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On properties of geodesic semilocal E-preinvex functions

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Abstract

The authors define a class of functions on Riemannian manifolds, which are called geodesic semilocal E-preinvex functions, as a generalization of geodesic semilocal E-convex and geodesic semi E-preinvex functions, and some of its properties are established. Furthermore, a nonlinear fractional multiobjective programming is considered, where the functions involved are geodesic E-η-semidifferentiability, sufficient optimality conditions are obtained. A dual is formulated and duality results are proved by using concepts of geodesic semilocal E-preinvex functions, geodesic pseudo-semilocal E-preinvex functions, and geodesic quasi-semilocal E-preinvex functions.

Keywords: Generalized convexity; Riemannian geometry; Duality

1 Introduction

Convexity and generalized convexity play a significant role in many fields, for example, in biological system, economy, optimization, and so on [1–5].

Generalized convex functions, labeled as semilocal convex functions, were introduced by Ewing [6] by using more general semilocal preinvexity and η-semidifferentiability. After that optimality conditions for weak vector minima were given [7]. Also, optimality conditions and duality results for a nonlinear fractional involving η-semidifferentiability were established [8].

Furthermore, some optimality conditions and duality results for semilocal E-convex programming were established [9]. E-convexity was extended to E-preinvexity [10]. Recently, semilocal E-preinvexity (SLEP) and some of its applications were introduced [11–13].

Generalized convex functions in manifolds, such as Riemannian manifolds, were studied by many authors; see [14–17]. Udrist [18] and Rapcsak [19] considered a generalization of convexity called geodesic convexity. In this setting, the linear space is replaced by a Riemannian manifold and the line segment by a geodesic one. In 2012, geodesic E-convex (GEC) sets and geodesic E-convex (GEC) functions on Riemannian manifolds were studied [20]. Moreover, geodesic semi E-convex (GsEC) functions were introduced [21]. Recently, geodesic strongly E-convex (GSEC) functions were introduced and some of their properties were discussed [22].

2 Geodesic semilocal E-preinvexity

Assume that $\aleph$ is a complete n-dimensional Riemannian manifold with Riemannian connection $\bigtriangledown$. Let $\kappa_1, \kappa_2 \in \aleph$ and $\gamma : [0,1] \longrightarrow \aleph$ be a geodesic joining the points κ_1 and κ_2, which means that $\gamma_{\kappa_1,\kappa_2}(0) = \kappa_2$ and $\gamma_{\kappa_1,\kappa_2}(1) = \kappa_1$.

Definition 2.1 A nonempty set $B \subset \aleph$ is said to be

1. a geodesic E-invex (GEI) with respect to η if there is exactly one geodesic $\gamma_{E(\kappa_1),E(\kappa_2)} : [0,1] \longrightarrow \aleph$ such that

$$\gamma_{E(\kappa_1),E(\kappa_2)}(0) = E(\kappa_2), \qquad \acute{\gamma}_{E(\kappa_1),E(\kappa_2)} = \eta\big(E(\kappa_1), E(\kappa_2)\big), \qquad \gamma_{E(\kappa_1),E(\kappa_2)}(t) \in B,$$

   $\forall \kappa_1, \kappa_2 \in B$ and $t \in [0,1]$.
2. a geodesic local E-invex (GLEI) with respect to η if there is $u(\kappa_1, \kappa_2) \in (0,1]$ such that $\forall t \in [0, u(\kappa_1, \kappa_2)]$,

$$\gamma_{E(\kappa_1),E(\kappa_2)}(t) \in B \quad \forall \kappa_1, \kappa_2 \in B. \tag{1}$$

3. a geodesic local starshaped E-convex if there is a map E such that, corresponding to each pair of points $\kappa_1, \kappa_2 \in A$, there is a maximal positive number $u(\kappa_1, \kappa_2) \leq 1$ such as

$$\gamma_{E(\kappa_1),E(\kappa_2)} \in A, \quad \forall t \in \big[0, u(\kappa_1, \kappa_2)\big]. \tag{2}$$

Definition 2.2 A function $f : A \subset \aleph \longrightarrow \mathbb{R}$ is said to be

1. a geodesic E-preinvex (GEP) on $A \subset \aleph$ with respect to η if A is a GEI set and

$$f\big(\gamma_{E(\kappa_1),E(\kappa_2)}(t)\big) \leq tf\big(E(\kappa_1)\big) + (1-t)f\big(E(\kappa_2)\big), \quad \forall \kappa_1, \kappa_2 \in A, t \in [0,1];$$

2. a geodesic semi E-preinvex (GSEP) on A with respect to η if A is a GEI set and

$$f\big(\gamma_{E(\kappa_1),E(\kappa_2)}(t)\big) \leq tf(\kappa_1) + (1-t)f(\kappa_2), \quad \forall \kappa_1, \kappa_2 \in A, t \in [0,1].$$

3. a geodesic local E-preinvex (GLEP) on $A \subset \aleph$ with respect to η if, for any $\kappa_1, \kappa_2 \in A$, there exists $0 < \nu(\kappa_1, \kappa_2) \leq u(\kappa_1, \kappa_2)$ such that A is a GLEI set and

$$f\big(\gamma_{E(\kappa_1),E(\kappa_2)}(t)\big) \leq tf\big(E(\kappa_1)\big) + (1-t)f\big(E(\kappa_2)\big), \quad \forall t \in \big[0, \nu(\kappa_1, \kappa_2)\big].$$

Definition 2.3 A function $f : \aleph \longrightarrow \mathbb{R}$ is a geodesic semilocal E-convex (GSLEC) on a geodesic local starshaped E-convex set $B \subset \aleph$ if, for each pair of $\kappa_1, \kappa_2 \in B$ (with a maximal positive number $u(\kappa_1, \kappa_2) \leq 1$ satisfying 2), there exists a positive number $\nu(\kappa_1, \kappa_2) \leq u(\kappa_1, \kappa_2)$ satisfying

$$f\big(\gamma_{E(\kappa_1),E(\kappa_2)}(t)\big) \leq tf(\kappa_1) + (1-t)f(\kappa_2), \quad \forall t \in \big[0, \nu(\kappa_1, \kappa_2)\big].$$

Remark 2.1 Every GEI set with respect to η is a GLEI set with respect to η, where $u(\kappa_1, \kappa_2) = 1$, $\forall \kappa_1, \kappa_2 \in \aleph$. On the other hand, their converses are not necessarily true, and we can see that in the next example.

