabla f|^p\,d\mu\right)^{\frac{1}{p}}.$$

The infimum in (2.1) is achieved by $f \in W^{1,p}$ satisfying the Euler–Lagrange

$$\int_M |\nabla f|^{p-2}\langle \nabla f, \nabla\psi\rangle\,d\mu - \lambda_1 \int_M |f|^{p-2}\langle f, \psi\rangle\,d\mu = 0 \tag{2.2}$$

for all $\psi \in C_0^\infty(M)$ in the sense of distribution.

We need to compute evolution of $\lambda_1(t)$ but we know that it is nonlinear in general. Even we do not know whether $\lambda_1(t)$ or its corresponding eigenfunction is C^1-differentiable along Ricci-harmonic flow. It has been pointed out that differentiability for the case $p = 2$

under geometric flow is a consequence of eigenvalue perturbation theory, see for instance [13]. To overcome this difficulty, we will use the approach of [7], see also Wu [25] and Wu, Wang, and Zheng [24], as used under the Ricci flow. The details will be discussed in Sect. 4.

3 Regularization procedure and basic lemma

Firstly, we introduce the linearized operator of the weighted p-Laplacian on function $h \in C^\infty(M)$ defined pointwise at the points $\nabla h \neq 0$ [23]

$$\begin{aligned}\mathcal{L}_\phi(\tilde{f}) &:= e^\phi \operatorname{div}\bigl(e^{-\phi}|\nabla h|^{p-2}G(\nabla\tilde{f})\bigr)\\ &= |\nabla h|^{p-2}\Delta_\phi\tilde{f} + (p-2)|\nabla h|^{p-2}\operatorname{Hess}\tilde{f}(\nabla h,\nabla h) + (p-2)\Delta_{p,\phi}h\frac{\langle\nabla h,\nabla\tilde{f}\rangle}{|\nabla h|^2}\\ &\quad + 2(p-2)|\nabla h|^{p-4}\operatorname{Hess}\tilde{f}\left(\nabla h, \nabla\tilde{f} - \frac{\nabla h}{|\nabla h|}\left\langle\frac{\nabla h}{|\nabla h|},\nabla\tilde{f}\right\rangle\right)\end{aligned}$$

for a smooth function $\tilde{f}$ on M, where G can be viewed as a tensor defined as

$$G := \mathrm{Id} + (p-2)\frac{\nabla h\otimes\nabla h}{|\nabla h|^2}.$$

Notice that $\mathcal{L}_\phi$ is positive definite for $p > 1$ and strictly elliptic in general at these points ($\nabla h \neq 0$), and that the sum of its second order part is

$$\mathbb{L}_\phi\tilde{f} := |\nabla h|^{p-2}\Delta_\phi\tilde{f} + (p-2)|\nabla h|^{p-2}\operatorname{Hess}\tilde{f}(\nabla h,\nabla h)$$

with

$$\mathbb{L}_\phi h = \Delta_{p,\phi}h.$$

When $p \neq 2$, the weighted p-Laplacian degenerates or is singular at points $\nabla f = 0$. In this case ε-regularization technique is usually applied by replacing the linearized operator with its approximate operator, see [14, 22] for examples. For $\varepsilon > 0$, we define an approximate operator $\mathcal{L}_{\phi,\varepsilon} := \Delta_{p,\phi,\varepsilon}$ for a smooth function f_ε by

$$\Delta_{p,\phi,\varepsilon}f_\varepsilon = e^\phi \operatorname{div}\bigl(e^{-\phi}A_\varepsilon^{\frac{p-2}{2}}\nabla f_\varepsilon\bigr)$$

with the following characterization:

$$\lambda_\varepsilon = \inf_f\left\{\int_M A_\varepsilon^{\frac{p}{2}}\,d\mu : \int_M |f_\varepsilon|^p\,d\mu = 1, \int_M |f_\varepsilon|^{p-2}f_\varepsilon\,d\mu = 0\right\},$$

where $A_\varepsilon = |\nabla f_\varepsilon|^2 + \varepsilon$.

It has been shown in [22] that the infimum above is achieved by a C^∞ eigenfunction f_ε satisfying

$$\Delta_{p,\phi,\varepsilon}f_\varepsilon = -\lambda_\varepsilon|f_\varepsilon|^{p-2}f_\varepsilon$$

with $\lambda_\varepsilon = \int_M A_\varepsilon^{\frac{p-2}{2}} |\nabla f_\varepsilon|^2 \, d\mu$ by using standard elliptic theory. Taking the limit as $\varepsilon \searrow 0$, we then obtain a continuous weak solution $\lambda_1 = \lim_{\varepsilon\searrow 0} \lambda_\varepsilon$ and $f = \lim_{\varepsilon\searrow 0} f_\varepsilon$.

The following basic evolution formulas will be used in the proof of evolution of $\lambda_1(t)$ under the Ricci-harmonic flow.

