Shen et al. *Journal of Inequalities and Applications* (2018) 2018:356
https://doi.org/10.1186/s13660-018-1951-0

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Parameter estimation for Ornstein–Uhlenbeck processes driven by fractional Lévy process

Guangjun Shen[1,2*], Yunmeng Li[2] and Zhenlong Gao[3]

*Correspondence: gjshen@163.com
[1]School of Mathematics and Finance, Chuzhou University, Chuzhou, China
[2]Department of Statistics, Anhui Normal University, Wuhu, China
Full list of author information is available at the end of the article

Abstract

We study the minimum Skorohod distance estimation θ_ε^* and minimum L_1-norm estimation $\widetilde{\theta_\varepsilon}$ of the drift parameter θ of a stochastic differential equation $dX_t = \theta X_t\,dt + \varepsilon\,dL_t^d, X_0 = x_0$, where $\{L_t^d, 0 \le t \le T\}$ is a fractional Lévy process, $\varepsilon \in (0, 1]$. We obtain their consistency and limit distribution for fixed T, when $\varepsilon \to 0$. Moreover, we also study the asymptotic laws of their limit distributions for $T \to \infty$.

MSC: 60G18; 65C30; 93E24

Keywords: Fractional Lévy process; Minimum Skorohod distance estimation; Minimum L_1-norm estimation; Consistency; Limit distribution; Asymptotic law

1 Introduction

Statistical inference for stochastic equations is a main research direction in probability theory and its applications. The asymptotic theory of parametric estimation for diffusion processes with small noise is well developed. Genon-Catalot [8] and Laredo [17] considered the efficient estimation for drift parameters of small diffusions from discrete observations as $\epsilon \to 0$ and $n \to \infty$. Using martingale estimating function, Sørensen [27] obtained consistency and asymptotic normality of the estimators of drift and diffusion coefficient parameters as $\epsilon \to 0$ and n is fixed. Using a contrast function under suitable conditions on ϵ and n, Sørensen and Uchida [28] and Gloter and Sørensen [9] considered the efficient estimation for unknown parameters in both drift and diffusion coefficient functions. Long [20], Ma [21] studied parameter estimation for Ornstein–Uhlenbeck processes driven by small Lévy noises for discrete observations when $\epsilon \to 0$ and $n \to \infty$ simultaneously. Shen and Yu [26] obtained consistency and the asymptotic distribution of the estimator for Ornstein–Uhlenbeck processes with small fractional Lévy noises.

Recently, Diop and Yode [4] obtained the minimum Skorohod distance estimate for the parameter θ of a stochastic differential equation with a centered Lévy processes $\{Z_t, 0 \le t \le T\}$, $\epsilon \in (0, 1]$,

$$dX_t = \theta X_t\,dt + \epsilon\,dZ_t, \qquad X_0 = x_0.$$

When $\{Z_t, 0 \le t \le T\}$ is a Brownian motion, Millar [24] obtained the asymptotic behavior of the estimator of the parameter θ. The minimum uniform metric estimate of parameters of diffusion-type processes was considered in Kutoyants and Pilibossian [14, 15]. Hénaff [10] considered the asymptotics of a minimum distance estimator of the parameter of the Ornstein–Uhlenbeck process. Prakasa Rao [25] studied the minimum L_1-norm estimates of the drift parameter of Ornstein–Uhlenbeck process driven by fractional Brownian motion and investigated the asymptotic properties following Kutoyants and Pilibossian [14, 15]. Some surveys on the parameter estimates of fractional Ornstein–Uhlenbeck process can be found in Hu and Nualart [11], El Onsy, Es-Sebaiy and Ndiaye [5], Xiao, Zhang and Xu [29], Jiang and Dong [12], Liu and Song [19].

Motivated by the above results, in this paper we consider the minimum Skorohod distance estimation θ_ε^* and minimum L_1-norm estimation $\widetilde{\theta_\varepsilon}$ of the drift parameter θ for Ornstein–Uhlenbeck processes driven by the fractional Lévy process $\{L_t^d, 0 \leq t \leq T\}$ which satisfies the following stochastic differential equation:

$$dX_t = \theta X_t \, dt + \epsilon \, dL_t^d, \quad X_0 = x_0, \tag{1}$$

where the shift parameter $\theta \in \Theta = (\theta_1, \theta_2) \subseteq R$ is unknown, $\varepsilon \in (0,1]$. Denote by θ_0 the true value of the unknown parameter θ. Note that

$$X_t(\theta) = x_t(\theta) + \varepsilon e^{\theta t} \int_0^t e^{-\theta s} \, dL_s^d,$$

where $x_t(\theta) = x_0 e^{\theta t}$ is a solution of (1) with $\varepsilon = 0$.

Recall that fractional Lévy processes is a natural generalization of the integral representation of fractional Brownian motion. Analogously to Mandelbrot and Van Ness [22] for fractional Brownian motion we introduce the following definition.

Definition 1.1 (Marquardt [23]) Let $L = (L(t), t \in R)$ be a zero-mean two-sided Lévy process with $E[L(1)^2] < \infty$ and without a Brownian component. For $d \in (0, \frac{1}{2})$, a stochastic process

$$L_t^d := \frac{1}{\Gamma(d+1)} \int_{-\infty}^{\infty} \left[(t-s)_+^d - (-s)_+^d\right] L(ds), \quad t \in R, \tag{2}$$

is called a fractional Lévy process (fLp), where

$$L(t) = L_1(t), \quad t \geq 0, \qquad L(t) = -L_2(-t_-), \quad t < 0. \tag{3}$$

$\{L_1(t), t \geq 0\}$ and $\{L_2(t), t \geq 0\}$ are two independent copies of a one-side Lévy process.