Example 2.1 Put $A = [-4,-1) \cup [1,4]$,

$$E(\kappa) = \begin{cases} \kappa^2 & \text{if } |\kappa| \leq 2, \\ -1 & \text{if } |\kappa| > 2; \end{cases}$$

$$\eta(\kappa,\iota) = \begin{cases} \kappa - \iota & \text{if } \kappa \geq 0, \iota \geq 0 \text{ or } \kappa \leq 0, \iota \leq 0, \\ -1-\iota & \text{if } \kappa > 0, \iota \leq 0 \text{ or } \kappa \geq 0, \iota < 0, \\ 1-\iota & \text{if } \kappa < 0, \iota \geq 0 \text{ or } \kappa \leq 0, \iota > 0; \end{cases}$$

$$\gamma_{\kappa,\iota}(t) = \begin{cases} \iota + t(\kappa - l) & \text{if } \kappa \geq 0, \iota \geq 0 \text{ or } \kappa \leq 0, \iota \leq 0, \\ \iota + t(-1-\iota) & \text{if } \kappa > 0, \iota \leq 0 \text{ or } \kappa \geq 0, \iota < 0, \\ \iota + t(1-\iota) & \text{if } \kappa < 0, \iota \geq 0 \text{ or } \kappa \leq 0, \iota > 0. \end{cases}$$

Hence A is a GLEI set with respect to η. However, when $\kappa = 3$, $\iota = 0$, there is $t_1 \in [0,1]$ such that $\gamma_{E(\kappa),E(\iota)}(t_1) = -t_1$, then if $t_1 = 1$, we obtain $\gamma_{E(\kappa),E(\iota)}(t_1) \notin A$.

Definition 2.4 A function $f : \aleph \longrightarrow \mathbb{R}$ is GSLEP on $B \subset \aleph$ with respect to η if, for any $\kappa_1, \kappa_2 \in B$, there is $0 < \nu(\kappa_1,\kappa_2) \leq u(\kappa_1,\kappa_2) \leq 1$ such that B is a GLEI set and

$$f\bigl(\gamma_{E(\kappa_1),E(\kappa_2)}(t)\bigr) \leq tf(\kappa_1) + (1-t)f(\kappa_2), \quad \forall t \in \bigl[0, \nu(\kappa_1,\kappa_2)\bigr]. \tag{3}$$

If

$$f\bigl(\gamma_{E(\kappa_1),E(\kappa_2)}(t)\bigr) \geq tf(\kappa_1) + (1-t)f(\kappa_2), \quad \forall t \in \bigl[0, \nu(\kappa_1,\kappa_2)\bigr],$$

then f is GSLEP on B.

Remark 2.2 Any GSLEC function is a GSLEP function. Also, any GSEP function with respect to η is a GSLEP function. On the other hand, their converses are not necessarily true. The next example shows SLGEP, which is neither a GSLEC function nor a GSEP function.

Example 2.2 Assume that $E : \mathbb{R} \longrightarrow \mathbb{R}$ is given as

$$E(m) = \begin{cases} 0 & \text{if } m < 0, \\ 1 & \text{if } 1 < m \leq 2, \\ m & \text{if } 0 \leq m \leq 1 \text{ or } m > 2; \end{cases}$$

and the map $\eta : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is defined as

$$\eta(m,n) = \begin{cases} 0 & \text{if } m = n, \\ 1-m & \text{if } m \neq n; \end{cases}$$

also,

$$\gamma_{m,n}(t) = \begin{cases} n & \text{if } m = n, \\ n + t(1-m) & \text{if } m \neq n. \end{cases}$$

Assume that $h : \mathbb{R} \longrightarrow \mathbb{R}$, where

$$h(m) = \begin{cases} 0 & \text{if } 1 < m \leq 2, \\ 1 & \text{if } m > 2, \\ -m+1 & \text{if } 0 \leq m \leq 1, \\ -m+2 & \text{if } m < 0; \end{cases}$$

and since $\mathbb{R}$ is a geodesic local starshaped E-convex set and a geodesic local E-invex set with respect to η. Then h is a GSLEP on $\mathbb{R}$ with respect to η. However, when $m_0 = 2$, $n_0 = 3$ and for any $\nu \in (0,1]$, there is a sufficiently small $t_0 \in (0,\nu]$ such that

$$h(\gamma_{E(m_0),E(n_0)}(t_0)) = 1 > (1-t_0) = t_0 h(m_0) + (1-t_0)h(n_0).$$

Then $h(m)$ is not a GSLEC function on $\mathbb{R}$.

Similarly, taking $m_1 = 1$, $n_1 = 4$, we have

$$h(\gamma_{E(m_1),E(n_1)}(t_1)) = 1 > (1-t_1) = t_1 h(m_1) + (1-t_1)h(n_1)$$

for some $t_1 \in [0,1]$.

Hence, $h(m)$ is not a GSEP function on $\mathbb{R}$ with respect to η.