Lemma 3.1 *Suppose that $(M, g(t), \phi(t), d\mu)$, $t \in [0, T]$, $T < \infty$ solves RHF (1.1). Then, for any $f \in C^\infty(M)$, we have the following formulas:*

(1) $\frac{\partial}{\partial t}|\nabla f|^p = p|\nabla f|^{p-2}(S^{ij}\nabla_i f \nabla_j f + g^{ij}\nabla_i f \nabla_j f_t)$,

(2) $\frac{\partial}{\partial t}|\nabla f|^{p-2} = (p-2)|\nabla f|^{p-4}(S^{ij}\nabla_i f \nabla_j f + g^{ij}\nabla_i f \nabla_j f_t)$,

(3) $\frac{\partial}{\partial t}(\Delta_p f) = 2S^{ij}\nabla_i(Z\nabla_j f) + g^{ij}\nabla_i(Z_t\nabla_j f) + g^{ij}\nabla_i(Z\nabla_j f_t) - 2\alpha Z(\Delta\phi)g^{ij}\nabla_i\phi\nabla_j f$, *and*

(4)

$$\begin{aligned}\frac{\partial}{\partial t}(\Delta_{p,\phi} f) = {} & 2S^{ij}\nabla_i(Z\nabla_j f) + g^{ij}\nabla_i(Z_t\nabla_j f) + g^{ij}\nabla_i(Z\nabla_j f_t) \\ & - 2\alpha Z(\Delta\phi)g^{ij}\nabla_i\phi\nabla_j f - Z_t\langle\nabla\phi, \nabla f\rangle - Z\langle\nabla\phi_t, \nabla f\rangle, \\ & - Z\langle\nabla\phi, \nabla f_t\rangle - 2S^{ij}Z\nabla_i\phi\nabla_j f,\end{aligned}$$

where $Z := |\nabla f|^{p-2}$ and $f_t = \frac{\partial}{\partial t} f$.

Proof The proofs of formulas (1), (2), and (3) are contained in [1, Lemma 2.2]. For completeness, we sketch the proofs of (2) and (4) here.

Recall that $\frac{\partial}{\partial t} g^{ij} = 2S^{ij}$ (see [1]) and $A_\varepsilon := |\nabla f_\varepsilon|^2 + \varepsilon = g^{ij}\nabla_i f_\varepsilon \nabla_j f_\varepsilon + \varepsilon$. Thus

$$\begin{aligned}\frac{\partial}{\partial t} A_\varepsilon^{\frac{p-2}{2}} &= \frac{p-2}{2} A_\varepsilon^{\frac{p-4}{2}} \frac{\partial}{\partial t} A_\varepsilon \\ &= \frac{p-2}{2} A_\varepsilon^{\frac{p-4}{2}} \big(2S^{ij}\nabla_i f_\varepsilon \nabla_j f_\varepsilon + 2g^{ij}\nabla_i f_\varepsilon \nabla_j (f_\varepsilon)_t\big) \\ &= (p-2)A^{\frac{p-4}{2}} \big(S^{ij}\nabla_i f_\varepsilon \nabla_j f_\varepsilon + g^{ij}\nabla_i f_\varepsilon \nabla_j (f_\varepsilon)_t\big).\end{aligned}$$

Sending $\varepsilon \searrow 0$, we arrive at formula (2). To prove formula (4), we write

$$\Delta_{p,\phi,\varepsilon} f_\varepsilon = \Delta_{p,\varepsilon} f_\varepsilon - A_\varepsilon^{\frac{p-2}{2}} \langle\nabla\phi, \nabla f_\varepsilon\rangle.$$

Then

$$\frac{\partial}{\partial t}(\Delta_{p,\phi,\varepsilon} f_\varepsilon) = \frac{\partial}{\partial t}(\Delta_{p,\varepsilon} f_\varepsilon) - \frac{\partial}{\partial t}\big(A_\varepsilon^{\frac{p-2}{2}} g^{ij}\nabla_i\phi\nabla_j f_\varepsilon\big).$$

Note that $\frac{\partial}{\partial t}(\Delta_{p,\varepsilon} f_\varepsilon) = \frac{\partial}{\partial t}(\Delta_p f)$ as $\varepsilon \searrow 0$. This is formula (3). For the second term, we have

$$\begin{aligned}\frac{\partial}{\partial t}\big(A_\varepsilon^{\frac{p-2}{2}} g^{ij}\nabla_i\phi\nabla_j f_\varepsilon\big) &= \frac{\partial}{\partial t}\big(Z_\varepsilon g^{ij}\nabla_i\phi\nabla_j f_\varepsilon\big) \\ &= (Z_\varepsilon)_t\langle\nabla\phi, \nabla f_\varepsilon\rangle + Z_\varepsilon\langle\nabla\phi_t, \nabla f_\varepsilon\rangle + Z_\varepsilon\big\langle\nabla\phi, \nabla(f_\varepsilon)_t\big\rangle \\ &\quad + 2S^{ij}Z_\varepsilon\nabla_\phi\nabla_j f_\varepsilon,\end{aligned}$$

where $Z_\varepsilon = A_\varepsilon^{\frac{p-2}{2}}$. Combining the computations and letting $\varepsilon \searrow 0$, we arrive at the result. □

4 Evolution of λ_1 and monotonic quantities

In this section, we derive an evolution formula for the first nonzero eigenvalue of $\Delta_{p,\phi}$ and show that λ_1 is monotone nondecreasing along the Ricci-harmonic flow as a corollary. We also obtain some monotonic quantities involving λ_1 which are also nondecreasing along the flow. In order to do these, we need to compute time derivatives of λ_1 and its corresponding eigenfunction. Unfortunately, we do not know whether λ_1 or its corresponding eigenfunction ($p \neq 2$) is C^1 or not along the flow. As we remarked earlier, we adopt Cao's approach [8] (see also [25] and [24]) to assume that $\lambda_1(f(t),t) = \lambda_1(t)$ and that $f(t)$ and $\lambda_1(f(t),t)$ are smooth. Precisely, let $(M, g(t), \phi(t), d\mu)$, $t \in [0,T]$ be a smooth compact solution of (1.1). Define a general smooth function as follows:

$$\lambda_1\big(f(t),t\big) := \int_M f(t)\Delta_{p,\phi} f(t)\,d\mu = \int_M \big|\nabla f(t)\big|^p\,d\mu, \tag{4.1}$$

where $f(t)$ is a smooth function satisfying the normalization condition

$$\int_M \big|f(t)\big|^p\,d\mu = 1 \quad \text{and} \quad \int_M \big|f(t)\big|^{p-2} f(t)\,d\mu = 0. \tag{4.2}$$

