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General iterative methods for systems of variational inequalities with the constraints of generalized mixed equilibria and fixed point problem of pseudocontractions

Qian-Wen Wang[1], Jin-Lin Guan[1], Lu-Chuan Ceng[1*] and Bing Hu[2]

*Correspondence:
zenglc@hotmail.com
[1]Department of Mathematics, Shanghai Normal University, Shanghai, China
Full list of author information is available at the end of the article

Abstract

In this paper, we introduce two general iterative methods (one implicit method and one explicit method) for finding a solution of a general system of variational inequalities (GSVI) with the constraints of finitely many generalized mixed equilibrium problems and a fixed point problem of a continuous pseudocontractive mapping in a Hilbert space. Then we establish strong convergence of the proposed implicit and explicit iterative methods to a solution of the GSVI with the above constraints, which is the unique solution of a certain variational inequality. The results presented in this paper improve, extend, and develop the corresponding results in the earlier and recent literature.

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1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. We denote by P_C the metric projection of H onto C and by $\operatorname{Fix}(S)$ the set of fixed points of the mapping S. Recall that a mapping $T:C\to H$ is nonexpansive if $\|Tx-Ty\|\le\|x-y\|$, $\forall x,y\in C$. A mapping $T:C\to H$ is called pseudocontractive if

$$\langle Tx-Ty,x-y\rangle\le\|x-y\|^2,\quad\forall x,y\in C.$$

This inequality can be equivalently rewritten as

$$\|Tx-Ty\|^2\le\|x-y\|^2+\big\|(I-T)x-(I-T)y\big\|^2,\quad\forall x,y\in C,$$

where I is the identity mapping.

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$T : C \to H$ is said to be k-strictly pseudocontractive if there exists a constant $k \in [0,1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\big\|(I - T)x - (I - T)y\big\|^2, \quad \forall x, y \in C.$$

A mapping $V : C \to H$ is said to be l-Lipschitzian if there exists a constant $l \ge 0$ such that

$$\|Vx - Vy\| \le l\|x - y\|, \quad \forall x, y \in C.$$

A mapping $F : C \to H$ is called monotone if

$$\langle x - y, Fx - Fy\rangle \ge 0, \quad \forall x, y \in C,$$

and F is called α-inverse-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle x - y, Fx - Fy\rangle \ge \alpha\|Fx - Fy\|^2, \quad \forall x, y \in C.$$

If F is an α-inverse-strongly monotone mapping, then it is obvious that F is $\frac{1}{\alpha}$-Lipschitz continuous, that is, $\|Fx - Fy\| \le \frac{1}{\alpha}\|x - y\|$ for all $x, y \in C$.

A mapping $F : C \to H$ is called β-strongly monotone if there exists a constant $\beta > 0$ such that

$$\langle x - y, Fx - Fy\rangle \ge \beta\|x - y\|^2, \quad \forall x, y \in C.$$

A linear operator $A : H \to H$ is said to be strongly positive on H if there exists a constant $\bar{\gamma} > 0$ such that

$$\langle Ax, x\rangle \ge \bar{\gamma}\|x\|^2, \quad \forall x \in H.$$

Let $F : C \to H$ be a mapping. The classical variational inequality problem (VIP) is to find $x^* \in C$ such that

$$\big\langle Fx^*, x - x^*\big\rangle \ge 0, \quad \forall x \in C. \tag{1.1}$$

We denote the set of solutions of VIP (1.1) by $\mathrm{VI}(C, F)$.

In 2008, Ceng et al. [1] considered the following general system of variational inequalities (GSVI):

$$\begin{cases} \langle \lambda F_1 y^* + x^* - y^*, x - x^*\rangle \ge 0, & \forall x \in C, \\ \langle \nu F_2 x^* + y^* - x^*, x - y^*\rangle \ge 0, & \forall x \in C, \end{cases} \tag{1.2}$$

where F_1, F_2 are α-inverse-strongly monotone and β-inverse-strongly monotone, respectively, and $\lambda \in (0, 2\alpha)$ and $\nu \in (0, 2\beta)$ are two constants. Many iterative methods have been developed for solving GSVI (1.2); see [2–7] and the references therein.

Subsequently, Alofi et al. [8] also introduced two composite iterative algorithms based on the composite iterative methods in Ceng et al. [9] and Jung [10] for solving the problem of GSVI (1.2). Moreover, they showed strong convergence of the proposed algorithms to a common solution of these two problems.

Very recently, Kong et al. [11] established the strong convergence of two hybrid steepest-descent schemes to the same solution of GSVI (1.2), which is also a common solution of finitely many variational inclusions and a minimization problem.

Lemma 1.1 (see [12, Proposition 3.1]) *Let C be a nonempty closed convex subset of a real Hilbert space H. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of GSVI (1.3) for continuous monotone mappings F_1 and F_2 if and only if x^* is a fixed point of the composite $R = F_{1,\lambda}F_{2,\nu} : H \to C$ of nonexpansive mappings $F_{1,\lambda} : H \to C$ and $F_{2,\nu} : H \to C$, where $y^* = F_{2,\nu}x^*$,*

$$F_{1,\lambda}x = \left\{ z \in C : \langle y - z, F_1 z\rangle + \frac{1}{\lambda}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\},$$

and

$$F_{2,\nu}x = \left\{ z \in C : \langle y - z, F_2 z\rangle + \frac{1}{\nu}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\}.$$

For simplicity, we denote by $\mathrm{GSVI}(C, F_1, F_2)$ *the fixed point set of mapping R.*

In the meantime, inspired by Ceng et al. [1], Jung [12] introduced a general system of variational inequalities (GSVI) for two continuous monotone mappings F_1 and F_2 of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda F_1 x^* + x^* - y^*, x - x^*\rangle \geq 0, & \forall x \in C, \\ \langle \nu F_2 y^* + y^* - x^*, x - y^*\rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.3}$$

where $\lambda, \nu > 0$ are two constants. In order to find an element of $\mathrm{Fix}(R) \cap \mathrm{Fix}(T)$, he proposed one implicit algorithm generating a net $\{x_t\}$:

$$x_t = (I - \theta_t A)T_{r_t}Rx_t + \theta_t\bigl[t\gamma V x_t + (I - t\mu G)T_{r_t}Rx_t\bigr], \tag{1.4}$$

with $t \in (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\})$ and $\theta_t \in (0, \min\{\frac{1}{2}, \|A\|^{-1}\})$, and an explicit algorithm generating a sequence $\{x_n\}$:

$$\begin{cases} y_n = \alpha_n \gamma V x_n + (I - \alpha_n \mu G)T_{r_n}Rx_n, \\ x_{n+1} = (I - \beta_n A)T_{r_n}Rx_n + \beta_n y_n, & \forall n \geq 0, \end{cases} \tag{1.5}$$

with $\{\alpha_n\} \subset [0, 1]$, $\{\beta_n\} \subset (0, 1]$, $\{r_n\} \subset (0, \infty)$, and $x_0 \in C$ any initial guess, where $T_{r_t}x = \{z \in C : \langle y - z, Tz\rangle - \frac{1}{r_t}\langle y - z, (1 + r_t)z - x\rangle \leq 0, \forall y \in C\}$ for $r_t \in (0, \infty)$, and $T_{r_n}x = \{z \in C : \langle y - z, Tz\rangle - \frac{1}{r_n}\langle y - z, (1 + r_n)z - x\rangle \leq 0, \forall y \in C\}$ for $r_n \in (0, \infty)$. Moreover, he established strong convergence of the proposed iterative algorithms to an element $\widetilde{x} \in \mathrm{Fix}(R) \cap \mathrm{Fix}(T)$, which uniquely solves the variational inequality

$$\bigl\langle (A - I)\widetilde{x}, \widetilde{x} - p\bigr\rangle \leq 0, \quad \forall p \in \mathrm{Fix}(R) \cap \mathrm{Fix}(T).$$

On the other hand, the generalized mixed equilibrium problem (GMEP) is to find $x \in C$ such that

$$\Theta(x,y) + \varphi(y) - \varphi(x) + \langle Bx, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.6}$$

We denote the set of solutions of GMEP (1.6) by GMEP(Θ, φ, B). GMEP (1.6) is very general in the sense that it includes many problems as special cases, namely optimization problems, variational inequalities, minimax problems, Nash equilibrium problems in noncooperative games, and others. For different aspects and solution methods, we refer to [13–18] and the references therein.

In this paper, we introduce implicit and explicit iterative methods for finding a solution of GSVI (1.3) with solutions belonging also to the common solution set $\bigcap_{i=1}^N \mathrm{GMEP}(\Theta_i, \varphi_i, B_i)$ of finitely many generalized mixed equilibrium problems and the fixed point set of a continuous pseudocontractive mapping T. First, GSVI (1.3) and each generalized mixed equilibrium problem both are transformed into fixed point problems of nonexpansive mappings. Then we establish strong convergence of the proposed iterative methods to an element of $\bigcap_{i=1}^N \mathrm{GMEP}(\Theta_i, \varphi_i, B_i) \cap \mathrm{GSVI}(C, F_1, F_2) \cap \mathrm{Fix}(T)$, which is the unique solution of a certain variational inequality.

2 Preliminaries and lemmas

Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. We write $x_n \to x$ and $x_n \rightharpoonup x$ to indicate the strong convergence of the sequence $\{x_n\}$ to x and the weak convergence of the sequence $\{x_n\}$ to x, respectively.

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\|, \quad \forall y \in C.$$

P_C is called the metric projection of H onto C. It is well known that P_C is nonexpansive and is characterized by the property

$$u = P_C(x) \quad \Leftrightarrow \quad \langle x - u, u - y\rangle \geq 0, \quad \forall x \in H, y \in C. \tag{2.1}$$

In a Hilbert space H, the following equality holds:

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle, \quad \forall x, y \in H. \tag{2.2}$$

The following lemma is an immediate consequence of an inner product.

Lemma 2.1 *In a real Hilbert space H, there holds the following inequality*:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H.$$

Next we list some elementary conclusions for the MEP.

It is first assumed as in [19] that $\Theta : C \times C \to \mathbf{R}$ is a bifunction satisfying conditions (A1)–(A4) and $\varphi : C \to \mathbf{R}$ is a lower semicontinuous and convex function with restriction (B1) or (B2), where

(A1) $\Theta(x,x) = 0$ for all $x \in C$;
(A2) Θ is monotone, i.e., $\Theta(x,y) + \Theta(y,x) \leq 0$ for any $x,y \in C$;
(A3) Θ is upper-hemicontinuous, i.e., for each $x,y,z \in C$,

$$\limsup_{t \to 0^+} \Theta\big(tz + (1-t)x, y\big) \leq \Theta(x,y);$$

(A4) $\Theta(x,\cdot)$ is convex and lower semicontinuous for each $x \in C$;
(B1) for $\forall x \in H$ and $r > 0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C$ such that, for $\forall z \in C \setminus D_x$,

$$\Theta(z,y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0;$$

(B2) C is a bounded set.