Definition 2.5 A function $h : S \subset \aleph \longrightarrow \mathbb{R}$, where S is a GLEI set, is said to be a geodesic quasi-semilocal E-preinvex (GqSLEP) (with respect to η) if, for all $\kappa_1, \kappa_2 \in S$ satisfying $h(\kappa_1) \leq h(\kappa_2)$, there is a positive number $\nu(\kappa_1,\kappa_2) \leq u(\kappa_1,\kappa_2)$ such that

$$h(\gamma_{E(\kappa_1),E(\kappa_2)}(t)) \leq h(\kappa_2), \quad \forall t \in [0, \nu(\kappa_1,\kappa_2)].$$

Definition 2.6 A function $h : S \subset \aleph \longrightarrow \mathbb{R}$, where S is a GLEI set, is said to be a geodesic pseudo-semilocal E-preinvex (GpSLEP) (with respect to η) if, for all $\kappa_1, \kappa_2 \in S$ satisfying $h(\kappa_1) < h(\kappa_2)$, there are positive numbers $\nu(\kappa_1,\kappa_2) \leq u(\kappa_1,\kappa_2)$ and $w(\kappa_1,\kappa_2)$ such that

$$h(\gamma_{E(\kappa_1),E(\kappa_2)}(t)) \leq h(\kappa_2) - tw(\kappa_1,\kappa_2), \quad \forall t \in [0, \nu(\kappa_1,\kappa_2)].$$

Remark 2.3 Every GSLEP on a GLEI set with respect to η is both a GqELEP function and a GpSLEP function.

Definition 2.7 A function $h : S \longrightarrow \mathbb{R}$ is called a geodesic E-η- semidifferentiable at $\kappa^* \in S$, where $S \subset \aleph$ is a GLEI set with respect to η, if $E(\kappa^*) = \kappa^*$ and

$$h'_+(\gamma_{\kappa^*,E(\kappa)}(t)) = \lim_{t \longrightarrow 0^+} \frac{1}{t}\left[h(\gamma_{\kappa^*,E(\kappa)}(t)) - h(\kappa^*)\right]$$

exist for every $\kappa \in S$.

Remark 2.4

1. If $\aleph = \mathbb{R}^n$, then the geodesic E-η- semidifferentiable is E-η-semidifferentiable [11].

2. If $\aleph = \mathbb{R}^n$ and $E = I$, then the geodesic E-η-semidifferentiable is the η-semidifferentiability [23].
3. If $\aleph = \mathbb{R}^n$, $E = I$, and $\eta(\kappa,\kappa^*) = \kappa - \kappa^*$, then the geodesic E-$\eta$-semidifferentiable is the semidifferentiability [11].

Lemma 2.1

1. *Assume that h is a GSLEP (E-preconcave) and a geodesic E-η-semidifferentiable at $\kappa^* \in S \subset \aleph$, where S is a GLEI set with respect to η. Then*

$$h(\kappa) - h(\kappa^*) \geq (\leq) h'_+\bigl(\gamma_{\kappa^*,E(\kappa)}(t)\bigr), \quad \forall \kappa \in S.$$

2. *Let h be a GqSLEP (GpSLEP) and a geodesic E-η-semidifferentiable at $\kappa^* \in S \subset \aleph$, where S is a LGEI set with respect to η. Hence*

$$h(\kappa) \leq (<) h(\kappa^*) \quad \Rightarrow \quad h'_+\bigl(\gamma_{\kappa^*,E(\kappa)}(t)\bigr) \leq (<) 0, \quad \forall \kappa \in S.$$

The above lemma is proved directly by using definitions (Definition 2.4, Definition 2.5, Definition 2.6, and Definition 2.4).

Theorem 2.1 *Let $f : S \subset \aleph \longrightarrow \mathbb{R}$ be a GLEP function on a GLEI set S with respect to η, then f is a GSLEP function iff $f(E(\kappa)) \leq f(\kappa)$, $\forall \kappa \in S$.*

Proof Assume that f is a GSLEP function on set S with respect to η, then $\forall \kappa_1, \kappa_2 \in S$, there is a positive number $\nu(\kappa_1,\kappa_2) \leq u(\kappa_1,\kappa_2)$ where

$$f\bigl(\gamma_{E(\kappa_1),E(\kappa_2)}(t)\bigr) \leq tf(\kappa_2) + (1-t)f(\kappa_1), \quad t \in \bigl[0, \nu(\kappa_1,\kappa_2)\bigr].$$

By letting $t = 0$, then $f(E(\kappa_1)) \leq f(\kappa_1)$, $\forall \kappa_1 \in S$.

Conversely, consider that f is a GLEP function on a GLEI set S, then for any $\kappa_1, \kappa_2 \in S$, there exist $u(\kappa_1,\kappa_2) \in (0,1]$ (1) and $\nu(\kappa_1,\kappa_2) \in (0, u(\kappa_1,\kappa_2)]$ such that

$$f\bigl(\gamma_{E(\kappa_1),E(\kappa_2)}(t)\bigr) \leq tf\bigl(E(\kappa_1)\bigr) + (1-t)f\bigl(E(\kappa_2)\bigr), \quad t \in \bigl[0, \nu(\kappa_1,\kappa_2)\bigr].$$

Since $f(E(\kappa_1)) \leq f(\kappa_1)$, $\forall \kappa_1 \in S$, then

$$f\bigl(\gamma_{E(\kappa_1),E(\kappa_2)}(t)\bigr) \leq tf(\kappa_1) + (1-t)f(\kappa_2), \quad t \in \bigl[0, \nu(\kappa_1,\kappa_2)\bigr]. \qquad \square$$

Definition 2.8 The set $\omega = \{(\kappa,\alpha) : \kappa \in B \subset \aleph, \alpha \in \mathbb{R}\}$ is said to be a GLEI set with respect to η corresponding to $\aleph$ if there are two maps η, E and a maximal positive number $u((\kappa_1,\alpha_1),(\kappa_2,\alpha_2)) \leq 1$ for each $(\kappa_1,\alpha_1),(\kappa_2,\alpha_2) \in \omega$ such that

$$\bigl(\gamma_{E(\kappa_1),E(\kappa_2)}(t), t\alpha_1 + (1-t)\alpha_2\bigr) \in \omega, \quad \forall t \in \bigl[0, u\bigl((\kappa_1,\alpha_1),(\kappa_2,\alpha_2)\bigr)\bigr].$$

Theorem 2.2 *Let $B \subset \aleph$ be a GLEI set with respect to η. Then f is a GSLEP function on B with respect to η iff its epigraph*

$$\omega_f = \bigl\{(\kappa_1,\alpha) : \kappa_1 \in B, f(\kappa_1) \leq \alpha, \alpha \in \mathbb{R}\bigr\}$$

is a GLEI set with respect to η corresponding to $\aleph$.