By this, we claim that there exists a smooth function $f(t_0)$ at time $t = t_0 \in [0,T]$ satisfying (4.1). To see this claim, we first assume that at $t = t_0, f(t_0)$ is the eigenfunction corresponding to $\lambda_1(t_0)$ of $\Delta_{p,\phi}$, which implies

$$\int_M \big|f(t_0)\big|^p\,d\mu = 1 \quad \text{and} \quad \int_M \big|f(t_0)\big|^{p-2} f(t_0)\,d\mu = 0.$$

Then we consider the following smooth function:

$$u(t) = f(t_0)\left(\frac{\det(g(t_0))}{\det(g(t))}\right)^{\frac{1}{2(p-2)}} \tag{4.3}$$

under the Ricci-harmonic flow $g(t)$. We normalize this smooth function

$$f(t) = \frac{u(t)}{(\int_M |u(t)|^p\,d\mu_{g(t)})^{\frac{1}{p}}} \tag{4.4}$$

under the flow $g(t)$. By (4.4) we can easily check that $f(t)$ satisfies (4.2). Note that in general $\lambda_1(f,t)$ is not equal to $\lambda_1(t)$. But at time $t = t_0$, if $f(t_0)$ is the eigenfunction of the first eigenvalue $\lambda_1(t_0)$, then we conclude that

$$\lambda_1\big(f(t_0),t_0\big) = \lambda_1(t_0)$$

and that

$$\frac{d}{dt}\lambda_1\big(f(t_0),t_0\big) = \frac{d}{dt}\lambda_1(t_0)$$

at some time $t = t_0$.

We are now set to state the main results of this section.

Theorem 4.1 *Let $(M, g(t), \phi(t), d\mu)$, $t \in [0, T]$, $T < \infty$ solve RHF (1.1) on a closed Riemannian manifold M. Let $\lambda_1(t)$ be the first nonzero eigenvalue of the weighted p-Laplacian $\Delta_{p,\phi}$ and $f(x,t)$ its corresponding eigenfunction. Then $\lambda_1(t)$ evolves by*

$$\begin{aligned}\frac{d}{dt}\lambda_1(t) = \lambda_1(t)\int_M (S+\phi_t)|f|^p\,d\mu - \int_M (S+\phi_t)|\nabla f|^p\,d\mu \\ + p\int_M |\nabla f|^{p-2}S^{ij}\nabla_i f\nabla_j f\,d\mu \end{aligned} \tag{4.5}$$

for all time $t \in [0, T]$, $T < \infty$.

Corollary 4.2 *Under the assumption of Theorem 4.1. Furthermore, if $S_{ij} \geq \beta(S + \Delta\phi)g_{ij}$, $\beta > \frac{1}{p}$ is a constant. Then*

$$\lambda_1(t_2) \geq \lambda_1(t_1) + \int_{t_1}^{t_2} \Theta\big(g(t), f(x,t)\big)\,dt, \tag{4.6}$$

where

$$\Theta\big(g(t), f(x,t)\big) = \lambda_1(t)\int_M (S+\Delta\phi)|f|^p\,d\mu + (\beta p - 1)\int_M (S+\Delta\phi)|\nabla f|^p\,d\mu$$

for $t_1 < t_2$, $t_1, t_2 \in [0, T]$, $T < \infty$.

Note that (4.5) in the theorem above is a general formula to describe the evolution of $\lambda_1(t)$ under the Ricci-harmonic flow. Under some technical assumptions, we can obtain some monotonicity quantities.

Set $S_{\min}(0) = \min_{x\in M} S(x, 0)$, i.e., the minimum of $S(x,t)$ with respect to $g(t)$ and $f(x,t)$ at $t = 0$.

Theorem 4.3 *Let $(M, g(t), \phi(t), d\mu)$, $t \in [0, T]$, $T < \infty$ solve RHF (1.1) on a closed Riemannian manifold M. Let $\lambda_1(t)$ be the first nonzero eigenvalue of the weighted p-Laplacian $\Delta_{p,\phi}$ and $f(x,t)$ its corresponding eigenfunction. Suppose*

(i) *$S_{ij} \geq \beta(S + \Delta\phi)g_{ij}$, $\beta > \frac{1}{p}$ is a constant;*

(ii) *$\Delta\phi \geq 0$, i.e., ϕ is subharmonic;*

(iii) *$S \geq S_{\min}(0) \geq 0$, then*

$$\lambda_1(t_2) \geq \lambda_1(t_1)\exp\left(\beta p\int_{t_1}^{t_2} S_{\min}(t)\,dt\right) \tag{4.7}$$

and $\lambda_1(t)$ is monotonically nondecreasing along RHF (1.1);

(iv) *If instead of (iii), $S \geq S_{\min}(0) \neq 0$ (i.e., either $S_{\min}(0) > 0$ or $S_{\min}(0) < 0$), then $\lambda_1(t)(S_{\min}(0)^{-1} - \frac{2}{n}t)^{\frac{n\beta p}{2}}$ is monotonically nondecreasing along RHF (1.1).*

Furthermore, $\lambda_1(t)$ is differentiable almost everywhere along RHF (1.1).