Proposition 2.1 ([19]) *Assume that $\Theta : C \times C \to \mathbf{R}$ satisfies* (A1)–(A4), *and let $\varphi : C \to \mathbf{R}$ be a proper lower semicontinuous and convex function. Assume that either* (B1) *or* (B2) *holds. For $r > 0$ and $x \in H$, define a mapping $T_r^{(\Theta,\varphi)} : H \to C$ as follows*:

$$T_r^{(\Theta,\varphi)}(x) := \left\{ z \in C : \Theta(z,y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\}$$

for all $x \in H$. Then the following hold:
(i) *for each $x \in H$, $T_r^{(\Theta,\varphi)}(x)$ is nonempty and single-valued*;
(ii) *$T_r^{(\Theta,\varphi)}$ is firmly nonexpansive, that is, for any $x,y \in H$,*

$$\big\| T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y \big\|^2 \leq \big\langle T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y, x - y\big\rangle;$$

(iii) $\mathrm{Fix}(T_r^{(\Theta,\varphi)}) = \mathrm{MEP}(\Theta,\varphi)$;
(iv) $\mathrm{MEP}(\Theta,\varphi)$ *is closed and convex*;
(v) $\|T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\varphi)}x\|^2 \leq \frac{s-t}{s}\langle T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\varphi)}x, T_s^{(\Theta,\varphi)}x - x\rangle$ *for all $s,t > 0$ and $x \in H$.*

Proposition 2.2 *Let $F : C \to H$ be an α-inverse-strongly monotone mapping. Then, for all $x,y \in C$ and $\lambda > 0$, one has*

$$\big\|(I - \lambda F)x - (I - \lambda F)y\big\|^2 \leq \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Fx - Fy\|^2.$$

In particular, if $\lambda \in (0, 2\alpha]$, $I - \lambda F : C \to H$ is a nonexpansive mapping.

We will use the following lemmas for the proof of our main results in the sequel.

Lemma 2.2 ([20]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \omega_n)s_n + \omega_n\delta_n + \gamma_n, \quad \forall n \geq 0,$$

where $\{\omega_n\}$, $\{\delta_n\}$, and $\{\gamma_n\}$ satisfy the following conditions:
(i) $\{\omega_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty}\omega_n = \infty$ *or, equivalently,* $\prod_{n=0}^{\infty}(1 - \omega_n) = 0$;

(ii) $\limsup_{n\to\infty} \delta_n \le 0$ *or* $\sum_{n=0}^{\infty} \omega_n |\delta_n| < \infty$;
(iii) $\gamma_n \ge 0$ $(n \ge 0)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$.
Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.3 (Demiclosedness principle [21]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S : C \to C$ be a nonexpansive mapping with* $\mathrm{Fix}(S) \neq \emptyset$. *Then the mapping $I - S$ is demiclosed. That is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x^*$ and $(I - S)x_n \to y$, then $(I - S)x^* = y$. Here I is the identity mapping of H.*

Lemma 2.4 ([22]) *Let H be a real Hilbert space. Let $A : H \to H$ be a strongly positive bounded linear operator with a constant $\bar{\gamma} > 1$. Then*

$$\bigl\langle (A - I)x - (A - I)y, x - y \bigr\rangle \ge (\bar{\gamma} - 1)\|x - y\|^2, \quad \forall x, y \in C.$$

That is, $A - I$ is strongly monotone with a constant $\bar{\gamma} - 1$.

Lemma 2.5 ([22]) *Assume that $A : H \to H$ is a strongly positive bounded linear operator with a coefficient $\bar{\gamma} > 0$ and $0 < \zeta \le \|A\|^{-1}$. Then $\|I - \zeta A\| \le 1 - \zeta \bar{\gamma}$.*

Lemma 2.6 ([23]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $G : C \to H$ be a ρ-Lipschitzian and η-strongly monotone mapping with constants $\rho, \eta > 0$. Let $0 < \mu < \frac{2\eta}{\rho^2}$ and $0 < t < \sigma \le 1$. Then $S := \sigma I - t\mu G : C \to H$ is a contractive mapping with constant $\sigma - t\tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\rho^2)}$.*

Lemma 2.7 ([24]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \to H$ be a continuous monotone mapping. Then, for $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$\langle y - z, Fz \rangle + \frac{1}{r} \langle y - z, z - x \rangle \ge 0, \quad \forall y \in C.$$

For $r > 0$ and $x \in H$, define $F_r : H \to C$ by

$$F_r x = \left\{ z \in C : \langle y - z, Fz \rangle + \frac{1}{r} \langle y - z, z - x \rangle \ge 0, \forall y \in C \right\}.$$

Then the following hold:
(i) *F_r is single-valued*;
(ii) *F_r is firmly nonexpansive, that is,*

$$\|F_r x - F_r y\|^2 \le \langle x - y, F_r x - F_r y \rangle, \quad \forall x, y \in H;$$

(iii) $\mathrm{Fix}(F_r) = \mathrm{VI}(C, F)$;
(iv) $\mathrm{VI}(C, F)$ *is a closed convex subset of C.*

Lemma 2.8 ([24]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to H$ be a continuous pseudocontractive mapping. Then, for $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$\langle y - z, Tz \rangle - \frac{1}{r} \bigl\langle y - z, (1 + r)z - x \bigr\rangle \le 0, \quad \forall y \in C.$$

For $r > 0$ and $x \in H$, define $T_r : H \to C$ by

$$T_r x = \left\{ z \in C : \langle y - z, Tz \rangle - \frac{1}{r} \langle y - z, (1 + r)z - x \rangle \leq 0, \forall y \in C \right\}.$$

Then the following hold:

(i) T_r *is single-valued*;

(ii) T_r *is firmly nonexpansive*, *that is*,

$$\| T_r x - T_r y \|^2 \leq \langle x - y, T_r x - T_r y \rangle, \quad \forall x, y \in H;$$

(iii) $\mathrm{Fix}(T_r) = \mathrm{Fix}(T)$;

(iv) $\mathrm{Fix}(T)$ *is a closed convex subset of* C.

3 Main results

Throughout this section, we always assume the following:

- $B_i : C \to H$ is a μ_i-inverse-strongly monotone mapping for each $i = 1, 2, \dots, N$;
- $\Theta_i : C \times C \to \mathbf{R}$ is a bifunction satisfying conditions (A1)–(A4) for each $i = 1, 2, \dots, N$;
- $\varphi_i : C \to \mathbf{R}$ is a proper lower semicontinuous and convex function with restriction (B1) or (B2) for each $i = 1, 2, \dots, N$;
- $A : H \to H$ is a strongly positive linear bounded self-adjoint operator with a constant $\bar{\gamma} \in (1, 2)$;
- $V : C \to C$ is l-Lipschitzian with constant $l \in [0, \infty)$;
- $G : C \to C$ is a ρ-Lipschitzian and η-strongly monotone mapping with constants $\rho > 0$ and $\eta > 0$;
- constants μ, l, τ, and γ satisfy $0 < \mu < \frac{2\eta}{\rho^2}$ and $0 \leq \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\rho^2)}$;
- $F_1, F_2 : C \to H$ are continuous monotone mappings and $T : C \to C$ is a continuous pseudocontractive mapping such that $\Omega := \bigcap_{i=1}^N \mathrm{GMEP}(\Theta_i, \varphi_i, B_i) \cap \mathrm{GSVI}(C, F_1, F_2) \cap \mathrm{Fix}(T) \neq \emptyset$;
- $R_t = F_{1,\lambda_t} F_{2,\nu_t} : H \to C$, where $F_{1,\lambda_t}, F_{2,\nu_t} : H \to C$ are defined as follows:

$$F_{1,\lambda_t} x = \left\{ z \in C : \langle y - z, F_1 z \rangle + \frac{1}{\lambda_t} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\},$$

$$F_{2,\nu_t} x = \left\{ z \in C : \langle y - z, F_2 z \rangle + \frac{1}{\nu_t} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\},$$

  for $\lambda_t, \nu_t \in (0, \infty)$, $t \in (0, 1)$, $\lim_{t \to 0} \lambda_t = \lambda > 0$, and $\lim_{t \to 0} \nu_t = \nu > 0$;
- $R_n = F_{1,\lambda_n} F_{2,\nu_n} : H \to C$, where $F_{1,\lambda_n}, F_{2,\nu_n} : H \to C$ are defined as follows:

$$F_{1,\lambda_n} x = \left\{ z \in C : \langle y - z, F_1 z \rangle + \frac{1}{\lambda_n} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\},$$

$$F_{2,\nu_n} x = \left\{ z \in C : \langle y - z, F_2 z \rangle + \frac{1}{\nu_n} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\},$$

  for $\lambda_n, \nu_n \in (0, \infty)$, $\lim_{n \to \infty} \lambda_n = \lambda > 0$, and $\lim_{n \to \infty} \nu_n = \nu > 0$;

- $T_{r_t}: H \to C$ is a mapping defined by

$$T_{r_t}x = \left\{z \in C : \langle y - z, Tz\rangle - \frac{1}{r_t}\bigl\langle y - z, (1 + r_t)z - x\bigr\rangle \geq 0, \forall y \in C\right\}$$

for $r_t \in (0, \infty)$, $t \in (0, 1)$, and $\liminf_{t\to 0} r_t > 0$;
- $T_{r_n}: H \to C$ is a mapping defined by

$$T_{r_n}x = \left\{z \in C : \langle y - z, Tz\rangle - \frac{1}{r_n}\bigl\langle y - z, (1 + r_n)z - x\bigr\rangle \geq 0, \forall y \in C\right\}$$

for $r_n \in (0, \infty)$, and $\liminf_{n\to \infty} r_n > 0$;
- $T_{r_{i,t}}^{(\Theta_i,\varphi_i)}: H \to C$ is a mapping defined by

$$T_{r_{i,t}}^{(\Theta_i,\varphi_i)}x = \left\{z \in C : \Theta_i(z, y) + \varphi_i(y) - \varphi_i(z) + \frac{1}{r_{i,t}}\langle y - z, z - x\rangle \geq 0, \forall y \in C\right\}$$

for $\{r_{i,t}\}_{t\in(0,1)} \subset [c_i, d_i] \subset (0, 2\mu_i)$ and $i \in \{1, 2, \ldots, N\}$;
- $T_{r_{i,n}}^{(\Theta_i,\varphi_i)}: H \to C$ is a mapping defined by

$$T_{r_{i,n}}^{(\Theta_i,\varphi_i)}x = \left\{z \in C : \Theta_i(z, y) + \varphi_i(y) - \varphi_i(z) + \frac{1}{r_{i,n}}\langle y - z, z - x\rangle \geq 0, \forall y \in C\right\}$$

for $\{r_{i,n}\}_{n=1}^{\infty} \subset [c_i, d_i] \subset (0, 2\mu_i)$ and $i \in \{1, 2, \ldots, N\}$.