Lemma 1.1 (Marquardt [23]) *Let $g \in H$, H is the completion of $L^1(R) \cap L^2(R)$ with respect to the norm $\|g\|_H^2 = E[L(1)^2] \int_R (I_-^d g)^2(u) \, du$, then*

$$\int_R g(s) \, dL_s^d = \int_R \left(I_-^d g\right)(u) \, dL(u), \tag{4}$$

where the equality holds in the L^2 sense and $I_-^d g$ denotes the Riemann–Liouville fractional integral defined by

$$\left(I_-^d g\right)(x) = \frac{1}{\Gamma(d)} \int_x^{\infty} g(t)(t-x)^{d-1} \, dt.$$

Lemma 1.2 (Marquardt [23]) *Let* $|f|, |g| \in H$. *Then*

$$E\left[\int_R f(s)\,dL_s^d \int_R g(s)\,dL_s^d\right] = \frac{\Gamma(1-2d)E[L(1)^2]}{\Gamma(d)\Gamma(1-d)} \int_R\int_R f(t)g(s)|t-s|^{2d-1}\,ds\,dt. \tag{5}$$

Lemma 1.3 (Bender et al. [2]) *Let* L_t^d *be a fLp. Then for every* $p \geq 2$ *and* $\delta > 0$ *such that* $d + \delta < \frac{1}{2}$ *there exists a constant* $C_{p,\delta,d}$ *independent of the driving Lévy process L such that for every* $T \geq 1$

$$E\Bigl(\sup_{0\leq t\leq T}\bigl|L_t^d\bigr|^p\Bigr) \leq C_{p,\delta,d}E\bigl(\bigl|L(1)\bigr|^p\bigr)T^{p(d+1/2+\delta)}.$$

For the study of fLp see Bender et al. [3], Fink and Klüppelberg [7], Lin and Cheng [18], Benassi et al. [1], Lacaux [16], Engelke [6] and the references therein.

The rest of this paper is organized as follows. In Sect. 2, we consider the minimum Skorohod distance estimation θ_ε^* of the drift parameter θ, its consistency and limit distribution are studied for fixed T, when $\varepsilon \to 0$. Moreover, the asymptotic law of its limit distribution are also studied for $T \to \infty$. The similar problems for minimum L_1-norm estimation $\widetilde{\theta}_\varepsilon$ of the drift parameter θ were studied in Sect. 3.

2 Minimum Skorohod distance estimation

In this section, we consider the minimum Skorohod distance estimation which defined by

$$\theta_\varepsilon^* = \arg\min_{\theta\in\Theta} \rho\bigl(X, x(\theta)\bigr), \tag{6}$$

where

$$\rho(x,y) = \inf_{\mu\in\Lambda([0,T])}\Bigl(H(\mu) + \sup\bigl|x\bigl(\mu(t)\bigr) - y(t)\bigr|\Bigr) \tag{7}$$

on the Skorohod space $D([0,T],R)$ consists of càdlàg functions on $[0,T]$, $\Lambda([0,T])$ is the set of functions μ defined on $[0,T]$ with values in $[0,T]$, continuous, strictly increasing such that $\mu(0) = 0$ and $\mu(T) = T$, and

$$H(\mu) = \sup_{s,t\in[0,T],s\neq t}\left|\log\left(\frac{\mu(s)-\mu(t)}{s-t}\right)\right| < \infty.$$

Let

$$\eta_T = \arg\min_{u\in R} \rho\bigl(Y(\theta_0), u\dot{x}(\theta_0)\bigr), \tag{8}$$

where $\dot{x}(\theta_0) = x_0 te^{\theta_0 t}$ is the derivative of $x_t(\theta_0)$ with respect to θ_0 and

$$Y_t(\theta_0) = e^{\theta_0 t}\int_0^t e^{\theta_0 s}\,dL_s^d. \tag{9}$$

Let

$$f(\kappa) = \inf_{|\theta-\theta_0|>\kappa}\bigl|X_t - x(\theta_0)\bigr|_\infty = \inf_{|\theta-\theta_0|>\kappa}\sup_{0\leq t\leq T}\bigl|X_t - x(\theta_0)\bigr|, \quad \kappa > 0 \tag{10}$$

and $P_{\theta_0}^{(\varepsilon)}$ denotes the probability measure induced by the process X_t for fixed ε.

Theorem 2.1 (Consistency) *For every $p \geq 2$ and $\kappa > 0$ such that, for every $T \geq 1$, we have*

$$P_{\theta_0}^{(\varepsilon)}\left(\left|\theta_\varepsilon^* - \theta_0\right| > \kappa\right) \leq C_{p,\kappa,d} E\left(\left|L(1)\right|^p\right) T^{p(d+1/2+\kappa)} \left(\frac{2\varepsilon e^{|\theta_0| T}}{f(\kappa)}\right)^p = O\left(\left(f(\kappa)\right)^{-p} \varepsilon^p\right), \tag{11}$$

where constant $C_{p,\kappa,d}$ is only dependent on p, κ, d.

Proof Fixed $\kappa > 0$ and let

$$\mathcal{I}_0 = \left\{\omega : \inf_{|\theta-\theta_0|<\kappa} \rho\left(X, x(\theta)\right) > \inf_{|\theta-\theta_0|>\kappa} \rho\left(X, x(\theta)\right)\right\}.$$

Then we can obtain $\mathcal{I}_0 = \{|\theta_\varepsilon^* - \theta_0| > \kappa\}$. In fact, for $\omega \in \mathcal{I}_0$, we have

$$\inf_{|\theta-\theta_0|<\kappa} \rho\left(X(\omega), x(\theta)\right) \geq \inf_{\theta\in\Theta} \rho\left(X(\omega), x(\theta)\right) = \rho\left(X(\omega), x\left(\theta_\varepsilon^*\right)\right),$$

thus, $|\theta_\varepsilon^*(\omega) - \theta_0| > \kappa$. On the other hand, assume that $|\theta_\varepsilon^*(\omega) - \theta_0| > \kappa$,

$$\rho\left(X(\omega), x\left(\theta_\varepsilon^*\right)\right) = \inf_{|\theta-\theta_0|>\kappa} \rho\left(X(\omega), x(\theta)\right) < \inf_{|\theta-\theta_0|<\kappa} \rho\left(X(\omega), x(\theta)\right).$$

For any $\kappa > 0$, we have

$$\begin{aligned}
P_{\theta_0}^{(\varepsilon)}(\mathcal{I}_0) &= P_{\theta_0}^{(\varepsilon)}\left(\inf_{|\theta-\theta_0|<\kappa} \rho\left(X, x(\theta)\right) > \inf_{|\theta-\theta_0|>\kappa} \rho\left(X, x(\theta)\right)\right) \\
&\leq P_{\theta_0}^{(\varepsilon)}\left(\inf_{|\theta-\theta_0|<\kappa} \rho\left(X, x(\theta)\right) > \inf_{|\theta-\theta_0|>\kappa} \left|\rho\left(X, x(\theta)\right) - \rho\left(x(\theta_0), x(\theta)\right)\right|\right) \\
&\leq P_{\theta_0}^{(\varepsilon)}\left(\inf_{|\theta-\theta_0|<\kappa} \rho\left(X, x(\theta)\right) > \inf_{|\theta-\theta_0|>\kappa} \rho\left(x(\theta_0), x(\theta)\right) - \rho\left(X, x(\theta_0)\right)\right) \\
&\leq P_{\theta_0}^{(\varepsilon)}\left(\inf_{|\theta-\theta_0|<\kappa} \rho\left(x(\theta), x(\theta_0)\right) + 2\rho\left(X, x(\theta_0)\right) > \inf_{|\theta-\theta_0|>\kappa} \rho\left(x(\theta_0), x(\theta)\right)\right) \\
&\leq P_{\theta_0}^{(\varepsilon)}\left(\left\|X - x(\theta_0)\right\|_\infty > \frac{f(\kappa)}{2}\right).
\end{aligned}$$