Corollary 4.4 *Let $(M, g(t), \phi(t), d\mu)$, $t \in [0, T]$, $T < \infty$ solve RHF (1.1) on a compact Riemannian surface (M^2, g_0). Let $\lambda_1(t)$ be the first nonzero eigenvalue of $\Delta_{p,\phi}$ with $f(x,t)$ being its corresponding eigenfunction. Assume ϕ is subharmonic (i.e., $\Delta\phi \geq 0$).*

(A) *Suppose $Rc \leq \epsilon\nabla\phi\otimes\nabla\phi$, where $\epsilon \leq \frac{2\alpha(1-\beta)}{1-2\beta}$, $\beta > \frac{1}{2}$.*

(i) *If* $S_{\min}(0) \geq 0$, *then* $\lambda_1(t)$ *is monotonically nondecreasing along RHF* (1.1) *for all* $t \in [0, T]$, $T < \infty$.
(ii) *If* $S_{\min}(0) \neq 0$, *then the quantity* $\lambda_1(t)(S_{\min}(0)^{-1} - t)^{\beta p}$ *is monotonically nondecreasing along RHF* (1.1) *for all* $t \in [0, T]$, $T < \infty$.

(B) *Suppose that* $\nabla\phi \otimes \nabla\phi \leq \frac{1}{2}|\nabla\phi|^2 g_{ij}$.
(i) *If* $S_{\min}(0) \geq 0$, *then* $\lambda_1(t)$ *is monotonically nondecreasing along RHF* (1.1) *for all* $t \in [0, T]$, $T < \infty$.
(ii) *If* $S_{\min}(0) \neq 0$, *then the quantity* $\lambda_1(t)(S_{\min}(0)^{-1} - t)^{\beta p}$ *is monotonically nondecreasing along RHF* (1.1) *for all* $t \in [0, T]$, $T < \infty$.

Remark 4.5 Assuming the weight function $\phi = \psi(x)$ is t-independent, i.e., $d\mu = e^{-\psi(x)}\,dv$, then (4.5) of Theorem 4.1 becomes

$$\frac{d}{dt}\lambda_1(t) = \lambda_1(t)\int_M S|f|^p\,d\mu - \int_M S|\nabla f|^p\,d\mu + p\int_M |\nabla f|^{p-2} Sc(\nabla f, \nabla f)\,d\mu. \tag{4.8}$$

Then, scaling $\phi(x,t)$ by taking $\phi = \sqrt{\frac{2}{\alpha}}u(x,t)$, RHF (1.1) becomes

$$\begin{cases} \frac{\partial}{\partial t}g = -2Rc + 4\nabla u \otimes \nabla u, \\ \frac{\partial}{\partial t}u = \Delta u \end{cases} \tag{4.9}$$

studied by Li [17]. Thus, under these conditions our results in Theorem 4.1, Theorem 4.3, and Corollary 4.4 reduce to Theorem 1.5, Theorem 1.6, and Corollary 1.7 of [17], respectively.

4.1 Proof of Theorem 4.1

Proof The proof follows by direct computation using evolution formula (4) in Lemma 3.1. Let $f(t_0)$ and $\lambda(t_0) = \lambda(f(t_0), t)$ be an eigenpair. Then, for a smooth function $f(t)$, we can set

$$\lambda\bigl(f(t), t\bigr) = -\int_M f(t)\Delta_{p,\phi} f(t)\,d\mu.$$

Then

$$\frac{d}{dt}\lambda_1(t)\bigg|_{t=t_0} = \frac{\partial}{\partial t}\lambda_1\bigl(f(t), t\bigr) = -\frac{\partial}{\partial t}\int_M f(t)\Delta_{p,\phi} f(t)\,d\mu. \tag{4.10}$$

Using evolution formula (4) of Lemma 3.1, we obtain

$$\begin{aligned} &\frac{\partial}{\partial t}\int_M f\Delta_{p,\phi} f\,d\mu \\ &\quad = \int_M \frac{\partial}{\partial t}(\Delta_{p,\phi} f) f\,d\mu + \int_M \Delta_{p,\phi} f \frac{\partial}{\partial t}(f\,d\mu) \\ &\quad = 2\int_M \mathcal{S}^{ij}\nabla_i(Z\nabla_j f) f\,d\mu + \int_M g^{ij}\nabla_i(Z_t\nabla_j f) f\,d\mu + \int_M g^{ij}\nabla_i(Z\nabla_j f_t) f\,d\mu \end{aligned}$$

$$- 2\alpha \int_M Z(\Delta\phi)g^{ij}\nabla_i\phi\nabla_j ff \, d\mu - \int_M Z_t\langle\nabla\phi, \nabla f\rangle f \, d\mu - \int_M Z\langle\nabla\phi_t, \nabla f\rangle f \, d\mu$$
$$- \int_M Z\langle\nabla\phi, \nabla f_t\rangle f \, d\mu - 2\int_M S^{ij}Z\nabla_i\phi\nabla_j f + \int_M \Delta_{p,\phi} f \frac{\partial}{\partial t}(f \, d\mu). \tag{4.11}$$

We now apply integration by parts formula on the first three terms on the right-hand side of (4.11). For the first term

$$2\int_M S^{ij}\nabla_i(Z\nabla_j f)f \, d\mu = -2\int_M Z\nabla_j f\nabla_j\big(S^{ij}fe^{-\phi}\big)\, d\nu = -2\int_M ZS^{ij}\nabla_i f\nabla_j f \, d\mu$$
$$- 2\int_M Z\nabla_i S^{ij}\nabla_j ff \, d\mu + 2\int_M ZS^{ij}\nabla_i f\nabla_j\phi f \, d\mu. \tag{4.12}$$

The second term on the right-hand side of (4.12) can be written as (see the computation in Lemma A.1 below):

$$-2\int_M Z\nabla_i S^{ij}\nabla_j ff \, d\mu = \int_M S(\Delta_{p,\phi} f)f \, d\mu + \int_M S|\nabla f|^p \, d\mu$$
$$+ 2\alpha\int_M Z\Delta\phi\langle\nabla\phi, \nabla f\rangle \, d\mu. \tag{4.13}$$