By Proposition 2.1 and Lemmas 2.7 and 2.8, we note that $T_{r_{i,t}}^{(\Theta_i,\varphi_i)}$, $T_{r_{i,n}}^{(\Theta_i,\varphi_i)}$, F_{1,λ_t}, F_{1,λ_n}, F_{2,ν_t}, F_{2,ν_n}, T_{r_t}, and T_{r_n} are nonexpansive, $\mathrm{GMEP}(\Theta_i, \varphi_i, B_i) = \mathrm{Fix}(T_{r_{i,t}}^{(\Theta_i,\varphi_i)}(I - r_{i,t}B_i)) = \mathrm{Fix}(T_{r_{i,n}}^{(\Theta_i,\varphi_i)}(I - r_{i,n}B_i))$, and $\mathrm{Fix}(T) = \mathrm{Fix}(T_{r_t}) = \mathrm{Fix}(T_{r_n})$. So it is known that the composite mappings $R_t = F_{1,\lambda_t}F_{2,\nu_t}$ and $R_n = F_{1,\lambda_n}F_{2,\nu_n}$ are nonexpansive. Also, we note that $\mathrm{GSVI}(C, F_1, F_2) = \mathrm{Fix}(R_t) = \mathrm{Fix}(R_n)$ by Lemma 1.1.

In this section, for $t \in (0, 1)$, $n \geq 1$ and $i \in \{1, 2, \ldots, N\}$, we put

$$\begin{aligned}
\Delta_t^i &= T_{r_{i,t}}^{(\Theta_i,\varphi_i)}(I - r_{i,t}B_i)T_{r_{i-1,t}}^{(\Theta_{i-1},\varphi_{i-1})}(I - r_{i-1,t}B_{i-1}) \cdots T_{r_{1,t}}^{(\Theta_1,\varphi_1)}(I - r_{1,t}B_1),\\
\Delta_n^i &= T_{r_{i,n}}^{(\Theta_i,\varphi_i)}(I - r_{i,n}B_i)T_{r_{i-1,n}}^{(\Theta_{i-1},\varphi_{i-1})}(I - r_{i-1,n}B_{i-1}) \cdots T_{r_{1,n}}^{(\Theta_1,\varphi_1)}(I - r_{1,n}B_1),
\end{aligned}$$

and $\Delta_t^0 = \Delta_n^0 = I$.

We now introduce the first general iterative scheme that generates a net $\{x_t\}$ in an implicit way:

$$x_t = P_C\bigl[(I - \theta_t A)T_{r_t}\Delta_t^N R_t x_t + \theta_t\bigl(t\gamma Vx_t + (I - t\mu G)T_{r_t}\Delta_t^N R_t x_t\bigr)\bigr], \tag{3.1}$$

where $t \in (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\})$ and $\theta_t \in (0, \min\{\frac{1}{2}, \|A\|^{-1}\})$.

We prove the strong convergence of $\{x_t\}$ as $t \to 0$ to a point $\widetilde{x} \in \Omega$, which is a unique solution to the VI

$$\bigl\langle (A - I)\widetilde{x}, p - \widetilde{x}\bigr\rangle \geq 0, \quad \forall p \in \Omega. \tag{3.2}$$

In the meantime, we also propose the second general iterative scheme that generates a sequence $\{x_n\}$ in an explicit way:

$$\begin{cases} w_n = \alpha_n \gamma V x_n + (I - \alpha_n \mu G) T_{r_n} \Delta_n^N R_n x_n, \\ x_{n+1} = P_C[(I - \beta_n A) T_{r_n} \Delta_n^N R_n x_n + \beta_n w_n], \quad \forall n \geq 1, \end{cases} \tag{3.3}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and $x_0 \in C$ is an arbitrary initial guess, and establish the strong convergence of $\{x_n\}$ as $n \to \infty$ to the same point $\widetilde{x} \in \Omega$, which is the unique solution to VI (3.2).

Next, for $t \in (0, \min\{1, \frac{2-\bar{\gamma}}{\tau - \gamma l}\})$ and $\theta_t \in (0, \min\{\frac{1}{2}, \|A\|^{-1}\})$, consider a mapping $Q_t : C \to C$ defined by

$$Q_t x = P_C\big[(I - \theta_t A) T_{r_t} \Delta_t^N R_t x + \theta_t\big(t\gamma V x + (I - t\mu G) T_{r_t} \Delta_t^N R_t x\big)\big], \quad \forall x \in C.$$

It is easy to see that Q_t is a contractive mapping with constant $1 - \theta_t(\bar{\gamma} - 1 + t(\tau - \gamma l))$. Indeed, by Propositions 2.1 and 2.2 and Lemmas 2.5 and 2.6, we have

$$\begin{aligned} \|Q_t x - Q_t y\| &\leq \big\|(I - \theta_t A) T_{r_t} \Delta_t^N R_t x + \theta_t\big(t\gamma V x + (I - t\mu G) T_{r_t} \Delta_t^N R_t x\big) \\ &\quad - (I - \theta_t A) T_{r_t} \Delta_t^N R_t y - \theta_t\big(t\gamma V y + (I - t\mu G) T_{r_t} \Delta_t^N R_t y\big)\big\| \\ &\leq \big\|(I - \theta_t A) T_{r_t} \Delta_t^N R_t x - (I - \theta_t A) T_{r_t} \Delta_t^N R_t y\big\| \\ &\quad + \theta_t \big\|\big(t\gamma V x + (I - t\mu G) T_{r_t} \Delta_t^N R_t x\big) - \big(t\gamma V y + (I - t\mu G) T_{r_t} \Delta_t^N R_t y\big)\big\| \\ &\leq (1 - \theta_t \bar{\gamma})\big\|T_{r_t} \Delta_t^N R_t x - T_{r_t} \Delta_t^N R_t y\big\| + \theta_t\big[t\gamma \|V x - V y\| \\ &\quad + \big\|(I - t\mu G) T_{r_t} \Delta_t^N R_t x - (I - t\mu G) T_{r_t} \Delta_t^N R_t y\big\|\big] \\ &\leq (1 - \theta_t \bar{\gamma})\|x - y\| + \theta_t\big[t\gamma l\|x - y\| + (1 - t\tau)\|x - y\|\big] \\ &= \big[1 - \theta_t\big(\bar{\gamma} - 1 + t(\tau - \gamma l)\big)\big]\|x - y\|. \end{aligned}$$

Since $\bar{\gamma} \in (1,2)$, $\tau - \gamma l > 0$ and $0 < t < \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\} \leq \frac{2-\bar{\gamma}}{\tau-\gamma l}$, it follows that $0 < \bar{\gamma} - 1 + t(\tau - \gamma l) < 1$, which together with $0 < \theta_t < \min\{\frac{1}{2}, \|A\|^{-1}\} < 1$ yields $0 < 1 - \theta_t(\bar{\gamma} - 1 + t(\tau - \gamma l)) < 1$. Hence Q_t is a contractive mapping. By the Banach contraction principle, Q_t has a unique fixed point, denoted by x_t, which uniquely solves the fixed point equation (3.1).

We summarize the basic properties of $\{x_t\}$.

Theorem 3.1 *Let $\{x_t\}$ be defined via (3.1). Then*

(i) *$\{x_t\}$ is bounded for $t \in (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\})$;*

(ii) *$\lim_{t\to 0} \|x_t - R_t x_t\| = 0$, $\lim_{t\to 0} \|x_t - \Delta_t^N x_t\| = 0$, and $\lim_{t\to 0} \|x_t - T_{r_t} x_t\| = 0$ provided $\lim_{t\to 0} \theta_t = 0$;*

(iii) *$x_t : (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\}) \to H$ is locally Lipschitzian provided*
$\theta_t : (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\}) \to (0, \min\{\frac{1}{2}, \|A\|^{-1}\})$ is locally Lipschitzian,
$r_t, \lambda_t, \nu_t : (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\}) \to (0, \infty)$ are locally Lipschitzian, and
$r_{i,t} : (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\}) \to [c_i, d_i]$ is locally Lipschitzian for each $i = 1, 2, \ldots, N$;

(iv) *x_t defines a continuous path from $(0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\})$ into H provided*
$\theta_t : (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\}) \to (0, \min\{\frac{1}{2}, \|A\|^{-1}\})$ is continuous,
$r_t, \lambda_t, \nu_t : (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\}) \to (0, \infty)$ are continuous, and
$r_{i,t} : (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\}) \to [c_i, d_i]$ is continuous for each $i = 1, 2, \ldots, N$.

Proof Let $z_t = R_t x_t$, $u_t = \Delta_t^N z_t$, and $v_t = T_{r_t} u_t$. Take $p \in \Omega$. Then $p = T_{r_t} p$ by Lemma 2.8(iii), $p = \Delta_t^i p$ $(= T_{r_{i,t}}^{(\Theta_i,\varphi_i)}(I - r_{i,t}B_i)p)$ by Proposition 2.1(iii), and $p = R_t p$ by Lemma 1.1.

(i) Utilizing Proposition 2.1(ii) and Proposition 2.2, we have

$$\begin{aligned} \|u_t - p\| &= \left\| T_{r_{N,t}}^{(\Theta_N,\varphi_N)}(I - r_{N,t}B_N)\Delta_t^{N-1} z_t - T_{r_{N,t}}^{(\Theta_N,\varphi_N)}(I - r_{N,t}B_N)\Delta_t^{N-1} p \right\| \\ &\le \left\| (I - r_{N,t}B_N)\Delta_t^{N-1} z_t - (I - r_{N,t}B_N)\Delta_t^{N-1} p \right\| \\ &\le \left\| \Delta_t^{N-1} z_t - \Delta_t^{N-1} p \right\| \\ &\le \cdots \\ &\le \left\| \Delta_t^0 z_t - \Delta_t^0 p \right\| = \|z_t - p\|. \end{aligned} \tag{3.4}$$

Moreover, it is easy from the nonexpansivity of R_t to see that

$$\|z_t - p\| = \|R_t x_t - R_t p\| \le \|x_t - p\|,$$

which together with the nonexpansivity of T_{r_t} and (3.4) implies that

$$\|v_t - p\| = \|T_{r_t} u_t - T_{r_t} p\| \le \|u_t - p\| \le \|z_t - p\| \le \|x_t - p\|. \tag{3.5}$$