Proof Suppose that f is a GSLEP on B with respect to η and $(\kappa_1,\alpha_1),(\kappa_2,\alpha_2) \in \omega_f$, then $\kappa_1,\kappa_2 \in B$, $f(\kappa_1) \leq \alpha_1$, $f(\kappa_2) \leq \alpha_2$. By applying Definition 2.1, we obtain $\gamma_{E(\kappa_1),E(\kappa_2)}(t) \in B$, $\forall t \in [0,u(\kappa_1,\kappa_2)]$.

Moreover, there is a positive number $\nu(\kappa_1,\kappa_2) \leq u(\kappa_1,\kappa_2)$ such that

$$f\bigl(\gamma_{E(\kappa_1),E(\kappa_2)}(t), t\alpha_1 + (1-t)\alpha_2\bigr) \in \omega_f, \quad \forall t \in \bigl[0,\nu(\kappa_1,\kappa_2)\bigr].$$

Conversely, if ω_f is a GLEI set with respect to η corresponding to $\aleph$, then for any points $(\kappa_1,f(\kappa_1)),(\kappa_2,f(\kappa_2)) \in \omega_f$, there is a maximal positive number $u((\kappa_1,f(\kappa_1)),(\kappa_2,f(\kappa_2)) \leq 1$ such that

$$\bigl(\gamma_{E(\kappa_1),E(\kappa_2)}(t), tf(\kappa_1) + (1-t)f(\kappa_2)\bigr) \in \omega_f, \quad \forall t \in \bigl[0,u\bigl(\bigl(\kappa_1,f(\kappa_1)\bigr),\bigl(\kappa_2,f(\kappa_2)\bigr)\bigr)\bigr].$$

That is, $\gamma_{E(\kappa_1),E(\kappa_2)}(t) \in B$,

$$f\bigl(\gamma_{E(\kappa_1),E(\kappa_2)}(t)\bigr) \leq tf(\kappa_1) + (1-t)f(\kappa_2), \quad t \in \bigl[0,u\bigl(\bigl(\kappa_1,f(\kappa_1)\bigr),\bigl(\kappa_2,f(\kappa_2)\bigr)\bigr)\bigr].$$

Thus, B is a GLEI set and f is a GSLEP function on B. □

Theorem 2.3 *If f is a GSLEP function on a GLEI set $B \subset \aleph$ with respect to η, then the level $K_\alpha = \{\kappa_1 \in B : f(\kappa_1) \leq \alpha\}$ is a GLEI set for any $\alpha \in \mathbb{R}$.*

Proof For any $\alpha \in \mathbb{R}$ and $\kappa_1,\kappa_2 \in K_\alpha$, then $\kappa_1,\kappa_2 \in B$ and $f(\kappa_1) \leq \alpha$, $f(\kappa_2) \leq \alpha$. Since B is a GLEI set, then there is a maximal positive number $u(\kappa_1,\kappa_2) \leq 1$ such that

$$\gamma_{E(\kappa_1),E(\kappa_2)}(t) \in B, \quad \forall t \in \bigl[0,u(\kappa_1,\kappa_2)\bigr].$$

In addition, since f is GSLEP, there is a positive number $\nu(\kappa_1,\kappa_2) \leq u(y_1,y_2)$ such that

$$\begin{aligned} f\bigl(\gamma_{E(\kappa_1),E(\kappa_2)}(t)\bigr) &\leq tf(\kappa_1) + (1-t)f(\kappa_2) \\ &\leq t\alpha + (1-t)\alpha \\ &= \alpha, \quad \forall t \in \bigl[0,\nu(\kappa_1,\kappa_2)\bigr]. \end{aligned}$$

That is, $\gamma_{E(\kappa_1),E(\kappa_2)}(t) \in K_\alpha$, $\forall t \in [0,\nu(\kappa_1,\kappa_2)]$. Therefore, K_α is a GLEI set with respect to η for any $\alpha \in \mathbb{R}$. □

Theorem 2.4 *Let $f : B \subset \aleph \longrightarrow \mathbb{R}$, where B is a GLEI. Then f is a GSLEP function with respect to η iff for each pair of points $\kappa_1,\kappa_2 \in B$, there is a positive number $\nu(\kappa_1,\kappa_2) \leq u(\kappa_1,\kappa_2) \leq 1$ such that*

$$f\bigl(\gamma_{E(\kappa_1),E(\kappa_2)}(t)\bigr) \leq t\alpha + (1-t)\beta, \quad \forall t \in \bigl[0,\nu(\kappa_1,\kappa_2)\bigr].$$

Proof Let $\kappa_1,\kappa_2 \in B$ and $\alpha,\beta \in \mathbb{R}$ such that $f(\kappa_1) < \alpha$ and $f(\kappa_2) < \beta$. Since B is GLEI, there is a maximal positive number $u(\kappa_1,\kappa_2) \leq 1$ such that

$$\gamma_{E(\kappa_1),E(\kappa_2)}(t) \in B, \quad \forall t \in \bigl[0,u(\kappa_1,\kappa_2)\bigr].$$

In addition, there is a positive number $\nu(\kappa_1,\kappa_2) \leq u(\kappa_1,\kappa_2)$, where

$$f\big(\gamma_{E(\kappa_1),E(\kappa_2)}(t)\big) \leq t\alpha + (1-t)\beta, \quad \forall t \in \big[0, \nu(\kappa_1,\kappa_2)\big].$$