Besides, since the process X_t satisfies the stochastic differential Eqs. (1), it follows that

$$X_t - x_t(\theta_0) = x_0 + \theta_0 \int_0^t X_s \, ds + \varepsilon L_t^d - x_t(\theta_0) = \theta_0 \int_0^t \left(X_s - x_s(\theta_0)\right) ds + \varepsilon L_t^d. \tag{12}$$

Then

$$\left|X_t - x_t(\theta_0)\right| = \left|\theta_0 \int_0^t \left(X_s - x_s(\theta_0)\right) ds + \varepsilon L_t^d\right| \leq |\theta_0| \int_0^t \left|X_s - x_s(\theta_0)\right| ds + \varepsilon \left|L_t^d\right|. \tag{13}$$

Hence, we have

$$\left\|X - x(\theta_0)\right\|_\infty = \sup_{0\leq t\leq T} \left|X_t - x_t(\theta_0)\right| \leq \varepsilon e^{|\theta_0 T|} \sup_{0\leq t\leq T} \left|L_t^d\right| \tag{14}$$

because of the Gronwall–Bellman lemma. Thus,

$$P_{\theta_0}^{(\varepsilon)}\left(\left\|X - x(\theta_0)\right\|_\infty > \frac{f(\kappa)}{2}\right) \leq P\left(\sup_{0\leq t\leq T} \left|L_t^d\right| \geq \frac{f(\kappa)}{2\varepsilon e^{|\theta_0 T|}}\right). \tag{15}$$

According to Lemma 1.3 and Chebyshev's inequality, for all $p \geq 2$, we get

$$\begin{aligned} P_{\theta_0}^{(\varepsilon)}\left(\left|\theta_\varepsilon^* - \theta_0\right| > \kappa\right) &\leq E\Big(\sup_{0\leq t\leq T}\left|L_t^d\right|\Big)^p \left(\frac{2\varepsilon e^{|\theta_0 T|}}{f(\kappa)}\right)^p \\ &\leq C_{p,\kappa,d} E\left(\left|L(1)\right|^p\right) T^{p(d+1/2+\kappa)} 2^p e^{|\theta_0 T|p}\left(f(\kappa)\right)^{-p}\varepsilon^p \\ &= O\left(\left(f(\kappa)\right)^{-p}\varepsilon^p\right). \end{aligned} \tag{16}$$

This completes the proof. □

Remark 2.1 As a consequence of the above theorem, we obtain the result that θ_ε^* converges in probability to θ_0 under $P_{\theta_0}^{(\varepsilon)}$-measure as $\varepsilon \to 0$. Furthermore, the rate of convergence is of order $O(\varepsilon^p)$ for every $p \geq 2$.

Theorem 2.2 (Limit distribution) *For any $h \in D([0,T],R)$ satisfying $h(0) = 0$, $\phi_h^\alpha = \rho(h, u\cdot a)$, $a(t) = te^{\alpha t}$, $\alpha \in R$, $u \in R$ admits a unique minimum at u. Then we have, as $\varepsilon \to 0$, $\varepsilon^{-1}(\theta_\varepsilon^* - \theta_0) \xrightarrow{d} \zeta_T$, where the notation "$\xrightarrow{d}$" denotes "convergence in distribution".*

Remark 2.2 ϕ_h^α is a convex function and $\phi_h^\alpha \to +\infty$ when $|u| \to +\infty$, so ϕ_h^α admits a minimum.

The following lemma due to Diop and Yode [4] which is vital for our proof of Theorem 2.2.

Lemma 2.1 *Let $\{K_\varepsilon\}_{\varepsilon>0}$ be a sequence of continuous functions on R and K_0 be a convex function which admits a unique minimum η on R. Let $\{L_\varepsilon\}_{\varepsilon>0}$ be a sequence of positive numbers such that $L_\varepsilon \to +\infty$ as $\varepsilon \to 0$. We suppose that*

$$\lim_{\varepsilon\to 0}\sup_{|u|\leq L_\varepsilon}\left|K_\varepsilon(u) - K_0(u)\right| = 0.$$

Then

$$\lim_{\varepsilon\to 0}\arg\min_{|u|\leq L_\varepsilon} K_\varepsilon(u) = \eta,$$

where if there are several minima of K_ε, we choose one of them arbitrarily.

Proof of Theorem 2.2 We introduce the following notations:

$$\begin{aligned} K_\varepsilon(u) &= \rho\left(Y, \frac{1}{\varepsilon}\left(x(\theta_0 + \varepsilon u) - x(\theta_0)\right)\right), \\ K_0(u) &= \rho\left(Y, u\dot{x}(\theta_0)\right). \end{aligned}$$

Since

$$\begin{aligned} \left|K_\varepsilon(u) - K_0(u)\right| = \bigg| &\inf_{\mu\in\Lambda([0,T])}\left(H(\mu) + \left\|Y_\mu - \frac{1}{\varepsilon}\left(x(\theta_0+\varepsilon u) - x(\theta_0)\right)\right\|_\infty\right) \\ &- \inf_{\mu\in\Lambda([0,T])}\left(H(\mu) + \left\|Y_\mu - u\dot{x}(\theta_0)\right\|_\infty\right)\bigg| \end{aligned}$$

$$= \left| \inf_{\mu \in \Lambda([0,T])} \left(H(\mu) + \left\| Y_\mu - u\dot{x}(\theta_0) - \frac{1}{2}\varepsilon u^2 \ddot{x}(\widetilde{\theta}) \right) \right\|_\infty) \right.$$
$$\left. - \inf_{\mu \in \Lambda([0,T])} \left(H(\mu) + \left\| Y_\mu - u\dot{x}(\theta_0) \right\|_\infty \right) \right|$$

with $\widetilde{\theta} = \widetilde{\theta}_{\varepsilon,u,t} \in (\theta_0, \theta_0 + \varepsilon u)$, where the second equality is because of the Taylor expansion. If we take $L_\varepsilon = \varepsilon^{\delta-1}$ with $\delta \in (1/2, 1)$, we get