By (3.5), we have

$$\begin{aligned} \|x_t - p\| &\le \left\| (I - \theta_t A)v_t + \theta_t\bigl(t\gamma V x_t + (I - t\mu G)v_t\bigr) - p \right\| \\ &= \left\| (I - \theta_t A)v_t - (I - \theta_t A)p + \theta_t\bigl(t\gamma V x_t + (I - t\mu G)v_t - p\bigr) + \theta_t(I - A)p \right\| \\ &\le \left\| (I - \theta_t A)v_t - (I - \theta_t A)p \right\| + \theta_t \left\| t\gamma V x_t + (I - t\mu G)v_t - p \right\| + \theta_t \left\| (I - A)p \right\| \\ &= \left\| (I - \theta_t A)v_t - (I - \theta_t A)p \right\| \\ &\quad + \theta_t \left\| (I - t\mu G)v_t - (I - t\mu G)p + t(\gamma V x_t - \mu G p) \right\| + \theta_t \left\| (I - A)p \right\| \\ &\le (1 - \theta_t \bar{\gamma})\|v_t - p\| + \theta_t\bigl[\left\| (I - t\mu G)v_t - (I - t\mu G)p \right\| \\ &\quad + t\bigl(\gamma \|V x_t - V p\| + \|\gamma V p - \mu G p\|\bigr)\bigr] + \theta_t \left\| (I - A)p \right\| \\ &\le (1 - \theta_t \bar{\gamma})\|x_t - p\| + \theta_t\bigl[(1 - t\tau)\|x_t - p\| + t\bigl(\gamma l \|x_t - p\| + \left\| (\gamma V - \mu G)p \right\|\bigr)\bigr] \\ &\quad + \theta_t \|I - A\| \|p\| \\ &= \bigl[1 - \theta_t\bigl(\bar{\gamma} - 1 + t(\tau - \gamma l)\bigr)\bigr]\|x_t - p\| + \theta_t\bigl[\|I - A\| \|p\| + t\left\| (\gamma V - \mu G)p \right\|\bigr]. \end{aligned}$$

So, it follows that

$$\|x_t - p\| \le \frac{\|I - A\| \|p\| + t\|(\gamma V - \mu G)p\|}{\bar{\gamma} - 1 + t(\tau - \gamma l)} \le \frac{\|I - A\| \|p\| + \|(\gamma V - \mu G)p\|}{\bar{\gamma} - 1}.$$

Hence $\{x_t\}$ is bounded and so are $\{V x_t\}$, $\{u_t\}$, $\{v_t\}$, $\{z_t\}$, and $\{G v_t\}$.

(ii) By the definition of $\{x_t\}$, we have

$$\begin{aligned} \|x_t - v_t\| &= \left\| P_C\bigl[(I - \theta_t A)v_t + \theta_t\bigl(t\gamma V x_t + (I - t\mu G)v_t\bigr)\bigr] - v_t \right\| \\ &\le \left\| (I - \theta_t A)v_t + \theta_t\bigl(t\gamma V x_t + (I - t\mu G)v_t\bigr) - v_t \right\| \end{aligned}$$

$$\begin{aligned}
&= \bigl\|\theta_t\bigl[(I-A)v_t + t(\gamma Vx_t - \mu Gv_t)\bigr]\bigr\| \\
&= \theta_t\bigl\|(I-A)v_t + t(\gamma Vx_t - \mu Gv_t)\bigr\| \\
&\le \theta_t\|I-A\|\|v_t\| + t\|\gamma Vx_t - \mu Gv_t\| \to 0 \quad \text{as } t \to 0,
\end{aligned}$$

using the boundedness of $\{Vx_t\}$, $\{v_t\}$, and $\{Gv_t\}$ in the proof of assertion (i). That is,

$$\lim_{t\to 0}\|x_t - v_t\| = 0. \tag{3.6}$$

In view of (3.5) and Lemma 2.7(ii), we get

$$\begin{aligned}
\|v_t - p\|^2 &\le \|z_t - p\|^2 = \|R_t x_t - R_t p\|^2 \\
&= \|F_{1,\lambda_t}F_{2,\nu_t}x_t - F_{1,\lambda_t}F_{2,\nu_t}p\|^2 \\
&\le \langle F_{2,\nu_t}x_t - F_{2,\nu_t}p, F_{1,\lambda_t}F_{2,\nu_t}x_t - F_{1,\lambda_t}F_{2,\nu_t}p\rangle \\
&= \langle F_{2,\nu_t}x_t - F_{2,\nu_t}p, z_t - p\rangle \\
&\le \frac{1}{2}\bigl[\|F_{2,\nu_t}x_t - F_{2,\nu_t}p\|^2 + \|z_t - p\|^2 - \bigl\|(F_{2,\nu_t}x_t - F_{2,\nu_t}p) - (z_t - p)\bigr\|^2\bigr] \\
&\le \frac{1}{2}\bigl[\|x_t - p\|^2 + \|x_t - p\|^2 - \bigl\|(F_{2,\nu_t}x_t - F_{2,\nu_t}p) - (z_t - p)\bigr\|^2\bigr] \\
&= \|x_t - p\|^2 - \frac{1}{2}\bigl\|(F_{2,\nu_t}x_t - F_{2,\nu_t}p) - (z_t - p)\bigr\|^2,
\end{aligned}$$

which immediately yields

$$\frac{1}{2}\bigl\|(F_{2,\nu_t}x_t - F_{2,\nu_t}p) - (z_t - p)\bigr\|^2 \le \|x_t - p\|^2 - \|v_t - p\|^2 \le \bigl(\|x_t - p\| + \|v_t - p\|\bigr)\|x_t - v_t\|.$$

From (3.6) and the boundedness of $\{x_t\}$ and $\{v_t\}$, we have

$$\lim_{t\to 0}\bigl\|(F_{2,\nu_t}x_t - F_{2,\nu_t}p) - (z_t - p)\bigr\| = 0. \tag{3.7}$$

Again from (3.5) and Lemma 2.7(ii), we obtain

$$\begin{aligned}
\|v_t - p\|^2 &\le \|z_t - p\|^2 = \|R_t x_t - R_t p\|^2 \\
&\le \|F_{2,\nu_t}x_t - F_{2,\nu_t}p\|^2 \\
&\le \langle x_t - p, F_{2,\nu_t}x_t - F_{2,\nu_t}p\rangle \\
&\le \frac{1}{2}\bigl[\|x_t - p\|^2 + \|F_{2,\nu_t}x_t - F_{2,\nu_t}p\|^2 - \bigl\|(x_t - p) - (F_{2,\nu_t}x_t - F_{2,\nu_t}p)\bigr\|^2\bigr] \\
&\le \frac{1}{2}\bigl[\|x_t - p\|^2 + \|x_t - p\|^2 - \bigl\|(x_t - p) - (F_{2,\nu_t}x_t - F_{2,\nu_t}p)\bigr\|^2\bigr] \\
&= \|x_t - p\|^2 - \frac{1}{2}\bigl\|(x_t - p) - (F_{2,\nu_t}x_t - F_{2,\nu_t}p)\bigr\|^2,
\end{aligned}$$

which hence leads to

$$\frac{1}{2}\bigl\|(x_t - p) - (F_{2,\nu_t}x_t - F_{2,\nu_t}p)\bigr\|^2 \le \|x_t - p\|^2 - \|v_t - p\|^2 \le \bigl(\|x_t - p\| + \|v_t - p\|\bigr)\|x_t - v_t\|.$$

Again from (3.6) and the boundedness of $\{x_t\}$ and $\{v_t\}$, we have

$$\lim_{t\to 0}\big\|(x_t-p)-(F_{2,v_t}x_t-F_{2,v_t}p)\big\|=0. \tag{3.8}$$

So it follows from (3.7) and (3.8) that

$$\|x_t-z_t\|\le\big\|(x_t-p)-(F_{2,v_t}x_t-F_{2,v_t}p)\big\|+\big\|(F_{2,v_t}x_t-F_{2,v_t}p)-(z_t-p)\big\|\to 0\quad\text{as } t\to 0.$$

That is,

$$\lim_{t\to 0}\|x_t-z_t\|=0. \tag{3.9}$$

Furthermore, from (3.5) and Proposition 2.1(ii) and Proposition 2.2, it follows that

$$\begin{aligned}
\|v_t-p\|^2\le\|u_t-p\|^2&=\big\|\Delta_t^N z_t-p\big\|^2\\
&\le\big\|\Delta_t^i z_t-p\big\|^2\\
&=\big\|T_{r_{i,t}}^{(\Theta_i,\varphi_i)}(I-r_{i,t}B_i)\Delta_t^{i-1}z_t-T_{r_{i,t}}^{(\Theta_i,\varphi_i)}(I-r_{i,t}B_i)p\big\|^2\\
&\le\big\|(I-r_{i,t}B_i)\Delta_t^{i-1}z_t-(I-r_{i,t}B_i)p\big\|^2\\
&\le\big\|\Delta_t^{i-1}z_t-p\big\|^2+r_{i,t}(r_{i,t}-2\mu_i)\big\|B_i\Delta_t^{i-1}z_t-B_ip\big\|^2\\
&\le\|z_t-p\|^2+r_{i,t}(r_{i,t}-2\mu_i)\big\|B_i\Delta_t^{i-1}z_t-B_ip\big\|^2\\
&\le\|x_t-p\|^2+r_{i,t}(r_{i,t}-2\mu_i)\big\|B_i\Delta_t^{i-1}z_t-B_ip\big\|^2,
\end{aligned}$$

which together with $\{r_{i,t}\}_{t\in(0,1)}\subset[c_i,d_i]\subset(0,2\mu_i)$ for $i\in\{1,2,\dots,N\}$ implies that

$$\begin{aligned}
c_i(2\mu_i-d_i)\big\|B_i\Delta_t^{i-1}z_t-B_ip\big\|^2&\le r_{i,t}(2\mu_i-r_{i,t})\big\|B_i\Delta_t^{i-1}z_t-B_ip\big\|^2\\
&\le\|x_t-p\|^2-\|v_t-p\|^2\le\big(\|x_t-p\|+\|v_t-p\|\big)\|x_t-v_t\|.
\end{aligned}$$