Conversely, let $(\kappa_1,\alpha) \in \omega_f$ and $(\kappa_2,\beta) \in \omega_f$, then $\kappa_1,\kappa_2 \in B$, $f(\kappa_1) < \alpha$, and $f(\kappa_2) < \beta$. Hence, $f(\kappa_1) < \alpha + \varepsilon$ and $f(\kappa_2) < \beta + \varepsilon$ hold for any $\varepsilon > 0$. According to the hypothesis for $\kappa_1,\kappa_2 \in B$, there is a positive number $\nu(\kappa_1,\kappa_2) \leq u(\kappa_1,\kappa_2) \leq 1$ such that

$$f\big(\gamma_{E(\kappa_1),E(\kappa_2)}(t)\big) \leq t\alpha + (1-t)\beta + \varepsilon, \quad \forall t \in \big[0, \nu(\kappa_1,\kappa_2)\big].$$

Let $\varepsilon \longrightarrow 0^+$, then

$$f\big(\gamma_{E(\kappa_1),E(\kappa_2)}(t)\big) \leq t\alpha + (1-t)\beta, \quad \forall t \in \big[0, \nu(\kappa_1,\kappa_2)\big].$$

That is, $(\gamma_{E(\kappa_1),E(\kappa_2)}(t), t\alpha + (1-t)\beta) \in \omega_f$, $\forall t \in [0, \nu(\kappa_1,\kappa_2)]$.

Therefore, ω_f is a GLEI set corresponding to $\aleph$. From Theorem 2.2 it follows that f is a GSLEP on B with respect to η. □

3 Optimality criteria

In this section, let us consider the nonlinear fractional multiobjective programming problem

$$(\text{VFP}) \begin{cases} \text{minimize } \frac{f(\kappa)}{g(\kappa)} = (\frac{f_1(\kappa)}{g_1(\kappa)}, \ldots, \frac{f_p(\kappa)}{g_p(\kappa)}), \\ \text{subject to } h_j(\kappa) \leq 0, \quad j \in Q = 1,2,\ldots,q \\ \kappa \in K_0; \end{cases}$$

where $K_0 \subset \aleph$ is a GLEI set and $g_i(\kappa) > 0$, $\forall \kappa \in K_0$, $i \in P = 1,2,\ldots,p$.

Let $f = (f_1, f_2, \ldots, f_p)$, $g = (g_1, g_2, \ldots, g_p)$, and $h = (h_1, h_2, \ldots, h_q)$ and denote that $K = \{\kappa : h_j(\kappa) \leq 0, j \in Q, \kappa \in K_0\}$, the feasible set of problem (VFP). For $\kappa^* \in K$, we put

$$Q(\kappa^*) = \{j : h_j(\kappa^*) = 0, j \in Q\}, \qquad L(\kappa^*) = \frac{Q}{Q(\kappa^*)}.$$

We also formulate the nonlinear multiobjective programming problem as follows:

$$(\text{VFP}_\lambda) \begin{cases} \text{minimize } (f_1(\kappa) - \lambda_1 g_1(\kappa), \ldots f_p(\kappa) - \lambda_p g_p(\kappa)), \\ \text{subject to } h_j(\kappa) \leq 0, \quad j \in Q = 1,2,\ldots,q \\ \kappa \in K_0; \end{cases}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p) \in \mathbb{R}^p$.

The following lemma connects the weak efficient solutions for (VFP) and (VFP_λ).

Lemma 3.1 *A point κ^* is a weak efficient solution for* (VFP_λ) *iff κ^* is a weak efficient solution for* (VFP^*_λ), *where* $\lambda^* = (\lambda^*_1, \ldots, \lambda^*_p) = (\frac{f_1(\kappa^*)}{g_1(\kappa^*)}, \ldots, \frac{f_p(\kappa^*)}{g_p(\kappa^*)})$.

Proof Assume that there is a feasible point $\kappa \in K$, where

$$f_i(\kappa) - \lambda_i^* g_i(\kappa) < f_i(\kappa^*) - \lambda_i^* g_i(\kappa^*), \quad \forall i \in Q$$

$\Longrightarrow$

$$f_i(\kappa) < \frac{f_i(\kappa^*)}{g_i(\kappa^*)g_i(\kappa)}$$

$\Longrightarrow$

$$\frac{f_i(\kappa)}{g_i(\kappa)} < \frac{f_i(\kappa^*)}{g_i(\kappa^*)},$$

which is a contradiction to the weak efficiency of κ^* for (VFP).

Presently, let us take $\kappa \in K$ as a feasible point such that

$$\frac{f_i(\kappa)}{g_i(\kappa)} < \frac{f_i(\kappa^*)}{g_i(\kappa^*)} = \lambda_i^*,$$

then $f_i(\kappa) - \lambda_i^* g_i(\kappa) < 0 = f_i(\kappa^*) - \lambda_i^* g_i(\kappa^*)$, $\forall i \in Q$, which is again a contradiction to the weak efficiency of κ^* for (VFP_λ^*). □

Next, some sufficient optimality conditions for the problem (VFP) are established.

Theorem 3.1 *Let $\bar{\kappa} \in K$, $E(\bar{\kappa}) = \bar{\kappa}$ and f, h be GSLEP and g be a geodesic semilocal E-preincave, and they are all geodesic E-η- semidifferentiable at $\bar{\kappa}$. Further, assume that there are $\zeta^o = (\zeta_i^o, i = 1, \dots, p) \in \mathbb{R}^p$ and $\xi^o = (\xi_j^o, j = 1, \dots, m) \in \mathbb{R}^m$ such that*

$$\zeta_i^o f'_{i+}\big(\gamma_{\bar{\kappa},E(\widehat{\kappa})}(t)\big) + \xi_j^o h'_{j+}\big(\gamma_{\bar{\kappa},E(\widehat{\kappa})}(t)\big) \geq 0 \quad \forall \kappa \in K, t \in [0,1], \tag{4}$$

$$g'_{i+}\big(\gamma_{\bar{\kappa},E(\kappa)}(t)\big) \leq 0, \quad \forall \kappa \in K, i \in P, \tag{5}$$

$$\xi^o h(\bar{\kappa}) = 0 \tag{6}$$

$$\zeta^o \geq 0, \qquad \xi^o \geq 0. \tag{7}$$

Then $\bar{\kappa}$ is a weak efficient solution for (VFP).