$$\begin{aligned}
\sup_{|u| \le L_\varepsilon} \left| K_\varepsilon(u) - K_0(u) \right| &= \left| \inf_{\mu \in \Lambda([0,T])} \left(H(\mu) + \left\| Y_\mu - u\dot{x}(\theta_0) - \frac{1}{2}\varepsilon u^2 \ddot{x}(\widetilde{\theta}) \right\|_\infty \right) \right. \\
&\quad \left. - \inf_{\mu \in \Lambda([0,T])} \left(H(\mu) + \left\| Y_\mu - u\dot{x}(\theta_0) \right\|_\infty \right) \right| \\
&\le \sup_{|u| \le L_\varepsilon} \left[\frac{1}{2}\varepsilon u^2 \sup_{0 \le t \le T} \ddot{x}(\widetilde{\theta}) \right] \le \frac{\varepsilon L_\varepsilon^2}{2} |x_0| T^2 e^{(|\theta_0| + \varepsilon L_\varepsilon)T} \\
&= \frac{\varepsilon^{2\delta-1}}{2} |x_0| T^2 e^{(|\theta_0| + \varepsilon L_\varepsilon)T} \to 0 \quad (\varepsilon \to 0).
\end{aligned}$$

Therefore, we get the desired results by Lemma 2.1. □

In the following, we will consider the limiting behavior of η_T for $T \to +\infty$. Let us introduce the following notations:

$$\begin{aligned}
A_t &= \int_t^{+\infty} e^{-\theta_0 s} \, dL_s^d, \\
B_t &= \int_0^t e^{-\theta_0 s} \, dL_s^d.
\end{aligned}$$

From Theorem 3.6.6 of Jurek and Mason [13] and Lemma 4 of Diop and Yode [4], we can get the logarithmic moment condition is necessary and sufficient for the existence of the improper integral A_0.

Lemma 2.2 *Suppose that* $E(\log(1 + |L_1|)) < +\infty$. *Then*

$$A_t \stackrel{d}{=} e^{-\theta_0 s} A_0 \tag{17}$$

where "$\stackrel{d}{=}$*" denotes "identical distribution".*

Proof It is not hard to see,

$$\begin{aligned}
A_t &= \int_t^{+\infty} e^{-\theta_0 s} \, dL_s^d = \int_t^{+\infty} \left(I_-^d e^{-\theta_0 u} \right)(s) \, dL(s) \\
&= \int_t^{+\infty} \left(\frac{1}{\Gamma(d)} \int_s^\infty e^{-\theta_0 u} (u - s)^{d-1} \, du \right) dL(s) \\
&= \int_t^{+\infty} \left(\frac{1}{\Gamma(d)} \int_0^\infty e^{-\theta_0 (s+x)} x^{d-1} \, dx \right) dL(s) \\
&= \int_t^{+\infty} \left(\frac{1}{\Gamma(d)} e^{-\theta_0 s} \theta_0^{-d} \int_s^\infty e^{-\theta_0 x} (\theta_0 x)^{d-1} d(\theta_0 x) \right) dL(s) \\
&= \theta_0^{-d} \int_t^{+\infty} e^{-\theta_0 s} \, dL(s).
\end{aligned}$$

In a similar way,

$$A_0 = \theta_0^{-d} \int_0^{+\infty} e^{-\theta_0 s}\, dL(s).$$

From Lemma 4 of Diop and Yode [4], we have immediately

$$A_t \overset{d}{=} e^{-\theta_0 s} A_0. \qquad \square$$

The next theorem gives the asymptotic behavior of the limit distribution η_T for large T.

Theorem 2.3 *Suppose that* $\theta_0 > 0$ *and* $E(\log(1 + |L_1|)) < +\infty$. *Then* $\xi_T = x_0 T \eta_T$ *converges in distribution to* A_0 *as* $T \to +\infty$.

Proof Recall that

$$\eta_T = \arg\min_{u\in R} \rho\bigl(Y(\theta_0), u\dot{x}(\theta_0)\bigr).$$

By changing variable, we have

$$\xi_T = \arg\min_{\omega\in R} \rho\bigl(Y(\theta_0), M_t(\omega)\bigr) := \arg\min_{\omega\in R} N(\omega), \tag{18}$$

where $M_t(\omega) = \frac{\omega t e^{\theta_0 t}}{T}$ and $N(\cdot) = \rho(Y(\theta_0), M(\cdot))$.

We want to show that, for every $\Delta > 0$,

$$\lim_{T\to+\infty} P_{\theta_0}\bigl\{|\xi_T - A_0| > \Delta\bigr\} = 0. \tag{19}$$

Therefore, let us consider the set

$$V_\Delta = \bigl\{\omega : |\omega - A_0| > \Delta\bigr\},$$

where P_{θ_0} is the probability measure induced by the process X_t when θ_0 is the true parameter and $\varepsilon \to 0$. We can get

$$\begin{aligned}
N(A_0) &= \rho\bigl(Y(\theta_0), M(A_0)\bigr) \le \bigl\|Y(\theta_0) - M(A_0)\bigr\|_\infty \\
&= \left\| e^{\theta_0 t}\left(\int_0^t e^{-\theta_0 s}\, dL_s^d - \frac{A_0 t}{T} - A_0 + A_0\right)\right\|_\infty \\
&= \left\| e^{\theta_0 t}\left(\int_0^t e^{-\theta_0 s}\, dL_s^d - A_0 + \left(1 - \frac{t}{T}\right) A_0\right)\right\|_\infty \\
&= \left\| e^{\theta_0 t}\left(\int_0^t e^{-\theta_0 s}\, dL_s^d - A_0\right)\right\|_\infty + |A_0 t| \left\|\left(1 - \frac{t}{T}\right) e^{\theta_0 t}\right\|_\infty.
\end{aligned}$$

On the other hand, for $\omega \in V_\Delta$, we have

$$\begin{aligned}
N(\omega) &= \rho\bigl(Y(\theta_0), M(\omega)\bigr) \\
&\ge \rho\bigl(M(A_0), M(\omega)\bigr) - \rho\bigl(Y(\theta_0), M(A_0)\bigr)
\end{aligned}$$

$$= \left\| M(\omega) - M(A_0) \right\|_\infty - N(A_0)$$
$$= |\omega - A_0| \left\| \frac{te^{\theta_0 t}}{T} \right\|_\infty - N(A_0)$$
$$\geq \Delta \left\| \frac{te^{\theta_0 t}}{T} \right\|_\infty - N(A_0).$$