From (3.6) and the boundedness of $\{x_t\}$ and $\{v_t\}$, we have

$$\lim_{t\to 0}\big\|B_i\Delta_t^{i-1}z_t-B_ip\big\|=0. \tag{3.10}$$

Also, by Proposition 2.1(ii), we obtain that, for each $i=1,2,\dots,N$,

$$\begin{aligned}
&\big\|\Delta_t^i z_t-p\big\|^2\\
&\quad=\big\|T_{r_{i,t}}^{(\Theta_i,\varphi_i)}(I-r_{i,t}B_i)\Delta_t^{i-1}z_t-T_{r_{i,t}}^{(\Theta_i,\varphi_i)}(I-r_{i,t}B_i)p\big\|^2\\
&\quad\le\big\langle(I-r_{i,t}B_i)\Delta_t^{i-1}z_t-(I-r_{i,t}B_i)p,\Delta_t^i z_t-p\big\rangle\\
&\quad=\frac{1}{2}\Big[\big\|(I-r_{i,t}B_i)\Delta_t^{i-1}z_t-(I-r_{i,t}B_i)p\big\|^2+\big\|\Delta_t^i z_t-p\big\|^2\\
&\qquad-\big\|(I-r_{i,t}B_i)\Delta_t^{i-1}z_t-(I-r_{i,t}B_i)p-\big(\Delta_t^i z_t-p\big)\big\|^2\Big]\\
&\quad\le\frac{1}{2}\Big[\big\|\Delta_t^{i-1}z_t-p\big\|^2+\big\|\Delta_t^i z_t-p\big\|^2-\big\|\Delta_t^{i-1}z_t-\Delta_t^i z_t-r_{i,t}\big(B_i\Delta_t^{i-1}z_t-B_ip\big)\big\|^2\Big]\\
&\quad\le\frac{1}{2}\Big[\|x_t-p\|^2+\big\|\Delta_t^i z_t-p\big\|^2-\big\|\Delta_t^{i-1}z_t-\Delta_t^i z_t-r_{i,t}\big(B_i\Delta_t^{i-1}z_t-B_ip\big)\big\|^2\Big],
\end{aligned}$$

which immediately implies that

$$\left\|\varDelta_t^i z_t - p\right\|^2 \le \|x_t - p\|^2 - \left\|\varDelta_t^{i-1} z_t - \varDelta_t^i z_t - r_{i,t}\left(B_i \varDelta_t^{i-1} z_t - B_i p\right)\right\|^2.$$

This together with (3.5) leads to

$$\begin{aligned}\|v_t - p\|^2 \le \|u_t - p\|^2 = \left\|\varDelta_t^N z_t - p\right\|^2 \le \left\|\varDelta_t^i z_t - p\right\|^2 \\ \le \|x_t - p\|^2 - \left\|\varDelta_t^{i-1} z_t - \varDelta_t^i z_t - r_{i,t}\left(B_i \varDelta_t^{i-1} z_t - B_i p\right)\right\|^2,\end{aligned}$$

which hence implies

$$\begin{aligned}\left\|\varDelta_t^{i-1} z_t - \varDelta_t^i z_t - r_{i,t}\left(B_i \varDelta_t^{i-1} z_t - B_i p\right)\right\|^2 &\le \|x_t - p\|^2 - \|v_t - p\|^2 \\ &\le \left(\|x_t - p\| + \|v_t - p\|\right)\|x_t - v_t\|.\end{aligned}$$

From (3.6) and the boundedness of $\{x_t\}$ and $\{v_t\}$, we have

$$\lim_{t\to 0}\left\|\varDelta_t^{i-1} z_t - \varDelta_t^i z_t - r_{i,t}\left(B_i \varDelta_t^{i-1} z_t - B_i p\right)\right\| = 0,$$

which together with (3.10) implies that, for each $i = 1, 2, \ldots, N$,

$$\lim_{t\to 0}\left\|\varDelta_t^{i-1} z_t - \varDelta_t^i z_t\right\| = 0. \tag{3.11}$$

Note that

$$\|z_t - u_t\| \le \sum_{i=1}^{N}\left\|\varDelta_t^{i-1} z_t - \varDelta_t^i z_t\right\|.$$

From (3.11), it is easy to see that

$$\lim_{t\to 0}\|z_t - u_t\| = 0. \tag{3.12}$$

Also, observe that

$$\begin{aligned}\left\|x_t - \varDelta_t^N x_t\right\| &\le \|x_t - z_t\| + \left\|z_t - \varDelta_t^N z_t\right\| + \left\|\varDelta_t^N z_t - \varDelta_t^N x_t\right\| \\ &\le \|x_t - z_t\| + \left\|z_t - \varDelta_t^N z_t\right\| + \|z_t - x_t\| \\ &= 2\|x_t - z_t\| + \|z_t - u_t\|.\end{aligned}$$

From (3.9) and (3.12), it is easy to see that

$$\lim_{t\to 0}\left\|x_t - \varDelta_t^N x_t\right\| = 0. \tag{3.13}$$

In the meantime, again from (3.5) and Lemma 2.7(ii), we obtain

$$\|v_t - p\|^2 = \|T_{r_t} u_t - T_{r_t} p\|^2$$

$$
\begin{aligned}
&\leq \langle u_t - p, T_{r_t}u_t - T_{r_t}p\rangle = \langle u_t - p, v_t - p\rangle \\
&= \frac{1}{2}\bigl[\|u_t - p\|^2 + \|v_t - p\|^2 - \bigl\|u_t - p - (v_t - p)\bigr\|^2\bigr] \\
&\leq \frac{1}{2}\bigl[\|x_t - p\|^2 + \|x_t - p\|^2 - \|u_t - v_t\|^2\bigr] \\
&= \|x_t - p\|^2 - \frac{1}{2}\|u_t - v_t\|^2,
\end{aligned}
$$

which immediately yields

$$
\frac{1}{2}\|u_t - v_t\|^2 \leq \|x_t - p\|^2 - \|v_t - p\|^2 \leq \bigl(\|x_t - p\| + \|v_t - p\|\bigr)\|x_t - v_t\|.
$$

From (3.6) and the boundedness of $\{x_t\}$ and $\{v_t\}$, we have

$$
\lim_{t\to 0}\|u_t - v_t\| = 0. \tag{3.14}
$$

Taking into account that

$$
\begin{aligned}
\|x_t - T_{r_t}x_t\| &\leq \|x_t - u_t\| + \|u_t - T_{r_t}u_t\| + \|T_{r_t}u_t - T_{r_t}x_t\| \\
&\leq \|x_t - u_t\| + \|u_t - T_{r_t}u_t\| + \|u_t - x_t\| \\
&= 2\|x_t - u_t\| + \|u_t - v_t\| \\
&\leq 2\bigl(\|x_t - z_t\| + \|z_t - u_t\|\bigr) + \|u_t - v_t\|,
\end{aligned}
$$

we deduce from (3.9), (3.12), and (3.14) that

$$
\lim_{t\to 0}\|x_t - T_{r_t}x_t\| = 0. \tag{3.15}
$$

(iii) Let $t, t_0 \in (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\})$. Since $v_t = T_{r_t}u_t$ and $v_{t_0} = T_{r_{t_0}}u_{t_0}$, we get

$$
\bigl\langle y - v_t, (I-T)v_t\bigr\rangle + \frac{1}{r_t}\langle y - v_t, v_t - u_t\rangle \geq 0, \quad \forall y \in C, \tag{3.16}
$$

and

$$
\bigl\langle y - v_{t_0}, (I-T)v_{t_0}\bigr\rangle + \frac{1}{r_{t_0}}\langle y - v_{t_0}, v_{t_0} - u_{t_0}\rangle \geq 0, \quad \forall y \in C. \tag{3.17}
$$

Putting $y = v_{t_0}$ in (3.16) and $y = v_t$ in (3.17), we obtain

$$
\bigl\langle v_{t_0} - v_t, (I-T)v_t\bigr\rangle + \frac{1}{r_t}\langle v_{t_0} - v_t, v_t - u_t\rangle \geq 0 \tag{3.18}
$$

and

$$
\bigl\langle v_t - v_{t_0}, (I-T)v_{t_0}\bigr\rangle + \frac{1}{r_{t_0}}\langle v_t - v_{t_0}, v_{t_0} - u_{t_0}\rangle \geq 0. \tag{3.19}
$$

Adding up (3.18) and (3.19), we have

$$-\langle v_t - v_{t_0}, (I-T)v_t - (I-T)v_{t_0}\rangle + \left\langle v_{t_0} - v_t, \frac{v_t - u_t}{r_t} - \frac{v_{t_0} - u_{t_0}}{r_{t_0}}\right\rangle \geq 0.$$

Since T is pseudocontractive, we know that $I - T$ is a monotone mapping such that

$$\left\langle v_{t_0} - v_t, \frac{v_t - u_t}{r_t} - \frac{v_{t_0} - u_{t_0}}{r_{t_0}}\right\rangle \geq 0,$$

and hence

$$\left\langle v_t - v_{t_0}, v_{t_0} - v_t + v_t - u_{t_0} - \frac{r_{t_0}}{r_t}(v_t - u_t)\right\rangle \geq 0. \tag{3.20}$$

Taking into account that $\liminf_{t\to 0} r_t > 0$, without loss of generality, we may assume that $r_t > b > 0$ $\forall t \in (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\})$ for some $b > 0$. Then from (3.20) we have

$$\begin{aligned}
\|v_t - v_{t_0}\|^2 &\leq \left\langle v_t - v_{t_0}, v_t - u_t + u_t - u_{t_0} - \frac{r_{t_0}}{r_t}(v_t - u_t)\right\rangle \\
&= \left\langle v_t - v_{t_0}, u_t - u_{t_0} + \left(1 - \frac{r_{t_0}}{r_t}\right)(v_t - u_t)\right\rangle \\
&\leq \|v_t - v_{t_0}\| \left\| u_t - u_{t_0} + \left(1 - \frac{r_{t_0}}{r_t}\right)(v_t - u_t)\right\| \\
&\leq \|v_t - v_{t_0}\| \left\{ \|u_t - u_{t_0}\| + \left|1 - \frac{r_{t_0}}{r_t}\right| \|v_t - u_t\| \right\},
\end{aligned}$$

which immediately yields

$$\begin{aligned}
\|v_t - v_{t_0}\| &\leq \|u_t - u_{t_0}\| + \frac{1}{r_t}|r_t - r_{t_0}|\|v_t - u_t\| \\
&\leq \|u_t - u_{t_0}\| + \frac{\tilde{L}_1}{b}|r_t - r_{t_0}|,
\end{aligned} \tag{3.21}$$

where $\tilde{L}_1 = \sup\{\|v_t - u_t\| : t \in (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\})\}$.