Substituting (4.13) into (4.12) we have

$$2\int_M S^{ij}\nabla_i(Z\nabla_j f)f \, d\mu = -2\int_M ZS^{ij}\nabla_i f\nabla_j f \, d\mu + \int_M S(\Delta_{p,\phi} f)f \, d\mu$$
$$+ \int_M S|\nabla f|^p \, d\mu + 2\alpha\int_M Z\Delta\phi\langle\nabla\phi, \nabla f\rangle \, d\mu$$
$$+ 2\int_M ZS^{ij}\nabla_i f\nabla_j\phi f \, d\mu. \tag{4.14}$$

For the second term on the right-hand side of (4.11), integration by parts implies

$$\int_M g^{ij}\nabla_i(Z_t\nabla_j f)f \, d\mu = -\int_M Z_t\nabla_j f\nabla^i\big(fe^{-\phi}\big)\, d\nu$$
$$= -\int_M Z_t|\nabla f|^2 \, d\mu + \int Z_t\langle\nabla f, \nabla\phi\rangle f \, d\mu. \tag{4.15}$$

Similarly, the third term on the right-hand side of (4.11) implies

$$\int_M g^{ij}\nabla_i(Z\nabla_j f_t)f \, d\mu = -\int_M Z\langle\nabla f_t, \nabla f\rangle \, d\mu + \int_m Z\langle\nabla f_t, \nabla\phi\rangle f \, d\mu. \tag{4.16}$$

Putting (4.14), (4.15), and (4.16) into (4.11), we obtain

$$\frac{\partial}{\partial t}\int_M f\Delta_{p,\phi} f \, d\mu = -2\int_M ZS^{ij}\nabla_i f\nabla_j f \, d\mu + \int_M S\Delta_{p,\phi} ff \, d\mu$$
$$+ \int_M S|\nabla f|^p \, d\mu - \int_M Z_t|\nabla f|^2 \, d\mu - \int_M Z\langle\nabla f_t, \nabla f\rangle \, d\mu$$
$$- \int_M Z\langle\nabla\phi_t, \nabla f\rangle f \, d\mu + \int_M \Delta_{p,\phi} f \frac{\partial}{\partial t}(f \, d\mu). \tag{4.17}$$

Using the evolution formula (2) of Lemma 3.1 into (4.17) yields

$$\begin{aligned}\frac{\partial}{\partial t}\int_M f\Delta_{p,\phi}f\,d\mu = {}& -p\int_M ZS^{ij}\nabla_i f\nabla_j f\,d\mu + \int_M S\Delta_{p,\phi}ff\,d\mu \\ &+ \int_M S|\nabla f|^p\,d\mu - (p-1)\int_M Z\langle\nabla f_t,\nabla f\rangle\,d\mu \\ &- \int_M Z\langle\nabla\phi_t,\nabla f\rangle f\,d\mu + \int_M \Delta_{p,\phi}f\frac{\partial}{\partial t}(f\,d\mu).\end{aligned} \tag{4.18}$$

A straightforward computation also yields

$$\begin{aligned}&-(p-1)\int_M Z\langle\nabla f_t,\nabla f\rangle\,d\mu \\ &\quad= (p-1)\int_M \nabla_i\big(Z\nabla_j fe^{-\phi}\big)f_t\,dv \\ &\quad= (p-1)\int_M \nabla_i(Z\nabla_j f)\,d\mu - (p-1)\int_M Z\langle\nabla f,\nabla\phi\rangle f_t\,d\mu \\ &\quad= (p-1)\int_M \Delta_{p,\phi}ff_t\,d\mu\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}\int_M Z\langle\nabla\phi_t,\nabla f\rangle f\,d\mu &= -\int_M \phi_t\nabla_i\big(Z\nabla_j ffe^{-\phi}\big)\,dv \\ &= -\int_M \phi_t\Delta_{p,\phi}f\,d\mu - \int_M \phi_t|\nabla f|^p\,d\mu.\end{aligned} \tag{4.20}$$

Putting (4.19) and (4.20) into (4.18) yields

$$\begin{aligned}&\frac{\partial}{\partial t}\int_M f\Delta_{p,\phi}f\,d\mu \\ &\quad= -p\int_M ZS^{ij}\nabla_i f\nabla_j f\,d\mu + \int_M S\Delta_{p,\phi}ff\,d\mu \\ &\qquad+ \int_M S|\nabla f|^p\,d\mu + (p-1)\int_M \Delta_{p,\phi}ff_t\,d\mu + \int_M \phi_t\Delta_{p,\phi}ff\,d\mu \\ &\qquad+ \int_M \phi_t|\nabla f|^p\,d\mu + \int_M \Delta_{p,\phi}f\frac{\partial}{\partial t}(f\,d\mu) \\ &\quad= -p\int_M ZS^{ij}\nabla_i f\nabla_j f\,d\mu + \int_M S\Delta_{p,\phi}ff\,d\mu + \int_M S|\nabla f|^p\,d\mu \\ &\qquad+ \int_M \phi_t\Delta_{p,\phi}ff\,d\mu + \int_M \phi_t|\nabla f|^p\,d\mu + \int_M \Delta_{p,\phi}f\Big((p-1)f_t\,d\mu - \frac{\partial}{\partial t}(f\,d\mu)\Big).\end{aligned}$$