Hence, we have

$$\frac{N(\omega)}{N(A_0)} \geq \frac{\Delta \| \frac{te^{\theta_0 t}}{T} \|_\infty}{N(A_0)} - 1,$$
$$\frac{\inf_{\omega \in V_\Delta} N(\omega)}{N(A_0)} \geq \Delta \left[\frac{T \| e^{\theta_0 t} (\int_0^t e^{-\theta_0 s}\, dL_s^d - A_0) \|_\infty}{\| te^{\theta_0 t} \|_\infty} + \frac{|A_0| \| (T-t) e^{\theta_0 t} \|_\infty}{\| te^{\theta_0 t} \|_\infty} \right]^{-1} - 1$$
$$= \Delta \left[\frac{T \| e^{\theta_0 t} (B_t - A_0) \|_\infty}{\| te^{\theta_0 t} \|_\infty} + \frac{|R_0| \| (T-t) e^{\theta_0 t} \|_\infty}{\| te^{\theta_0 t} \|_\infty} \right]^{-1} - 1$$
$$= \left[e^{-\theta_0 T} \left\| e^{\theta_0 t} A_t \right\|_\infty + \frac{|A_0|}{T\theta_0 e} \right]^{-1} - 1,$$

where we get the maximum value of the function $(T-t)e^{\theta_0 t}$ by taking the derivative.

We obtain

$$\frac{|A_0|}{T\theta_0 e} \to 0 \quad \text{a.s. as } T \to +\infty. \tag{20}$$

Using Lemma 2.2 we have

$$P_{\theta_0} \left(e^{-\theta_0 T} \left\| e^{\theta_0 t} A_t \right\|_\infty > \Delta \right) = P_{\theta_0} \left(|A_0| > e^{\theta_0 T} \Delta \right) \leq e^{-\theta_0 T} \frac{E_{\theta_0}(|A_0|)}{\Delta} \to 0, \quad T \to +\infty. \tag{21}$$

By (20) and (21), we obtain

$$\frac{\inf_{\omega \in V_\Delta} N(\omega)}{N(A_0)} \xrightarrow{P} +\infty, \quad T \to +\infty. \tag{22}$$

In addition, using (18), $\xi_T \in V_\Delta$, we have

$$N(\xi_T) = \inf_{\omega \in V_\Delta} N(\omega) \leq N(A_0). \tag{23}$$

We can get the desired result (19) by Eqs. (22) and (23). □

3 Minimum L_1-norm estimation

In this section, we will study the minimum L_1-norm estimation $\widetilde{\theta}_\varepsilon$ of the drift parameter θ. Let

$$D_T(\theta) = \int_0^T \left| X_t - x_t(\theta) \right| dt. \tag{24}$$

It is well known that $\widetilde{\theta}_\varepsilon$ is the minimum L_1-norm estimator if there exists a measurable selection $\widetilde{\theta}_\varepsilon$ such that

$$D_T(\widetilde{\theta}_\varepsilon) = \inf_{\theta \in \Theta} D_T(\theta). \tag{25}$$

Suppose that there exists a measurable selection $\widetilde{\theta}_\varepsilon$ satisfying the above equation. We can also define the estimator $\widetilde{\theta}_\varepsilon$ by the relation

$$\widetilde{\theta}_\varepsilon = \arg\inf_{\theta\in\Theta} \int_0^T \left|X_t - x_t(\theta)\right| dt. \tag{26}$$

For any $\kappa > 0$, we define

$$\widetilde{f}(\kappa) = \inf_{|\theta-\theta_0|>\kappa} \int_0^T \left|X_t(\theta) - x_t(\theta_0)\right| dt > 0, \quad \text{for any } \kappa > 0. \tag{27}$$

Theorem 3.1 (Consistency) *For any $p \geq 2$, there exists a constant $C_{p,\kappa,d}$ (only depending the p, κ, d), such that, for every $\kappa > 0$, we have*

$$P_{\theta_0}^{(\varepsilon)}\left(|\widetilde{\theta}_\varepsilon - \theta_0| > \kappa\right) \leq C_{p,\kappa,d} E\left(\left|L(1)\right|^p\right) T^{p(d+1/2+\kappa)} \left(\frac{2\varepsilon e^{|\theta_0|T}}{f(\kappa)}\right)^p = O\left(\left(\widetilde{f}(\kappa)\right)^{-p} \varepsilon^p\right). \tag{28}$$

Proof Set $\|\cdot\|$ denotes the L_1-norm, then we have

$$\begin{aligned}
P_{\theta_0}^{(\varepsilon)}\left(|\widetilde{\theta}_\varepsilon - \theta_0| > \kappa\right) &= P_{\theta_0}^{(\varepsilon)}\left\{\inf_{|\theta-\theta_0|\leq\kappa} \left\|X - x(\theta)\right\| > \inf_{|\theta-\theta_0|>\kappa} \left\|X - x(\theta)\right\|\right\} \\
&\leq P_{\theta_0}^{(\varepsilon)}\left\{\inf_{|\theta-\theta_0|\leq\kappa} \left(\left\|X - x(\theta_0)\right\| + \left\|x(\theta) - x(\theta_0)\right\|\right)\right. \\
&\quad \left. > \inf_{|\theta-\theta_0|>\kappa} \left(\left\|x(\theta) - x(\theta_0)\right\| - \left\|X - x(\theta_0)\right\|\right)\right\} \\
&= P_{\theta_0}^{(\varepsilon)}\left\{2\left\|X - x(\theta)\right\| > \inf_{|\theta-\theta_0|>\kappa} \left\|x(\theta) - x(\theta_0)\right\|\right\} \\
&= P_{\theta_0}^{(\varepsilon)}\left\{\left\|X - x(\theta)\right\| > \frac{1}{2}\widetilde{f}(\kappa)\right\}.
\end{aligned}$$

Since the process X_t satisfies the stochastic differential equation (1), it follows that

$$X_t - x_t(\theta_0) = x_0 + \theta_0 \int_0^t X_s\, ds + \varepsilon L_t^d - x_t(\theta_0) = \theta_0 \int_0^t \left(X_s - x_s(\theta_0)\right) ds + \varepsilon L_t^d, \tag{29}$$

where $x_t(\theta) = x_0 e^{\theta t}$.