Also, taking into account that $\lim_{t\to 0} \lambda_t = \lambda > 0$ and $\lim_{t\to 0} \nu_t = \nu > 0$, without loss of generality, we may assume that $\min\{\lambda_t, \nu_t\} > a > 0$ $\forall t \in (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\})$ for some $a > 0$. Since $z_t = F_{1,\lambda_t} y_t$ and $z_{t_0} = F_{1,\lambda_{t_0}} y_{t_0}$, where $y_t = F_{2,\nu_t} x_t$ and $y_{t_0} = F_{2,\nu_{t_0}} x_{t_0}$ for $t, t_0 \in (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\})$, by using arguments similar to those of (3.21), we get

$$\|z_t - z_{t_0}\| \leq \|y_t - y_{t_0}\| + \frac{1}{a}|\lambda_t - \lambda_{t_0}|\tilde{L}_2 \tag{3.22}$$

and

$$\|y_t - y_{t_0}\| \leq \|x_t - x_{t_0}\| + \frac{1}{a}|\nu_t - \nu_{t_0}|\tilde{L}_2, \tag{3.23}$$

where $\widetilde{L}_2 = \sup\{\|z_t - y_t\| + \|y_t - x_t\| : t \in (0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\})\}$. Substituting (3.23) for (3.22), we obtain

$$\|z_t - z_{t_0}\| \le \|x_t - x_{t_0}\| + \frac{\widetilde{L}_2}{a}\bigl(|\lambda_t - \lambda_{t_0}| + |\nu_t - \nu_{t_0}|\bigr). \tag{3.24}$$

In the meantime, by Proposition 2.1(ii), (v) and Proposition 2.2, we deduce that

$$\begin{aligned}
\|u_t - u_{t_0}\| &= \bigl\|\Delta_t^N z_t - \Delta_{t_0}^N z_{t_0}\bigr\| \\
&= \bigl\|T_{r_{N,t}}^{(\Theta_N,\varphi_N)}(I - r_{N,t}B_N)\Delta_t^{N-1}z_t - T_{r_{N,t_0}}^{(\Theta_N,\varphi_N)}(I - r_{N,t_0}B_N)\Delta_{t_0}^{N-1}z_{t_0}\bigr\| \\
&\le \bigl\|T_{r_{N,t}}^{(\Theta_N,\varphi_N)}(I - r_{N,t}B_N)\Delta_t^{N-1}z_t - T_{r_{N,t_0}}^{(\Theta_N,\varphi_N)}(I - r_{N,t_0}B_N)\Delta_t^{N-1}z_t\bigr\| \\
&\quad + \bigl\|T_{r_{N,t_0}}^{(\Theta_N,\varphi_N)}(I - r_{N,t_0}B_N)\Delta_t^{N-1}z_t - T_{r_{N,t_0}}^{(\Theta_N,\varphi_N)}(I - r_{N,t_0}B_N)\Delta_{t_0}^{N-1}z_{t_0}\bigr\| \\
&\le \bigl\|T_{r_{N,t}}^{(\Theta_N,\varphi_N)}(I - r_{N,t}B_N)\Delta_t^{N-1}z_t - T_{r_{N,t_0}}^{(\Theta_N,\varphi_N)}(I - r_{N,t}B_N)\Delta_t^{N-1}z_t\bigr\| \\
&\quad + \bigl\|T_{r_{N,t_0}}^{(\Theta_N,\varphi_N)}(I - r_{N,t}B_N)\Delta_t^{N-1}z_t - T_{r_{N,t_0}}^{(\Theta_N,\varphi_N)}(I - r_{N,t_0}B_N)\Delta_t^{N-1}z_t\bigr\| \\
&\quad + \bigl\|(I - r_{N,t_0}B_N)\Delta_t^{N-1}z_t - (I - r_{N,t_0}B_N)\Delta_{t_0}^{N-1}z_{t_0}\bigr\| \\
&\le \frac{|r_{N,t} - r_{N,t_0}|}{r_{N,t}}\bigl\|T_{r_{N,t}}^{(\Theta_N,\varphi_N)}(I - r_{N,t}B_N)\Delta_t^{N-1}z_t - (I - r_{N,t}B_N)\Delta_t^{N-1}z_t\bigr\| \\
&\quad + |r_{N,t} - r_{N,t_0}|\bigl\|B_N\Delta_t^{N-1}z_t\bigr\| + \bigl\|\Delta_t^{N-1}z_t - \Delta_{t_0}^{N-1}z_{t_0}\bigr\| \\
&= |r_{N,t} - r_{N,t_0}|\biggl[\bigl\|B_N\Delta_t^{N-1}z_t\bigr\| + \frac{1}{r_{N,t}}\bigl\|T_{r_{N,t}}^{(\Theta_N,\varphi_N)}(I - r_{N,t}B_N)\Delta_t^{N-1}z_t \\
&\quad - (I - r_{N,t}B_N)\Delta_t^{N-1}z_t\bigr\|\biggr] + \bigl\|\Delta_t^{N-1}z_t - \Delta_{t_0}^{N-1}z_{t_0}\bigr\| \\
&\le \cdots \\
&\le |r_{N,t} - r_{N,t_0}|\biggl[\bigl\|B_N\Delta_t^{N-1}z_t\bigr\| + \frac{1}{r_{N,t}}\bigl\|T_{r_{N,t}}^{(\Theta_N,\varphi_N)}(I - r_{N,t}B_N)\Delta_t^{N-1}z_t \\
&\quad - (I - r_{N,t}B_N)\Delta_t^{N-1}z_t\bigr\|\biggr] + \cdots + |r_{1,t} - r_{1,t_0}|\biggl[\bigl\|B_1\Delta_t^0 z_t\bigr\| \\
&\quad + \frac{1}{r_{1,t}}\bigl\|T_{r_{1,t}}^{(\Theta_1,\varphi_1)}(I - r_{1,t}B_1)\Delta_t^0 z_t - (I - r_{1,t}B_1)\Delta_t^0 z_t\bigr\|\biggr] + \bigl\|\Delta_t^0 z_t - \Delta_{t_0}^0 z_{t_0}\bigr\| \\
&\le \widetilde{L}_3\sum_{i=1}^N |r_{i,t} - r_{i,t_0}| + \|z_t - z_{t_0}\|,
\end{aligned} \tag{3.25}$$

where

$$\sup_{t\in(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma l}\})}\Biggl\{\sum_{i=1}^N\biggl[\bigl\|B_i\Delta_t^{i-1}z_t\bigr\| + \frac{1}{r_{i,t}}\bigl\|T_{r_{i,t}}^{(\Theta_i,\varphi_i)}(I - r_{i,t}B_i)\Delta_t^{i-1}z_t - (I - r_{i,t}B_i)\Delta_t^{i-1}z_t\bigr\|\biggr]\Biggr\} \le \widetilde{L}_3$$

for some $\widetilde{L}_3 > 0$. This together with (3.21) and (3.24) implies that

$$\|v_t - v_{t_0}\| \le \|u_t - u_{t_0}\| + \frac{\widetilde{L}_1}{b}|r_t - r_{t_0}|$$

$$
\begin{aligned}
&\le \widetilde{L}_3 \sum_{i=1}^{N} |r_{i,t} - r_{i,t_0}| + \|z_t - z_{t_0}\| + \frac{\tilde{L}_1}{b} |r_t - r_{t_0}| \\
&\le \widetilde{L}_3 \sum_{i=1}^{N} |r_{i,t} - r_{i,t_0}| + \|x_t - x_{t_0}\| + \frac{\tilde{L}_2}{a} \big(|\lambda_t - \lambda_{t_0}| + |\nu_t - \nu_{t_0}| \big) + \frac{\tilde{L}_1}{b} |r_t - r_{t_0}| \\
&\le \|x_t - x_{t_0}\| + \left(\frac{\tilde{L}_1}{b} + \frac{\tilde{L}_2}{a} \right) \big(|\lambda_t - \lambda_{t_0}| + |\nu_t - \nu_{t_0}| + |r_t - r_{t_0}| \big) \\
&\quad + \widetilde{L}_3 \sum_{i=1}^{N} |r_{i,t} - r_{i,t_0}|.
\end{aligned}
$$

Taking into account that both $\theta_{t_0} \in (0, \min\{\frac{1}{2}, \|A\|^{-1}\})$ and $0 \le \gamma l < \tau = 1 - \sqrt{1 - \mu(2\eta - \mu\rho^2)}$ imply

$$
0 < 1 - \theta_{t_0}(\bar{\gamma} - 1 + t_0 \tau) < 1,
$$

we calculate from (3.1)

$$
\begin{aligned}
\|x_t - x_{t_0}\| &\le \big\| (I - \theta_t A) T_{r_t} \Delta_t^N R_t x_t + \theta_t \big(t\gamma V x_t + (I - t\mu G) T_{r_t} \Delta_t^N R_t x_t \big) \\
&\quad - (I - \theta_{t_0} A) T_{r_{t_0}} \Delta_{t_0}^N R_{t_0} x_{t_0} + \theta_{t_0} \big(t_0 \gamma V x_{t_0} + (I - t_0 \mu G) T_{r_{t_0}} \Delta_{t_0}^N R_{t_0} x_{t_0} \big) \big\| \\
&= \big\| (I - \theta_t A) \nu_t + \theta_t \big(t\gamma V x_t + (I - t\mu G) \nu_t \big) - (I - \theta_{t_0} A) \nu_{t_0} \\
&\quad + \theta_{t_0} \big(t_0 \gamma V x_{t_0} + (I - t_0 \mu G) \nu_{t_0} \big) \big\| \\
&\le \big\| (I - \theta_t A) \nu_t - (I - \theta_{t_0} A) \nu_t \big\| + \big\| (I - \theta_{t_0} A) \nu_t - (I - \theta_{t_0} A) \nu_{t_0} \big\| \\
&\quad + |\theta_t - \theta_{t_0}| \big\| t\gamma V x_t + (I - t\mu G) \nu_t \big\| + \theta_{t_0} \big\| \big[t\gamma V x_t + (I - t\mu G) \nu_t \big] \\
&\quad - \big[t_0 \gamma V x_{t_0} + (I - t_0 \mu G) \nu_{t_0} \big] \big\| \\
&\le |\theta_t - \theta_{t_0}| \|A\| \|\nu_t\| + (1 - \theta_{t_0} \bar{\gamma}) \|\nu_t - \nu_{t_0}\| + |\theta_t - \theta_{t_0}| \big\| t\gamma V x_t + (I - t\mu G) \nu_t \big\| \\
&\quad + \theta_{t_0} \big\| (t - t_0) \gamma V x_t + t_0 \gamma (V x_t - V x_{t_0}) - (t - t_0) \mu G \nu_t + (I - t_0 \mu G) \nu_t \\
&\quad - (I - t_0 \mu G) \nu_{t_0} \big\| \\
&\le |\theta_t - \theta_{t_0}| \|A\| \|\nu_t\| + (1 - \theta_{t_0} \bar{\gamma}) \|\nu_t - \nu_{t_0}\| + |\theta_t - \theta_{t_0}| \big[\|\nu_t\| \\
&\quad + t \big(\gamma \|V x_t\| + \mu \|G \nu_t\| \big) \big] \\
&\quad + \theta_{t_0} \big[\big(\gamma \|V x_t\| + \mu \|G \nu_t\| \big) |t - t_0| + t_0 \gamma l \|x_t - x_{t_0}\| + (1 - t_0 \tau) \|\nu_t - \nu_{t_0}\| \big] \\
&\le |\theta_t - \theta_{t_0}| \big[\|\nu_t\| + \|A\| \|\nu_t\| + \gamma \|V x_t\| + \mu \|G \nu_t\| \big] + \theta_{t_0} t_0 \gamma l \|x_t - x_{t_0}\| \\
&\quad + \big[1 - \theta_{t_0} (\bar{\gamma} - 1 + t_0 \tau) \big] \\
&\quad \times \left\{ \|x_t - x_{t_0}\| + \left(\frac{\tilde{L}_1}{b} + \frac{\tilde{L}_2}{a} \right) \big(|\lambda_t - \lambda_{t_0}| + |\nu_t - \nu_{t_0}| + |r_t - r_{t_0}| \big) \right. \\
&\quad \left. + \widetilde{L}_3 \sum_{i=1}^{N} |r_{i,t} - r_{i,t_0}| \right\} + \theta_{t_0} \big(\gamma \|V x_t\| + \mu \|G \nu_t\| \big) |t - t_0| \\
&= |\theta_t - \theta_{t_0}| \big[\|\nu_t\| + \|A\| \|\nu_t\| + \gamma \|V x_t\| + \mu \|G \nu_t\| \big] \\
&\quad + \big[1 - \theta_{t_0} (\bar{\gamma} - 1 + t_0 (\tau - \gamma l)) \big] \|x_t - x_{t_0}\|
\end{aligned}
$$