Proof By contradiction, let $\bar{\kappa}$ be not a weak efficient solution for (VFP), then there exists a point $\widehat{\kappa} \in K$ such that

$$\frac{f_i(\widehat{\kappa})}{g_i(\widehat{\kappa})} < \frac{f_i(\bar{\kappa})}{g_i(\bar{\kappa})}, \quad i \in P. \tag{8}$$

By the above hypotheses and Lemma 3.1, we have

$$f_i(\widehat{\kappa}) - f_i(\bar{\kappa}) \geq f'_{i+}\big(\gamma_{\bar{\kappa},E(\widehat{\kappa})}(t)\big), \quad i \in P \tag{9}$$

$$g_i(\widehat{\kappa}) - g_i(\bar{\kappa}) \leq g'_{i+}\big(\gamma_{\bar{\kappa},E(\widehat{\kappa})}(t)\big), \quad i \in P \tag{10}$$

$$h_i(\widehat{\kappa}) - h_i(\bar{\kappa}) \geq h'_{j+}\big(\gamma_{\bar{\kappa},E(\widehat{\kappa})}(t)\big), \quad j \in Q. \tag{11}$$

Multiplying (9) by ζ_i^o and (11) by ξ_j^o, we get

$$\sum_{i=1}^{p} \zeta_i^o \big(f_i(\widehat{\kappa}) - f_i(\bar{\kappa})\big) + \sum_{j=1}^{m} \xi_j^o \big(h_j(\widehat{\kappa}) - h_j(\bar{\kappa})\big)$$
$$\geq \zeta_i^o f'_{i+}\big(\gamma_{\bar{\kappa},E(\widehat{\kappa})}(t)\big) + \xi_j^o h'_{j+}\big(\gamma_{\bar{\kappa},E(\widehat{\kappa})}(t)\big) \geq 0. \tag{12}$$

Since $\widehat{\kappa} \in K, \xi^o \geq 0$ by (6) and (12), we have

$$\sum_{i=1}^{p} \zeta_i^o \big(f_i(\widehat{\kappa}) - f_i(\bar{\kappa})\big) \geq 0. \tag{13}$$

Utilizing (7) and (13), there is at least i_0 $(1 \leq i_0 \leq p)$ such that

$$f_{i_0}(\widehat{\kappa}) \geq f_{i_0}(\bar{\kappa}). \tag{14}$$

On the other hand, (5) and (10) imply

$$g_i(\widehat{\kappa}) \leq g_i(\bar{\kappa}), \quad i \in P. \tag{15}$$

By using (14), (15), and $g > 0$, we have

$$\frac{f_{i_0}(\widehat{\kappa})}{g_{i_0}(\widehat{\kappa})} \geq \frac{f_{i_0}(\bar{\kappa})}{g_{i_0}(\bar{\kappa})}, \tag{16}$$

which is a contradiction to 8, then the proof of the theorem is completed. □

Similarly we can prove the next theorem.

Theorem 3.2 *Consider that $\bar{\kappa} \in B$, $E(\bar{\kappa}) = \bar{\kappa}$ and f, h are geodesic E-η-semidifferentiable at $\bar{\kappa}$. If there exist $\zeta^o \in \mathbb{R}^n$ and $\xi^o \in \mathbb{R}^m$ such that conditions (4)–(7) hold and $\zeta^o f(x) + \xi^o h(x)$ is a GSLEP function, then $\bar{\kappa}$ is a weak efficient solution for* (VFP).

Theorem 3.3 *Consider that $\bar{\kappa} \in B$, $E(\bar{\kappa}) = \bar{\kappa}$ and $\lambda_i^o = \frac{f_i(\bar{\kappa})}{g_i(\bar{\kappa})}$ $(i \in P)$ are all pSLGEP functions and $h_j(\kappa)$ $(j \in \aleph(\bar{\kappa}))$ are all GqSLEP functions and f, g, h are all geodesic E-η-semidifferentiable at $\bar{\kappa}$. If there are $\zeta^o \in \mathbb{R}^p$ and $\xi^o \in \mathbb{R}^m$ such that*

$$\sum_{i=1}^{p} \zeta_i^o \big(f'_{i+}\big(\gamma_{\bar{\kappa},E(\kappa)}(t)\big) - \lambda_i^o g'_{i+}\big(\gamma_{\bar{\kappa},E(\kappa)}(t)\big)\big) + \xi^o h'_{i+}\big(\gamma_{\bar{\kappa},E(\kappa)}(t)\big) \geq 0 \tag{17}$$

$$\xi^o h(\bar{\kappa}) = 0, \tag{18}$$

$$\zeta^o \geq 0, \qquad \xi^o \geq 0, \tag{19}$$

then $\bar{\kappa}$ is a weak efficient solution for (VFP).

Proof Assume that $\bar{\kappa}$ is not a weak efficient solution for (VFP). Therefore, there exists $\kappa^* \in B$, which yields

$$\frac{f_i(\kappa^*)}{g_i(\kappa^*)} < \frac{f_i(\bar{\kappa})}{g_i(\bar{\kappa})}.$$

Then

$$f_i(\kappa^*) - \lambda_i^o g_i(\kappa^*) < 0, \quad i \in P,$$

which means that

$$f_i(\kappa^*) - \lambda_i^o g_i(\kappa^*) < f_i(\bar{\kappa}) - \lambda_i^o g_i(\bar{\kappa}) < 0, \quad i \in P.$$

By the pSLGEP of $(f_i(\kappa) - \lambda_i^o g_i(\kappa))$ $(i \in P)$ and Lemma 2.1, we have

$$f'_{i+}(\gamma_{\bar{\kappa},E(\kappa)}(t)) - \lambda_i^o g'_{i+}(\gamma_{\bar{\kappa},E(\kappa)}(t)), \quad i \in P.$$