Similar to the proof of Theorem 2.1, we have

$$\sup_{0\leq t\leq T} \left|X_t - x_t(\theta_0)\right| \leq \varepsilon e^{|\theta_0 T|} \sup_{0\leq t\leq T} \left|L_t^d\right|. \tag{30}$$

Thus,

$$P_{\theta_0}^{(\varepsilon)}\left\{\left\|X - x(\theta)\right\| > \frac{1}{2}\widetilde{f}(\kappa)\right\} \leq P\left(\sup_{0\leq t\leq T} \left|L_t^d\right| \geq \frac{\widetilde{f}(\kappa)}{2\varepsilon e^{|\theta_0 T|}}\right). \tag{31}$$

Applying Lemma 1.3 to the estimate obtained above, we have

$$P_{\theta_0}^{(\varepsilon)}\left(|\widetilde{\theta}_\varepsilon - \theta_0| > \kappa\right) \leq E\left(\sup_{0\leq t\leq T} \left|L_t^d\right|\right)^p \left(\frac{2\varepsilon e^{|\theta_0 T|}}{\widetilde{f}(\kappa)}\right)^p$$

$$\leq C_{p,\kappa,d}E\bigl(\bigl|L(1)\bigr|^p\bigr)T^{p(d+1/2+\kappa)}2^p e^{|\theta_0 T|p}\bigl(\widetilde{f}(\kappa)\bigr)^{-p}\varepsilon^p$$
$$= O\bigl(\bigl(\widetilde{f}(\kappa)\bigr)^{-p}\varepsilon^p\bigr).$$

This completes the proof. □

Remark 3.1 It follows from Theorem 3.1 that we have $\widetilde{\theta}_\varepsilon$ converges in probability to θ_0 under $P_{\theta_0}^{(\varepsilon)}$-measure as $\varepsilon \to 0$. Furthermore, the rate of convergence is of order $O(\varepsilon^p)$ for every $p \geq 2$.

Theorem 3.2 (Limit distribution) *As* $\varepsilon \to 0$, $\varepsilon^{-1}(\widetilde{\theta}_\varepsilon - \theta_0) \xrightarrow{d} \xi$, ξ *has the same probability distribution as* $\widetilde{\eta}$ *under* $P_{\theta_0}^{(\varepsilon)}$

$$\widetilde{\eta} = \arg \inf_{-\infty<u<+\infty} \int_0^T \bigl|Y_t(\theta) - utx_0 e^{\theta_0 t}\bigr|\,dt. \tag{32}$$

Proof Let

$$Z_\varepsilon(u) = \bigl\|Y - \varepsilon^{-1}\bigl(x(\theta_0 + \varepsilon u) - x(\theta_0)\bigr)\bigr\| \tag{33}$$

and

$$Z_0(u) = \bigl\|Y - u\dot{x}(\theta_0)\bigr\|. \tag{34}$$

Furthermore, let

$$A_\varepsilon = \bigl\{\omega : |\widetilde{\theta}_\varepsilon - \theta_0| < \delta_\varepsilon\bigr\}, \qquad \delta_\varepsilon = \varepsilon^\tau \tau, \tau \in \left(\frac{1}{2}, 1\right), \qquad L_\varepsilon = \varepsilon^{\tau-1}. \tag{35}$$

It is easy to see that the random variable $\widetilde{u}_\varepsilon = \varepsilon^{-1}(\widetilde{\theta}_\varepsilon - \theta_0)$ satisfies the equation

$$Z_\varepsilon(\widetilde{u}_\varepsilon) = \inf_{|u|<L_\varepsilon} Z_\varepsilon(u), \quad \omega \in A_\varepsilon. \tag{36}$$

Define

$$\widetilde{\eta}_\varepsilon = \arg \inf_{|u|<L_\varepsilon} Z_0(u). \tag{37}$$

Observe that, with probability one,

$$\begin{aligned}\sup_{|u|<L_\varepsilon} \bigl|Z_\varepsilon(u) - Z_0(u)\bigr| &= \bigl|\bigl\|Y - u\dot{x}(\theta_0) - 1/2\varepsilon u^2\ddot{x}(\widetilde{\theta})\bigr\| - \bigl\|Y - u\dot{x}(\theta_0)\bigr\|\bigr| \\ &\leq \frac{\varepsilon}{2}L_\varepsilon^2 \sup_{|\theta-\theta_0|<\delta_\varepsilon} \int_0^T \bigl|\ddot{x}(\theta)\bigr|\,dt \leq C\varepsilon^{2\tau-1} \to 0, \quad \varepsilon \to 0,\end{aligned} \tag{38}$$

where $\widetilde{\theta} = \theta_0 + \alpha(\theta - \theta_0)$ for some $\alpha \in (0,1]$. Note that the last term in the above inequality tends to zero as $\varepsilon \to 0$. This follows from the arguments given in Theorem 2 of Kutoyants and Pilibossian [14, 15]. In addition, we can choose the interval $[-L, L]$ such that

$$P_{\theta_0}^{(\varepsilon)}\bigl\{u_\varepsilon^* \in (-L, L)\bigr\} \geq 1 - \beta\widetilde{f}(L)^{-p} \tag{39}$$

and

$$P\{u^* \in (-L, L)\} \geq 1 - \beta \widetilde{f}(L)^{-p}, \quad \beta > 0. \tag{40}$$

Note that $\widetilde{f}(L)$ increases as L increases. The process $\{Z_\varepsilon(u), u \in [-L, L]\}$ and $\{Z_0(u), u \in [-L, L]\}$ satisfy the Lipschitz conditions and $Z_\varepsilon(u)$ converges uniformly to $Z_0(u)$ over $u \in [-L, L]$. Hence the minimizer of $Z_\varepsilon(\cdot)$ converges to the minimizer of $Z_0(u)$. This completes the proof. □

Although the distribution of $\widetilde{\eta}$ is not clear, we can consider its limiting behaviors as $T \to +\infty$.

Theorem 3.3 (Asymptotic law) *Suppose that* $\theta_0 > 0$ *and* $E(\log(1 + |L_1|)) < +\infty$. *Then*

$$\widetilde{\xi}_T = x_0 T \widetilde{\eta}_T \xrightarrow{d} A_0, \quad T \to +\infty,$$

where L_1, A_0 *and other notations in the following are the same as Theorem* 2.3.

Proof Recall that

$$\widetilde{\eta}_T = \arg \inf_{u \in R} \int_0^T \left| Y_t(\theta_0) - u t x_0 e^{\theta_0 t} \right| dt.$$

Let $\|\cdot\|$ denote the L_1-norm. By changing variable, we have the following:

$$\widetilde{\xi}_T = \arg \inf_{\omega \in R} \left\| Y - \widetilde{M}(\omega) \right\| := \arg \inf_{\omega \in R} \widetilde{N}(\omega), \tag{41}$$

where $\widetilde{M}_t(\omega) = \frac{\omega t e^{\theta_0 t}}{T}$ and $\widetilde{N}(\cdot) = \|Y - \widetilde{M}(\cdot)\|$.