Using the facts that

$$\frac{\partial}{\partial t}\int_M f\Delta_{p,\phi}f\,d\mu = -\frac{\partial}{\partial t}\lambda_1\big(f(t),t\big)\Big|_{t=t_0},$$

$$\Delta_{p,\phi}f = -\lambda_1(t_0)|f|^{p-2}f$$

and the normalization condition $\int_M |f|^p \, d\mu = 1$, which implies

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}\left(\int_M |f|^p \, d\mu\right) = \frac{\partial}{\partial t}\left(\int_M |f|^{p-1} f \, d\mu\right) \\ &= \int_M |f|^{p-2} f\left[(p-1) f_t \, d\mu + |f|^{p-1} \frac{\partial}{\partial t}(f \, d\mu)\right], \end{aligned}$$

we arrive at

$$\begin{aligned} \frac{d}{dt}\lambda_1\big(f(t), t\big)\bigg|_{t=t_0} = {}& \lambda_1(t_0) \int_M S|f|^p \, d\mu - \int_M S|\nabla f|^p \, d\mu + \lambda_1(t_0) \int_M \phi_t |f|^p \, d\mu \\ & - \int_M \phi_t |\nabla f|^p \, d\mu + p \int_M |\nabla f|^{p-2} S^{ij} \nabla_i f \nabla_j f \, d\mu, \end{aligned}$$

which is what we wanted to prove. □

4.2 Proof of Corollary 4.2

Proof Using the condition $S_{ij} \geq \beta(S + \Delta\phi) g_{ij}$ in (4.5) of Theorem 4.1, we have

$$\frac{d}{dt}\lambda_1(t) \geq \lambda_1(t) \int_M (S + \Delta\phi)|f|^p \, d\mu + (\beta p - 1) \int_M (S + \Delta\phi)|\nabla f|^p \, d\mu. \tag{4.21}$$

Denote

$$\Theta\big(g(t), f(x,t)\big) := \lambda_1(t) \int_M (S + \Delta\phi)|f|^p \, d\mu + (\beta p - 1) \int_M (S + \Delta\phi)|\nabla f|^p \, d\mu. \tag{4.22}$$

Integrating (4.21) from t_1 to t_2, $t_1 < t_2$, where $t_1, t_2 \in [0, T]$, $T < \infty$ yields

$$\lambda_1(t_2) - \lambda_1(t_1) \geq \int_{t_1}^{t_2} \Theta\big(g(t), f(x,t)\big) \, dt,$$

which is the desired result. □

4.3 Proof of Theorem 4.3

Proof For $t_1 < t_2$, $t_1, t_2 \in [0, T]$, $T < \infty$, for all time $t \in [0, T]$, $T < \infty$, we use (4.22)

$$\begin{aligned} &\Theta\big(g(t), f(x,t)\big) \\ &\quad = \lambda_1(t) \int_M (S + \Delta\phi)|f|^p \, d\mu + (\beta p - 1) \int_M (S + \Delta\phi)|\nabla f|^p \, d\mu \\ &\quad = \lambda_1(t) \int_M (S + \Delta\phi)|f|^p \, d\mu + (\beta p - 1) \int_M (S + \Delta\phi)|\nabla f|^{p-2} g^{ij} \nabla_i f \nabla_j f \, d\mu \\ &\quad = \lambda_1(t) \int_M (S + \Delta\phi)|f|^p \, d\mu - (\beta p - 1) \int_M (S + \Delta\phi) \operatorname{div}\big(e^{-\phi}|\nabla f|^{p-2} \nabla f\big) f \, dv \\ &\quad = \lambda_1(t) \int_M (S + \Delta\phi)|f|^p \, d\mu - (\beta p - 1) \int_M (S + \Delta\phi) \Delta_{p,\phi} f \cdot f \, d\mu \\ &\quad = \lambda_1(t) \beta p \int_M (S + \Delta\phi)|f|^p \, d\mu. \end{aligned}$$

Using the conditions $\Delta\phi \geq 0$, $S \geq S_{\min}(t)$ and $\int_M |f|^p \, d\mu = 1$, we have

$$\frac{d}{dt}\lambda_1(t) \geq \lambda_1(t)\beta p S_{\min}(t). \tag{4.23}$$

Integrating between t_1 and t_2, $t_1, t_2 \in [0, T]$, we have

$$\ln\lambda_1(t_2) \geq \ln\lambda_1(t_1) + \beta p \int_{t_1}^{t_2} S_{\min}(t)\, dt,$$

which yields (4.7).

Now, set $S_{\min}(0) = z_0 \neq 0$. Recall that S satisfies an evolution equation [18, 20]

$$\frac{\partial S}{\partial t} = \Delta S + 2|S_{ij}|^2 + 2\alpha|\Delta\phi|^2$$

and inequality $|S_{ij}|^2 \geq \frac{1}{n}S^2$. Then solving

$$\frac{\partial S}{\partial t} \geq \Delta S + \frac{2}{n}|S_{ij}|^2$$

by applying the maximum principle, where one can compare S with the solution of an ODE $z' = \frac{2}{n}z^2$, $z(0) = z_0 = S_{\min}(0)$, we have

$$S(t) \geq z(t) = \frac{1}{z_0^{-1} - \frac{2}{n}t}, \quad t \in [0, T], T < \frac{n}{2S_{\min}(0)} < \infty.$$

Applying (4.23), we obtain

$$\frac{d}{dt}\lambda_1(t) \geq \lambda_1(t)\beta p \cdot \frac{1}{z_0^{-1} - \frac{2}{n}t},$$

which implies

$$\lambda_1(t_2) \geq \lambda_1(t_1)\exp\left(\beta p \int_{t_1}^{t_2} \frac{dt}{z_0^{-1} - \frac{2}{n}t}\right). \tag{4.24}$$

It is easy to check by elementary calculus that

$$\int_{t_1}^{t_2} \frac{dt}{z_0^{-1} - \frac{2}{n}t} = \ln\left(\frac{z_0^{-1} - \frac{2}{n}t_1}{z_0^{-1} - \frac{2}{n}t_2}\right)^{\frac{n}{2}}. \tag{4.25}$$