$$+\big[1-\theta_{t_0}(\bar{\gamma}-1+t_0\tau)\big]\left\{\left(\frac{\widetilde{L}_1}{b}+\frac{\widetilde{L}_2}{a}\right)\big(|\lambda_t-\lambda_{t_0}|+|\nu_t-\nu_{t_0}|+|r_t-r_{t_0}|\big)\right.$$
$$\left.+\widetilde{L}_3\sum_{i=1}^N|r_{i,t}-r_{i,t_0}|\right\}+\theta_{t_0}\big(\gamma\|Vx_t\|+\mu\|G\nu_t\|\big)|t-t_0|.$$

This immediately implies that

$$\begin{aligned}\|x_t-x_{t_0}\| \le{}& \frac{\|\nu_t\|+\|A\|\|\nu_t\|+\gamma\|Vx_t\|+\mu\|G\nu_t\|}{\theta_{t_0}(\bar{\gamma}-1+t_0(\tau-\gamma l)}|\theta_t-\theta_{t_0}|+\frac{\gamma\|Vx_t\|+\mu\|G\nu_t\|}{\bar{\gamma}-1+t_0(\tau-\gamma l)}|t-t_0| \\ &+\frac{1-\theta_{t_0}(\bar{\gamma}-1+t_0\tau)}{\theta_{t_0}(\bar{\gamma}-1+t_0(\tau-\gamma l)}\left\{\left(\frac{\widetilde{L}_1}{b}+\frac{\widetilde{L}_2}{a}\right)\big(|\lambda_t-\lambda_{t_0}|+|\nu_t-\nu_{t_0}|+|r_t-r_{t_0}|\big)\right. \\ &\left.+\widetilde{L}_3\sum_{i=1}^N|r_{i,t}-r_{i,t_0}|\right\}.\end{aligned}$$

Since $\theta_t:(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma l}\})\to(0,\min\{\frac{1}{2},\|A\|^{-1}\})$ is locally Lipschitzian, $r_t,\lambda_t,\nu_t:(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma l}\})\to(0,\infty)$ are locally Lipschitzian, and $r_{i,t}:(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma l}\})\to[c_i,d_i]$ is locally Lipschitzian for each $i=1,2,\ldots,N$, we deduce that $x_t:(0,\min\{1,\frac{2-\bar{\gamma}}{\tau-\gamma l}\})\to H$ is locally Lipschitzian.

(iv) From the last inequality in (iii), the desired result follows immediately. □

We prove the following strong convergence theorem for the net $\{x_t\}$ as $t\to 0$, which guarantees the existence of solutions of the variational inequality (3.2).

Theorem 3.2 *Let the net $\{x_t\}$ be defined via (3.1). If $\lim_{t\to 0}\theta_t=0$, then x_t converges strongly to $\widetilde{x}\in\Omega$ as $t\to 0$, which solves VI (3.2). Equivalently, we have $P_\Omega(2I-A)\widetilde{x}=\widetilde{x}$.*

Proof We first note that the uniqueness of a solution of VI (3.2) is a consequence of the strong monotonicity of $A-I$ (due to Lemma 2.4). See [2, 4, 5] for this fact.

Next, we prove that $x_t\to\widetilde{x}$ as $t\to 0$. For simplicity, let $\nu_t=T_{r_t}u_t$, $u_t=\Delta_t^N z_t$, $y_t=F_{2,\nu_t}x_t$, and $z_t=R_tx_t=F_{1,\lambda_t}y_t$. For any given $p\in\Omega$, we observe that $T_{r_t}p=p$, $\Delta_t^Np=p$, and $R_tp=p$. From (3.1), we write

$$\begin{aligned}x_t-p&=x_t-w_t+w_t-p=x_t-w_t+(I-\theta_tA)\nu_t+\theta_t\big(t\gamma Vx_t+(I-t\mu G)\nu_t\big)-p \\ &=x_t-w_t+(I-\theta_tA)(\nu_t-p)+\theta_t\big[t(\gamma Vx_t-\mu Gp)+(I-t\mu G)\nu_t-(I-t\mu G)p\big] \\ &\quad+\theta_t(I-A)p,\end{aligned}$$

where $w_t=(I-\theta_tA)\nu_t+\theta_t(t\gamma Vx_t+(I-t\mu G)\nu_t)$. In terms of (2.1) and (3.5), we have

$$\begin{aligned}\|x_t-p\|^2&=\langle x_t-w_t,x_t-p\rangle+\big\langle(I-\theta_tA)(\nu_t-p),x_t-p\big\rangle+\theta_t\big[t\langle\gamma Vx_t-\mu Gp,x_t-p\rangle \\ &\quad+\big\langle(I-t\mu G)\nu_t-(I-t\mu G)p,x_t-p\big\rangle\big]+\theta_t\big\langle(I-A)p,x_t-p\big\rangle \\ &\le(1-\theta_t\bar{\gamma})\|x_t-p\|^2+\theta_t\big[(1-t\tau)\|x_t-p\|^2+t\gamma l\|x_t-p\|^2 \\ &\quad+t\big\langle(\gamma V-\mu G)p,x_t-p\big\rangle\big]+\theta_t\big\langle(I-A)p,x_t-p\big\rangle \\ &=\big[1-\theta_t\big(\bar{\gamma}-1+t(\tau-\gamma l)\big)\big]\|x_t-p\|^2+\theta_t\big(t\big\langle(\gamma V-\mu G)p,x_t-p\big\rangle \\ &\quad+\big\langle(I-A)p,x_t-p\big\rangle\big).\end{aligned}$$

Therefore,

$$\|x_t - p\|^2 \le \frac{1}{\bar{\gamma} - 1 + t(\tau - \gamma l)} \bigl(t\langle(\gamma V - \mu G)p, x_t - p\rangle + \langle(I - A)p, x_t - p\rangle\bigr). \tag{3.26}$$

Since $\{x_t\}$ is bounded as $t \to 0$ (due to Theorem 3.1(i)), there exists a subsequence $\{t_n\}$ in $(0, \min\{1, \frac{2-\bar{\gamma}}{\tau-\gamma l}\})$ such that $t_n \to 0$ and $x_{t_n} \rightharpoonup x^*$. We first show that $x^* \in \Omega$. To this end, we divide its proof into four steps.

Step 1. We claim that $\lim_{n\to\infty} \|x_{t_n} - z_{t_n}\| = 0$, $\lim_{n\to\infty} \|z_{t_n} - u_{t_n}\| = 0$, and $\lim_{n\to\infty} \|u_{t_n} - v_{t_n}\| = 0$, where $z_{t_n} = R_{t_n} x_{t_n}$, $u_{t_n} = \Delta_{t_n}^N z_{t_n}$, and $v_{t_n} = T_{r_{t_n}} u_{t_n}$. Indeed, according to (3.9), (3.12), and (3.14) in the proof of Theorem 3.1, we obtain the assertion.

Step 2. We claim that $x^* \in \mathrm{Fix}(T)$. In fact, from the definition of $v_{t_n} = T_{r_{t_n}} u_{t_n}$, we have

$$\langle y - v_{t_n}, (I - T)v_{t_n}\rangle + \left\langle y - v_{t_n}, \frac{v_{t_n} - u_{t_n}}{r_{t_n}}\right\rangle \ge 0, \quad \forall y \in C. \tag{3.27}$$

Set $w_t = tv + (1 - t)x^*$ for all $t \in (0, 1]$ and $v \in C$. Then $w_t \in C$. From (3.27) it follows that

$$\begin{aligned}\langle w_t - v_{t_n}, (I - T)w_t\rangle &\ge \langle w_t - v_{t_n}, (I - T)w_t\rangle - \langle w_t - v_{t_n}, (I - T)v_{t_n}\rangle - \left\langle w_t - v_{t_n}, \frac{v_{t_n} - u_{t_n}}{r_{t_n}}\right\rangle \\ &= \langle w_t - v_{t_n}, (I - T)w_t - (I - T)v_{t_n}\rangle - \left\langle w_t - v_{t_n}, \frac{v_{t_n} - u_{t_n}}{r_{t_n}}\right\rangle. \end{aligned} \tag{3.28}$$

By Step 1, we have $\frac{v_{t_n}-u_{t_n}}{r_{t_n}} \to 0$ as $n \to \infty$. Moreover, since $x_{t_n} \rightharpoonup x^*$, by Step 1 we have $v_{t_n} \rightharpoonup x^*$. Since $I - T$ is monotone, we also have that $\langle w_t - v_{t_n}, (I - T)w_t - (I - T)v_{t_n}\rangle \ge 0$. Thus, from (3.28) it follows that

$$0 \le \lim_{n\to\infty} \langle w_t - v_{t_n}, (I - T)w_t\rangle = \langle w_t - x^*, (I - T)w_t\rangle,$$

and hence

$$\langle v - x^*, (I - T)w_t\rangle \ge 0, \quad \forall v \in C.$$

Letting $t \to 0$, we know from the continuity of $I - T$ that

$$\langle v - x^*, (I - T)x^*\rangle \ge 0, \quad \forall v \in C.$$

Putting $v = Tx^*$, we get $\|(I - T)x^*\|^2 = 0$, which leads to $x^* \in \mathrm{Fix}(T)$.