We want to show that, for every $\Delta > 0$,

$$\lim_{T \to +\infty} P_{\theta_0} \{ |\widetilde{\xi}_T - A_0| > \Delta \} = 0. \tag{42}$$

Therefore, we consider the set

$$V_\Delta = \{ \omega : |\omega - A_0| > \Delta \},$$

where P_{θ_0} is the probability measure induced by the process X_t when θ_0 is the true parameter and $\varepsilon \to 0$.

Besides, we have

$$\begin{aligned}
\widetilde{N}(A_0) &= \left\| Y - \widetilde{N}(A_0) \right\| \\
&= \left\| e^{\theta_0 t} \left(\int_0^t e^{-\theta_0 s} \, dL_s^d - \frac{A_0 t}{T} - A_0 + A_0 \right) \right\| \\
&= \left\| e^{\theta_0 t} \left(\int_0^t e^{-\theta_0 s} \, dL_s^d - A_0 + \left(1 - \frac{t}{T}\right) A_0 \right) \right\| \\
&= \left\| e^{\theta_0 t} \left(\int_0^t e^{-\theta_0 s} \, dL_s^d - A_0 \right) \right\| + |A_0 t| \left\| \left(1 - \frac{t}{T}\right) e^{\theta_0 t} \right\|.
\end{aligned}$$

On the other hand, for $\omega \in V_\Delta$, we can get

$$\begin{aligned}
\widetilde{N}(\omega) &= \|Y - \widetilde{M}(\omega)\| \\
&\geq \|\widetilde{M}(A_0) - \widetilde{M}(\omega)\| - \|Y - \widetilde{M}(A_0)\| \\
&= \|\widetilde{M}(\omega) - \widetilde{M}(A_0)\| - \widetilde{N}(A_0) \\
&= |\omega - R_0| \left\| \frac{te^{\theta_0 t}}{T} \right\| - \widetilde{N}(A_0) \\
&\geq \Delta \left\| \frac{te^{\theta_0 t}}{T} \right\| - \widetilde{N}(A_0).
\end{aligned}$$

Obviously, we have

$$\begin{aligned}
&\frac{\widetilde{N}(\omega)}{\widetilde{N}(A_0)} \geq \frac{\Delta \| \frac{te^{\theta_0 t}}{T} \|}{\widetilde{N}(A_0)} - 1, \\
&\frac{\inf_{\omega \in V_\Delta} \widetilde{N}(\omega)}{\widetilde{N}(A_0)} \geq \Delta \left[\frac{T \| e^{\theta_0 t}(\int_0^t e^{-\theta_0 s} \, dM_s^d - A_0) \|}{\| te^{\theta_0 t} \|} + \frac{|A_0| \| (T - t) e^{\theta_0 t} \|}{\| te^{\theta_0 t} \|} \right]^{-1} - 1 \\
&\qquad = \Delta \left[\frac{T \| e^{\theta_0 t}(B_t - A_0) \|}{\| te^{\theta_0 t} \|} + \frac{|A_0| \| (T - t) e^{\theta_0 t} \|}{\| te^{\theta_0 t} \|} \right]^{-1} - 1 \\
&\qquad = \Delta (I_1 + I_2)^{-1} - 1,
\end{aligned}$$

with

$$\begin{aligned}
I_1 &= \frac{T \| e^{\theta_0 t}(B_t - A_0) \|}{\| te^{\theta_0 t} \|} = \frac{T \| e^{\theta_0 t} A_t \|}{\int_0^T te^{\theta_0 t} \, dt} = \frac{T \| e^{\theta_0 t} A_t \|}{\theta_0^{-1} Te^{\theta_0 T} - \theta_0^{-2} e^{\theta_0 T} + \theta_0^{-2}}, \\
I_2 &= \frac{|A_0| \| (T - t) e^{\theta_0 t} \|}{\| te^{\theta_0 t} \|} = \frac{|A_0| \int_0^T e^{\theta_0 t} \, dt}{\int_0^T te^{\theta_0 t} \, dt} = \frac{|A_0| (\theta_0^{-2} e^{\theta_0 T} - \theta_0 T - \theta_0^{-2})}{\theta_0^{-1} Te^{\theta_0 T} - \theta_0^{-2} e^{\theta_0 T} + \theta_0^{-2}}.
\end{aligned}$$

We obtain with probability one

$$\begin{aligned}
\lim_{T \to +\infty} I_2 &= \lim_{T \to +\infty} \frac{|A_0| (\theta_0^{-2} e^{\theta_0 T} - \theta_0 T - \theta_0^{-2})}{\theta_0^{-1} Te^{\theta_0 T} - \theta_0^{-2} e^{\theta_0 T} + \theta_0^{-2}} \\
&= \lim_{T \to +\infty} \frac{|A_0| \theta_0^{-2} e^{\theta_0 T}}{\theta_0^{-1} Te^{\theta_0 T}} = \lim_{T \to +\infty} \frac{|A_0|}{\theta_0 T} = 0.
\end{aligned} \tag{43}$$

Moreover, using Lemma 2.2 we obtain

$$\begin{aligned}
\lim_{T \to +\infty} P_{\theta_0}(I_1 > \Delta) &= \lim_{T \to +\infty} P_{\theta_0} \left(\frac{T \| e^{\theta_0 t} R_t \|}{\theta_0^{-1} Te^{\theta_0 T} - \theta_0^{-2} e^{\theta_0 T} + \theta_0^{-2}} > \Delta \right) \\
&= \lim_{T \to +\infty} P_{\theta_0} \left(\frac{T \| e^{\theta_0 t} R_t \|}{\theta_0^{-1} Te^{\theta_0 T}} > \Delta \right) \\
&= \lim_{T \to +\infty} P_{\theta_0} \left(|R_0| > \theta_0 e^{\theta_0 T} \Delta \right) \leq \lim_{T \to +\infty} \theta_0^{-1} e^{-\theta_0 T} \frac{E_{\theta_0}(|R_0|)}{\Delta} = 0.
\end{aligned} \tag{44}$$

By (43) and (44), we obtain as $T \to +\infty$

$$\frac{\inf_{\omega\in V_\Delta} \widetilde{N}(\omega)}{\widetilde{N}(A_0)} \xrightarrow{P} +\infty. \tag{45}$$

Using (41), $\widetilde{\xi}_T \in V_\Delta$ implies

$$\widetilde{N}(\xi_T) = \inf_{\omega\in V_\Delta} \widetilde{N}(\omega) \le \widetilde{N}(A_0). \tag{46}$$

Therefore, from Eqs. (45) and (46), we have the result (42). □

Remark 3.2 If L_t^d is a Brownian motion, then $\widetilde{\xi}_T$ is asymptotically Gaussian, this is treated by Kutoyants and Pilibossian [14, 15].