Substituting (4.25) into (4.24), we obtain

$$\ln\left(\frac{\lambda_1(t_2)}{\lambda_1(t_1)}\right) \geq \ln\left(\frac{z_0^{-1} - \frac{2}{n}t_1}{z_0^{-1} - \frac{2}{n}t_2}\right)^{\frac{n}{2}\beta p}$$

for any time $t_1 < t_2$. By this we have

$$\lambda_1(t_2)\left(z_0^{-1} - \frac{2}{n}t_2\right)^{\frac{n}{2}\beta p} \geq \lambda_1(t_1)\left(z_0^{-1} - \frac{2}{n}t_1\right)^{\frac{n}{2}\beta p},$$

which means that the quantity $\lambda_1(t)(z_0^{-1} - \frac{2}{n}t)^{\frac{n}{2}\beta p}$ is nondecreasing in the interval $[t_1, t_2]$ along the Ricci-harmonic flow. Notice that $(z_0^{-1} - \frac{2}{n}t)$ is decreasing in the interval $t_1 < t_2$, $t_1, t_2 \in [0, T)$. This means that $\lambda_1(t)$ is nondecreasing along the flow. This completes the proofs of monotonicity.

For differentiability of $\lambda_1(t)$, it is easy to see that $\lambda_1(t)$ is differentiable almost everywhere by the classical Lebesgue's theorem [19, Chap. 4] since $\lambda_1(t)$ is nondecreasing on the time interval $[0, T)$. □

Remark 4.6 By the maximum principle, the assumption that ϕ is subharmonic implies that the maximum of ϕ cannot be achieved in M since ϕ is not a constant function; however, its minimum can be achieved in M.

Remark 4.7 Our proofs of evolution and monotonicity of the first eigenvalue of $\Delta_{p,\phi}$ do not use any differentiability of the first eigenvalue or its corresponding eigenfunction under the Ricci-harmonic flow. In fact, it is not known whether they are differentiable in advance.

4.4 Proof of Corollary 4.4

Proof Note that on M^2 we have $R_{ij} = \frac{1}{2}Rg_{ij}$. Then we compute

$$\begin{aligned} S_{ij} - \beta S g_{ij} &= \frac{R}{2}g_{ij} - \alpha\nabla_i\phi\nabla_j\phi - \beta\big(R - \alpha|\nabla\phi|^2\big)g_{ij} \\ &= \left(\frac{1}{2} - \beta\right)Rg_{ij} - \alpha\nabla_i\phi\nabla_j\phi - \beta\alpha|\nabla\phi|^2 g_{ij}. \end{aligned}$$

For any nonzero vector $X = (X^i)$ and the condition $R_{ij} \leq \epsilon\nabla_i\phi\nabla_j\phi$, we have

$$\begin{aligned} (S_{ij} - \beta S g_{ij})X^iX^j &= \left(\frac{1}{2} - \beta\right)R|X|^2 - \alpha\langle\nabla\phi, X\rangle^2 + \beta\alpha|\nabla\phi|^2|X|^2 \\ &\geq \left[\left(\frac{1}{2} - \beta\right)\epsilon - (1-\beta)\alpha\right]|\nabla\phi|^2|X|^2. \end{aligned}$$

From here one can conclude that $[(\frac{1}{2} - \beta)\epsilon - (1-\beta)\alpha]|\nabla\phi|^2 \geq 0$ since $\epsilon \leq \frac{2\alpha(1-\beta)}{1-2\beta}$, $\beta > \frac{1}{2}$, which implies that the condition $S_{ij} - \beta S g_{ij} \geq 0$, $\beta > \frac{1}{2}$ holds on M^2. Then consequences $A(i)$ and $A(ii)$ of the corollary follow from Theorem 4.3.

On the other hand, we can show that $\nabla\phi \otimes \nabla\phi \leq \frac{1}{2}|\nabla\phi|^2 g_{ij}$ holds on the Riemann surface M^2. Note that

$$\begin{aligned} (S_{ij} - \beta S g_{ij})X^iX^j &= \left[\frac{R}{2}g_{ij} - \alpha\nabla_i\phi\nabla_j\phi - \beta\big(R - \alpha|\nabla\phi|^2\big)g_{ij}\right]X^iX^j \\ &= \frac{R}{2}|X|^2 - \alpha\langle\nabla\phi, X\rangle^2 - \beta\big(R - \alpha|\nabla\phi|^2\big)|X|^2 \\ &\geq \frac{R}{2}|X|^2 - \frac{\alpha}{2}||\nabla\phi|^2|X|^2 - \beta\big(R - \alpha|\nabla\phi|^2\big)|X|^2 \end{aligned}$$

meaning that

$$S_{ij}X^iX^j \geq \frac{1}{2}\big(R - \alpha|\nabla\phi|^2\big)|X|^2,$$

which implies $S_{ij} - \frac{1}{2}Sg_{ij} \geq 0$. Then consequences $B(i)$ and $B(ii)$ follow from Theorem 4.3. □

Appendix

In the following lemma we want to establish formula (4.13).