Utilizing $\zeta^o \geq 0$, then

$$\sum_{i=1}^{p} \zeta_i^o \bigl(f'_{i+}(\gamma_{\bar{\kappa},E(\kappa)}(t)) - \lambda_i^o g'_{i+}(\gamma_{\bar{\kappa},E(\kappa)}(t))\bigr) < 0. \tag{20}$$

For $h(\kappa^*) \leq 0$ and $h_j(\bar{\kappa}) = 0$, $j \in \aleph(\bar{\kappa})$, we have $h_j(\kappa^*) \leq h_j(\bar{\kappa})$, $\forall j \in \aleph(\bar{\kappa})$.
By the GqSLEP of h_j and Lemma 2.1, we have

$$h_{j+}(\gamma_{\bar{\kappa},E(\kappa)}(t)) \leq 0, \quad \forall j \in \aleph(\bar{\kappa}).$$

Considering $\xi^o \geq 0$ and $\xi_j^o = 0$, $j \in \aleph(\bar{\kappa})$, then

$$\sum_{j=1}^{m} \xi_j^o h'_{j+}(\gamma_{\bar{\kappa},E(\kappa^*)}(t)) \leq 0. \tag{21}$$

Hence, by (20) and (21), we have

$$\sum_{i=1}^{p} \zeta_i^o \bigl(f'_{i+}(\gamma_{\bar{\kappa},E(\kappa^*)}(t)) - \lambda_i^o g'_{i+}(\gamma_{\bar{\kappa},E(\kappa^*)}(t))\bigr) + \xi^o h'_{i+}(\gamma_{\bar{\kappa},E(\kappa^*)}(t)) < 0, \tag{22}$$

which is a contradiction to relation (17) at $\kappa^* \in B$. Therefore, $\bar{\kappa}$ is a weak efficient solution for (VFP). □

Theorem 3.4 *Consider $\bar{\kappa} \in B$, $E(\bar{\kappa}) = \bar{\kappa}$ and $\lambda_i^o = \frac{f_i(\bar{\kappa})}{g_i(\bar{\kappa})}(i \in P)$. Also, assume that f, g, h are geodesic E-η-semidifferentiable at $\bar{\kappa}$. If there are $\zeta^o \in \mathbb{R}^p$ and $\xi^o \in \mathbb{R}^m$ such that conditions* (17)–(19) *hold and $\sum_{i=1}^{p} \zeta_i^o(f_i(\kappa) - \lambda_i^o g_i(\kappa)) + \xi^o_{\aleph(\bar{\kappa})} h_{\aleph(\bar{\kappa})}(\kappa)$ is a GpSLEP function, then $\bar{\kappa}$ is a weak efficient solution for* (VFP).

Corollary 3.1 *Let $\bar{\kappa} \in B$, $E(\bar{\kappa}) = \bar{\kappa}$ and $\lambda_i^o = \frac{f_i(\bar{\kappa})}{g_i(\bar{\kappa})}(i \in P)$. Further, let $f, h_{\aleph(\bar{\kappa})}$ be all GSLEP functions, g be a geodesic semilocal E-preincave function, and f, g, h be all geodesic E-η-semidifferentiable at $\bar{\kappa}$. If there exist $\zeta^o \in \mathbb{R}^p$ and $\xi^o \in \mathbb{R}^m$ such that conditions* (17)–(19) *hold, then $\bar{\kappa}$ is a weak efficient solution for* (VFP).

The dual problem for (VFP) is formulated as follows:

$$(\text{VFD})\begin{cases}\text{minimize } (\zeta_i, i=1,2,\dots,p),\\ \text{subject to } \sum_{i=1}^{p}\alpha_i(f'_{i+}(\gamma_{\lambda,E(\kappa)}(t))-\zeta_i g'_{i+}(\gamma_{\lambda,E(\kappa)}(t)))+\sum_{j=1}^{m}\beta_j h'_{j+}(\gamma_{\lambda,E(\kappa)}(t))\geq 0\\ \quad \kappa\in K_0, t\in[0,1],\\ \quad f_i(\lambda)-\zeta_i g_i(\lambda)\geq 0,\quad i\in P,\qquad \beta_j h_j(\lambda)\geq 0,\quad j\in\aleph;\end{cases}$$

where $\zeta=(\zeta_i, i=1,2,\dots,p)\geq 0$, $\alpha=(\alpha_i, i=1,2,\dots,p)>0$, $\beta=(\beta_i, i=1,2,\dots,m)\geq 0$, $\lambda\in K_0$.

Denote the feasible set problem (VFD) by K'.

Theorem 3.5 (General weak duality) *Let $\kappa\in K$, $(\alpha,\beta,\lambda,\zeta)\in K'$, and $E(\lambda)=\lambda$. If $\sum_{i=1}^{p}\alpha_i(f_i-\zeta_i g_i)$ is a GpSLEP function and $\sum_{j=1}^{m}\beta_j h_j$ is a GqSLEP function and they are all geodesic E-η-semidifferentiable at λ, then $\frac{f(\kappa)}{g(\kappa)}\nleq\zeta$.*

Proof From $\alpha>0$ and $(\alpha,\beta,\lambda,\zeta)\in K'$, we have

$$\sum_{i=1}^{p}\alpha_i\big(f_i(\kappa)-\zeta_i g_i(\kappa)\big)<0\leq\sum_{i=1}^{p}\alpha_i\big(f_i(\lambda)-\zeta_i g_i(\lambda)\big).$$

By the GpSLEP of $\sum_{i=1}^{p}\alpha_i(f_i-\zeta_i g_i)$ and Lemma 2.1, we obtain

$$\left(\sum_{i=1}^{p}\alpha_i(f_i-\zeta_i g_i)\right)'_+\big(\gamma_{\lambda,E(\kappa)}(t)\big)<0,$$

that is,

$$\sum_{i=1}^{p}\alpha_i\big(f'_{i+}\big(\gamma_{\lambda,E(\kappa)}(t)\big)-\zeta_i g'_{i+}\big(\gamma_{\lambda,E(\kappa)}(t)\big]\big)<0.$$