Acknowledgements
The authors are grateful to the referee for carefully reading the manuscript and for providing some comments and suggestions which led to improvements in this paper.

Funding
This research is supported by the Distinguished Young Scholars Foundation of Anhui Province (1608085J06), the National Natural Science Foundation of China (11271020, 11601260), the Top Talent Project of University Discipline (Speciality) (gxbjZD03), the Natural Science Foundation of Chuzhou University (2016QD13), and Natural Science Foundation of Shandong Province (ZR2016AB01).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

Author details
[1]School of Mathematics and Finance, Chuzhou University, Chuzhou, China. [2]Department of Statistics, Anhui Normal University, Wuhu, China. [3]School of Statistics, Qufu Normal University, Qufu, China.

Publisher's Note

Received: 11 September 2018 Accepted: 18 December 2018 Published online: 29 December 2018

References

1. Benassi, A., Cohen, S., Istas, J.: Identification and properties of real harmonizable fractional Lévy motions. Bernoulli **8**, 97–115 (2002)
2. Bender, C., Knobloch, R., Oberacker, P.: Maximal inequalities for fractional Lévy and related processes. Stoch. Anal. Appl. **33**, 701–714 (2015)
3. Bender, C., Lindner, A., Schicks, M.: Finite variation of fractional Lévy processes. J. Theor. Probab. **25**, 594–612 (2012)
4. Diop, A., Yode, A.F.: Minimum distance parameter estimation for Ornstein–Uhlenbeck processes driven by Lévy process. Stat. Probab. Lett. **80**, 122–127 (2010)
5. El Onsy, B., Es-Sebaiy, K., Ndiaye, D.: Parameter estimation for discretely observed non-ergodic fractional Ornstein–Uhlenbeck processes of the second kind. Braz. J. Probab. Stat. **32**, 545–558 (2018)
6. Engelke, S.: A unifying approach to fractional Lévy processes. Stoch. Dyn. **13**, 1250017 (2013) 19 pages
7. Fink, H., Klüppelberg, C.: Fractional Lévy-driven Ornstein–Uhlenbeck processes and stochastic differential equations. Bernoulli **17**, 484–506 (2011)
8. Genon-Catalot, V.: Maximum contrast estimation for diffusion processes from discrete observations. Statistics **21**, 99–116 (1990)
9. Gloter, A., Sørensen, M.: Estimation for stochastic differential equations with a small diffusion coefficient. Stoch. Process. Appl. **119**, 679–699 (2009)
10. Hénaff, S.: Asymptotics of a minimum diatance estimator of the Ornstein–Uhlenbeck process. C. R. Acad. Sci., Sér. 1 Math. **325**, 911–914 (1997)
11. Hu, Y., Nualart, D.: Parameter estimation for fractional Ornstein–Uhlenbeck processes. Stat. Probab. Lett. **80**, 1030–1038 (2010)
12. Jiang, H., Dong, X.: Parameter estimation for the non-stationary Ornstein–Uhlenbeck process with linear drift. Stat. Pap. **56**, 257–268 (2015)
13. Jurek, Z.J., Mason, J.D.: Operator-Limit Distributions in Probability Theory. Wiley, New York (1993)

14. Kutoyants, Y., Pilibossian, P.: On minimum L_1-norm estimate of the parameter of the Ornstein–Uhlenbeck process. Stat. Probab. Lett. **20**, 117–123 (1994)
15. Kutoyants, Y., Pilibossian, P.: On minimum uniform metric estimate of parameters of diffusion-type processes. Stoch. Process. Appl. **51**, 259–267 (1994)
16. Lacaux, C.: Real harmonizable multifractional Lévy motions. Ann. Inst. Henri Poincaré **40**, 259–277 (2004)
17. Laredo, C.F.: A sufficient condition for asymptotic sufficiency of incomplete observations of a diffusion process. Ann. Stat. **18**, 1158–1171 (1990)
18. Lin, Z., Cheng, Z.: Existence and joint continuity of local time of multiparameter fractional Lévy processes. Appl. Math. Mech. **30**, 381–390 (2009)
19. Liu, Z., Song, N.: Minimum distance estimation for fractional Ornstein–Uhlenbeck type process. Adv. Differ. Equ. **2014**, 137 (2014)
20. Long, H.: Least squares estimator for discretely observed Ornstein–Uhlenbeck processes with small Lévy noises. Stat. Probab. Lett. **79**, 2076–2085 (2009)
21. Ma, C.: A note on "Least squares estimator for discretely observed Ornstein–Uhlenbeck processes with small Lévy noises". Stat. Probab. Lett. **80**, 1528–1531 (2010)
22. Mandelbrot, B.B., Van Ness, J.W.: Fractional Brownian motions, fractional noises and applications. SIAM Rev. **10**, 422–437 (1968)
23. Marquardt, T.: Fractional Lévy processes with an application to long memory moving average processes. Bernoulli **12**, 1009–1126 (2006)
24. Millar, P.W.: A general approach to the optimality of the minimum distance estimators. Trans. Am. Math. Soc. **286**, 377–418 (1984)
25. Prakasa Rao, B.L.S.: Minimum L_1-norm estimation for fractional Ornstein–Uhlenbeck type process. Theory Probab. Math. Stat. **71**, 181–189 (2005)
26. Shen, G., Yu, Q.: Least squares estimator for Ornstein–Uhlenbeck processes driven by fractional Lévy processes from discrete observations. Stat. Pap. (2017). https://doi.org/10.1007/s00362-017-0918-4
27. Sørensen, M.: Small dispersion asymptotics for diffusion martingale estimating functions. Preprint No. 2000-2, Department of Statistics and Operation Research, University of Copenhagen, Copenhagen (2000)
28. Sørensen, M., Uchida, M.: Small diffusion asymptotics for discretely sampled stochastic differential equations. Bernoulli **9**, 1051–1069 (2003)
29. Xiao, W., Zhang, W., Xu, W.: Parameter estimation for fractional Ornstein–Uhlenbeck processes at discrete observation. Appl. Math. Model. **35**, 4196–4207 (2011)