Lemma A.1 *Let* $(M, g, d\mu)$ *be a closed Riemannian manifold on which RHF* (1.1) *holds. Then*

$$-2\int_M Z\nabla_i S^{ij}\nabla_j ff\,d\mu = \int_M S(\Delta_{p,\phi}f)f\,d\mu + \int_M S|\nabla f|^p\,d\mu + 2\alpha\int_M Z\Delta\phi\langle\nabla\phi,\nabla f\rangle\,d\mu. \tag{A.1}$$

Proof Notice that

$$\begin{aligned}\nabla_i S^{ij} &= \nabla_i\bigl(R^{ij} - \alpha\nabla^i\phi\nabla^j\phi\bigr) = g^{ik}g^{jl}\bigl(\nabla_i R_{kl} - \alpha\nabla_i(\nabla_k\phi\nabla_l\phi)\bigr)\\ &= g^{ik}g^{jl}(\nabla_i R_{kl} - \alpha\nabla_i\nabla_k\phi\nabla_l\phi - \alpha\nabla_k\nabla_i\nabla_l\phi),\end{aligned}$$

and by the contracted second Bianchi identity,

$$g^{ik}\nabla_i R_{kl} = \frac{1}{2}\nabla_l R.$$

We now compute using the last two expressions:

$$\begin{aligned}-2\int_M Z\nabla_i S^{ij}\nabla_j ff\,d\mu &= -2\int_M Zg^{ik}g^{jl}\nabla_j f\nabla_i R_{kl}f\,d\mu + 2\alpha\int_M Zg^{ik}g^{jl}\nabla_j f\nabla_i\nabla_k\phi\nabla_l\phi f\,d\mu\\ &\quad + 2\alpha\int_M Zg^{ik}g^{jl}\nabla_j f\nabla_k\phi\nabla_i\nabla_l\phi f\,d\mu\\ &= -\int_M Zg^{jl}\nabla_j f\nabla_l Rf\,d\mu + 2\alpha\int_M Z\Delta\phi g^{jl}\nabla_j f\nabla_l\phi f\,d\mu\\ &\quad + 2\alpha\int_M Zg^{ik}g^{jl}\nabla_j f\nabla_k\phi\nabla_i\nabla_l\phi f\,d\mu\\ &= \int_M R\,\mathrm{div}\bigl(Z\nabla_j ffe^{-\phi}\bigr)\,dv + 2\alpha\int_M Z\Delta\phi\langle\nabla\phi,\nabla f\rangle\,d\mu\\ &\quad + 2\alpha\int_M Zg^{ik}g^{jl}\nabla_j f\nabla_k\phi\nabla_i\nabla_l\phi f\,d\mu\\ &= \int_M R(\Delta_{p,\phi}f)f\,d\mu + \int_M RZ|\nabla f|^2\,d\mu + 2\alpha\int_M Z\Delta\phi\langle\nabla\phi,\nabla f\rangle\,d\mu\\ &\quad + 2\alpha\int_M Zg^{ik}g^{jl}\nabla_j f\nabla_k\phi\nabla_i\nabla_l\phi f\,d\mu\\ &= \int_M R(\Delta_{p,\phi}f)f\,d\mu + \int_M R|\nabla f|^p\,d\mu + 2\alpha\int_M Z\Delta\phi\langle\nabla\phi,\nabla f\rangle\,d\mu\\ &\quad + 2\alpha\int_M Zg^{ik}g^{jl}\nabla_j f\nabla_k\phi\nabla_i\nabla_l\phi f\,d\mu,\end{aligned}$$

which implies

$$-2\int_M Z\nabla_i S^{ij}\nabla_j ff\,d\mu = \int_M R(\Delta_{p,\phi}f)f\,d\mu + 2\alpha\int_M Z\Delta\phi\langle\nabla\phi,\nabla f\rangle\,d\mu + \int_M R|\nabla f|^p\,d\mu + 2\alpha\int_M Zg^{ik}g^{jl}\nabla_j f\nabla_k\phi\nabla_i\nabla_l\phi f\,d\mu. \tag{A.2}$$

Similarly, by direct computation the last term of the last equation implies

$$\begin{aligned}
&2\alpha\int_M Zg^{ik}g^{jl}\nabla_j f\nabla_k\phi\nabla_i\nabla_l\phi f\,d\mu\\
&= -2\alpha\int_M \nabla_i\big(Z\nabla^l f\nabla^i\phi e^{-\phi}f\big)\,dv\\
&= 2\alpha\int_M Z\nabla^l f\nabla^i\phi\nabla_l\nabla_i\phi f\,d\mu - 2\alpha\int_M \nabla_l\big(Z\nabla^l f\nabla^i\phi f\big)\nabla_i\phi\,d\mu\\
&= -2\alpha\int_M \nabla_l\big(Z\nabla^l f\big)\nabla^i\phi\nabla_i\phi f\,d\mu - 2\alpha\int_M Z\nabla^l f\nabla_i\nabla^i\phi\nabla_i\phi f\,d\mu\\
&\quad - 2\alpha\int_M Z\nabla^l f\nabla^i\phi\nabla_i\phi\nabla_l f\,d\mu + 2\alpha\int_M Z\nabla^l f\nabla^i\phi\nabla_l\phi\nabla_i\phi f\,d\mu\\
&= -2\alpha\int_M \Delta_p f|\nabla\phi|^2\,d\mu - 2\alpha\int_M Zg^{ik}g^{jl}\nabla_j f\nabla_k\phi\nabla_i\nabla_l\phi f\,d\mu\\
&\quad - 2\alpha\int_M Z|\nabla f|^2|\nabla\phi|^2\,d\mu + 2\alpha\int_M |\nabla\phi|^2\langle\nabla\phi,\nabla f\rangle f\,d\mu,
\end{aligned}$$

which implies

$$2\alpha\int_M Zg^{ik}g^{jl}\nabla_j f\nabla_k\phi\nabla_i\nabla_l\phi f\,d\mu = -\alpha\int_M \Delta_{p,\phi}f|\nabla\phi|^2 f\,d\mu - \alpha\int_M |\nabla f|^p|\nabla\phi|^2\,d\mu. \tag{A.3}$$

Combining (A.2) and (A.3) yields (A.1). □

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