Also, from $\beta\geq 0$ and $\kappa\in K$, then

$$\sum_{j=1}^{m}\beta_j h_j(\kappa)\leq 0\leq\sum_{j=1}^{m}\beta_j h_j(\lambda).$$

Using the GqSLEP of $\sum_{j=1}^{m}\beta_j h_j$ and Lemma 2.1, one has

$$\left(\sum_{j=1}^{m}\beta_j h_j\right)'_+\big(\gamma_{\lambda,E(\kappa)}(t)\big)\leq 0.$$

Then

$$\sum_{j=1}^{m}\beta_j h'_{j+}\big(\gamma_{\lambda,E(\kappa)}(t)\big)\leq 0.$$

Therefore,

$$\sum_{i=1}^{p}\alpha_i\big(f'_{i+}\big(\gamma_{\lambda,E(\kappa)}(t)\big)-\zeta_i g'_{i+}\big(\gamma_{\lambda,E(\kappa)}(t)\big)\big)+\sum_{j=1}^{m}\beta_j h'_{j+}\big(\gamma_{\lambda,E(\kappa)}(t)\big)<0.$$

This is a contradiction to $(\alpha,\beta,\lambda,\zeta)\in K'$. □

Theorem 3.6 *Consider that* $\kappa\in K$, $(\alpha,\beta,\lambda,\zeta)\in K'$ *and* $E(\lambda)=\lambda$. *If* $\sum_{i=1}^{p}\alpha_i(f_i-\zeta_i g_i)+\sum_{j=1}^{m}\beta_j h_j$ *is a GpSLEP function and a geodesic E-η-semidifferentiable at* λ, *then* $\frac{f(\kappa)}{g(\kappa)}\nleq\zeta$.

Theorem 3.7 (General converse duality) *Let* $\bar\kappa\in K$ *and* $(\kappa^*,\alpha^*,\beta^*,\zeta^*)\in K'$, $E(\kappa^*)=\kappa^*$, *where* $\zeta^*=\frac{f(\kappa^*)}{g(\kappa^*)}=\frac{f(\bar\kappa)}{g(\bar\kappa)}=(\zeta_i^*,i=1,2,\dots,p)$. *If* $f_i-\zeta_i^*g_i(i\in P)$, $h_j(j\in\aleph)$ *are all GSLEP functions and all geodesic E-η-semidifferentiable at* κ^*, *then* $\bar\kappa$ *is a weak efficient solution for* (VFP).

Proof By using the hypotheses and Lemma 2.1, for any $\kappa\in K$, we obtain

$$\begin{aligned}&\big(f_i(\kappa)-\zeta_i^*g_i(\kappa)\big)-\big(f_i\big(\kappa^*\big)-\zeta_i^*g_i\big(\kappa^*\big)\big)\geq f'_{i+}\big(\gamma_{\kappa^*,E(\kappa)}(t)\big)-\zeta_i g'_{i+}\big(\gamma_{\kappa^*,E(\kappa)}(t)\big)\\&h_j(y)-h_j\big(\kappa^*\big)\geq h'_{j+}\big(\gamma_{\kappa^*,E(\kappa)}(t)\big).\end{aligned}$$

Utilizing the first constraint condition for (VFD), $\alpha^*>0,\beta^*\geq0$, $\zeta^*\geq0$, and the two inequalities above, we have

$$\begin{aligned}&\sum_{i=1}^{p}\alpha_i^*\big(\big(f_i(\kappa)-\zeta_i^*g_i(\kappa)\big)-\big(f_i\big(\kappa^*\big)-\zeta_i^*g_i\big(\kappa^*\big)\big)\big)+\sum_{j=1}^{m}\beta_j^*\big(h_j(\kappa)-h_j\big(\kappa^*\big)\big)\\&\quad\geq\sum_{i=1}^{p}\big(f'_{i+}\big(\gamma_{\kappa^*,E(\kappa)}(t)\big)-\zeta_i g'_{i+}\big(\gamma_{\kappa^*,E(\kappa)}(t)\big)\big)\\&\qquad+\sum_{j=1}^{m}\beta_j^*h'_{j+}\big(\gamma_{\kappa^*,E(\kappa)}(t)\big)\\&\quad\geq0.\end{aligned}\tag{23}$$

In view of $h_j(\kappa)\leq0$, $\beta_j^*\geq0,\beta_j^*h_j(\kappa^*)\geq(j\in\aleph)$, and $\zeta_i^*=\frac{f_i(\kappa^*)}{g_i(\kappa^*)}$ $(i\in P)$, then

$$\sum_{i=1}^{p}\alpha_i^*\big(f_i(\kappa)-\zeta_i^*g_i(\kappa)\big)\geq0\quad\forall y\in Y.\tag{24}$$

Consider that $\bar\kappa$ is not a weak efficient solution for (VFP). From $\zeta_i^*=\frac{f_i(\bar\kappa)}{g_i(\bar\kappa)}$ $(i\in P)$ and Lemma 3.1, it follows that $\bar\kappa$ is not a weak efficient solution for (VFP_{ζ^*}). Hence, $\tilde\kappa\in K$ such that

$$f_i(\tilde\kappa)-\zeta_i^*g_i(\tilde\kappa)<f_i(\bar\kappa)-\zeta_i^*g_i(\bar\kappa)=0,\quad i\in P.$$

Therefore $\sum_{i=1}^{p}\alpha_i^*(f_i(\tilde\kappa)-\zeta_i^*g_i(\tilde\kappa))<0$. This is a contradiction to inequality (24). The proof of the theorem is completed. □

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Authors' contributions
Both authors conceived of the study, participated in its design and coordination, drafted the manuscript and participated in the sequence alignment. All authors read and approved the final manuscript.

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