Step 3. We claim that $x^* \in \mathrm{GSVI}(C, F_1, F_2)$. Indeed, note that $\lim_{t\to0} \lambda_t = \lambda > 0$ and $\lim_{t\to0} \nu_t = \nu > 0$. For each $x \in C$, we put $x(t) := F_{1,\lambda_t} x$, $x(0) := F_{1,\lambda} x$, $y(t) := F_{2,\nu_t} x$, and $y(0) := F_{2,\nu} x$. Then, by Lemma 1.1, we have $\mathrm{GSVI}(C, F_1, F_2) = \mathrm{Fix}(R)$, where $R = F_{1,\lambda} F_{2,\nu}$ and R is nonexpansive. Moreover, it is easy to see that

$$\langle y - x(t), F_1 x(t)\rangle + \frac{1}{\lambda_t}\langle y - x(t), x(t) - x\rangle \ge 0, \quad \forall y \in C, \tag{3.29}$$

and

$$\langle y - x(0), F_1 x(0)\rangle + \frac{1}{\lambda}\langle y - x(0), x(0) - x\rangle \ge 0, \quad \forall y \in C. \tag{3.30}$$

Putting $y = x(0)$ in (3.29) and $y = x(t)$ in (3.30), we obtain

$$\langle x(0) - x(t), F_1 x(t)\rangle + \frac{1}{\lambda_t}\langle x(0) - x(t), x(t) - x\rangle \geq 0 \tag{3.31}$$

and

$$\langle x(t) - x(0), F_1 x(0)\rangle + \frac{1}{\lambda}\langle x(t) - x(0), x(0) - x\rangle \geq 0. \tag{3.32}$$

Adding up (3.31) and (3.32), we have

$$-\langle x(t) - x(0), F_1 x(t) - F_1 x(0)\rangle + \left\langle x(0) - x(t), \frac{x(t) - x}{\lambda_t} - \frac{x(0) - x}{\lambda}\right\rangle \geq 0.$$

Since F_1 is a monotone mapping, we know that

$$\left\langle x(0) - x(t), \frac{x(t) - x}{\lambda_t} - \frac{x(0) - x}{\lambda}\right\rangle \geq 0,$$

and hence

$$\left\langle x(t) - x(0), x(0) - x(t) + x(t) - x - \frac{\lambda}{\lambda_t}\bigl(x(t) - x\bigr)\right\rangle \geq 0.$$

So it follows that

$$\begin{aligned}\bigl\|x(t) - x(0)\bigr\|^2 &\leq \left\langle x(t) - x(0), x(t) - x - \frac{\lambda}{\lambda_t}\bigl(x(t) - x\bigr)\right\rangle \\ &= \left\langle x(t) - x(0), \left(1 - \frac{\lambda}{\lambda_t}\right)\bigl(x(t) - x\bigr)\right\rangle \\ &\leq \bigl\|x(t) - x(0)\bigr\| \cdot \frac{|\lambda_t - \lambda|}{\lambda_t}\bigl\|x(t) - x\bigr\|,\end{aligned}$$

which immediately yields

$$\|F_{1,\lambda_t}x - F_{1,\lambda}x\| \leq \frac{|\lambda_t - \lambda|}{\lambda_t}\|F_{1,\lambda_t}x - x\|. \tag{3.33}$$

By using arguments similar to those of (3.33), we have

$$\|F_{2,\nu_t}x - F_{2,\nu}x\| \leq \frac{|\nu_t - \nu|}{\nu_t}\|F_{2,\nu_t}x - x\|. \tag{3.34}$$

Now, putting $t = t_n$, $x = F_{2,\nu}x_{t_n}$ in (3.33), and $t = t_n$, $x = x_{t_n}$ in (3.34), respectively, we deduce that

$$\|F_{1,\lambda_{t_n}}F_{2,\nu}x_{t_n} - F_{1,\lambda}F_{2,\nu}x_{t_n}\| \leq \frac{|\lambda_{t_n} - \lambda|}{\lambda_{t_n}}\|F_{1,\lambda_{t_n}}F_{2,\nu}x_{t_n} - F_{2,\nu}x_{t_n}\|$$

and

$$\|F_{2,\nu_{t_n}}x_{t_n} - F_{2,\nu}x_{t_n}\| \leq \frac{|\nu_{t_n} - \nu|}{\nu_{t_n}}\|F_{2,\nu_{t_n}}x_{t_n} - x_{t_n}\|.$$

Since $\lim_{n\to\infty}\lambda_{t_n}=\lambda>0$ and $\lim_{n\to\infty}\nu_{t_n}=\nu>0$, it follows from the last two inequalities that

$$\lim_{n\to\infty}\|F_{1,\lambda_{t_n}}F_{2,\nu}x_{t_n}-F_{1,\lambda}F_{2,\nu}x_{t_n}\|=\lim_{n\to\infty}\|F_{2,\nu_{t_n}}x_{t_n}-F_{2,\nu}x_{t_n}\|=0. \tag{3.35}$$

Also, we observe that

$$\begin{aligned}
&\|Rx_{t_n}-x_{t_n}\|\\
&\quad\le\|F_{1,\lambda}F_{2,\nu}x_{t_n}-F_{1,\lambda_{t_n}}F_{2,\nu}x_{t_n}\|+\|F_{1,\lambda_{t_n}}F_{2,\nu}x_{t_n}-F_{1,\lambda_{t_n}}F_{2,\nu_{t_n}}x_{t_n}\|\\
&\qquad+\|F_{1,\lambda_{t_n}}F_{2,\nu_{t_n}}x_{t_n}-x_{t_n}\|\\
&\quad\le\|F_{1,\lambda}F_{2,\nu}x_{t_n}-F_{1,\lambda_{t_n}}F_{2,\nu}x_{t_n}\|+\|F_{2,\nu}x_{t_n}-F_{2,\nu_{t_n}}x_{t_n}\|+\|F_{1,\lambda_{t_n}}F_{2,\nu_{t_n}}x_{t_n}-x_{t_n}\|\\
&\quad=\|F_{1,\lambda}F_{2,\nu}x_{t_n}-F_{1,\lambda_{t_n}}F_{2,\nu}x_{t_n}\|+\|F_{2,\nu}x_{t_n}-F_{2,\nu_{t_n}}x_{t_n}\|+\|R_{t_n}x_{t_n}-x_{t_n}\|.
\end{aligned} \tag{3.36}$$

Since $R_{t_n}x_{t_n}-x_{t_n}\to 0$ (due to Step 1), from (3.35) and (3.36) we get

$$\lim_{n\to\infty}\|Rx_{t_n}-x_{t_n}\|=0. \tag{3.37}$$

Taking into account that $x_{t_n}\rightharpoonup x^*$ and $x_{t_n}-Rx_{t_n}\to 0$ (due to (3.37)), from Lemma 2.3 we get $x^*=Rx^*$, that is, $x^*\in\operatorname{Fix}(R)=\operatorname{GSVI}(C,F_1,F_2)$.

Step 4. We claim that $x^*\in\bigcap_{i=1}^N\operatorname{GMEP}(\Theta_i,\varphi_i,B_i)$. In fact, since $\Delta_{t_n}^i z_{t_n}=T_{r_{i,t_n}}^{(\Theta_i,\varphi_i)}(I-r_{i,t_n}B_i)\Delta_{t_n}^{i-1}z_{t_n}$, for each $i=1,2,\dots,N$, we have

$$\begin{aligned}
0&\le\Theta_i\big(\Delta_{t_n}^i z_{t_n},y\big)+\varphi_i(y)-\varphi_i\big(\Delta_{t_n}^i z_{t_n}\big)\\
&\quad+\big\langle B_i\Delta_{t_n}^{i-1}z_{t_n},y-\Delta_{t_n}^i z_{t_n}\big\rangle+\frac{1}{r_{i,t_n}}\big\langle y-\Delta_{t_n}^i z_{t_n},\Delta_{t_n}^i z_{t_n}-\Delta_{t_n}^{i-1}z_{t_n}\big\rangle.
\end{aligned}$$

By (A2), we have

$$\begin{aligned}
\Theta_i\big(y,\Delta_{t_n}^i z_{t_n}\big)&\le\varphi_i(y)-\varphi_i\big(\Delta_{t_n}^i z_{t_n}\big)+\big\langle B_i\Delta_{t_n}^{i-1}z_{t_n},y-\Delta_{t_n}^i z_{t_n}\big\rangle\\
&\quad+\frac{1}{r_{i,t_n}}\big\langle y-\Delta_{t_n}^i z_{t_n},\Delta_{t_n}^i z_{t_n}-\Delta_{t_n}^{i-1}z_{t_n}\big\rangle.
\end{aligned}$$

Let $w_t=tv+(1-t)x^*$ for all $t\in(0,1]$ and $v\in C$. This implies that $w_t\in C$. Then we have

$$\begin{aligned}
&\big\langle w_t-\Delta_{t_n}^i z_{t_n},B_iw_t\big\rangle\\
&\quad\ge\varphi_i\big(\Delta_{t_n}^i z_{t_n}\big)-\varphi_i(w_t)+\big\langle w_t-\Delta_{t_n}^i z_{t_n},B_iw_t\big\rangle-\big\langle w_t-\Delta_{t_n}^i z_{t_n},B_i\Delta_{t_n}^{i-1}z_{t_n}\big\rangle\\
&\qquad-\Big\langle w_t-\Delta_{t_n}^i z_{t_n},\frac{\Delta_{t_n}^i z_{t_n}-\Delta_{t_n}^{i-1}z_{t_n}}{r_{i,t_n}}\Big\rangle+\Theta_i\big(w_t,\Delta_{t_n}^i z_{t_n}\big)\\
&\quad=\varphi_i\big(\Delta_{t_n}^i z_{t_n}\big)-\varphi_i(w_t)+\big\langle w_t-\Delta_{t_n}^i z_{t_n},B_iw_t-B_i\Delta_{t_n}^i z_{t_n}\big\rangle\\
&\qquad+\big\langle w_t-\Delta_{t_n}^i z_{t_n},B_j\Delta_{t_n}^i z_{t_n}-B_i\Delta_{t_n}^{i-1}z_{t_n}\big\rangle\\
&\qquad-\Big\langle w_t-\Delta_{t_n}^i z_{t_n},\frac{\Delta_{t_n}^i z_{t_n}-\Delta_{t_n}^{i-1}z_{t_n}}{r_{i,t_n}}\Big\rangle+\Theta_i\big(w_t,\Delta_{t_n}^i z_{t_n}\big).
\end{aligned}$$

By the same arguments as in the proof of Theorem 3.1, we have $\|B_i\Delta^i_{t_n}z_{t_n} - B_i\Delta^{i-1}_{t_n}z_{t_n}\| \to 0$ as $n \to \infty$. In the meantime, by the monotonicity of B_i, we obtain $\langle w_t - \Delta^i_{t_n}z_{t_n}, B_iw_t - B_i\Delta^i_{t_n}z_{t_n}\rangle \geq 0$. Then by (A4) we get

$$\langle w_t - x^*, B_iw_t\rangle \geq \varphi_i(x^*) - \varphi_i(w_t) + \Theta_i(w_t, x^*).$$

Utilizing (A1), (A4), and the last inequality, we obtain

$$\begin{aligned} 0 &= \Theta_i(w_t, w_t) + \varphi_i(w_t) - \varphi_i(w_t) \\ &\leq t\Theta_i(w_t, v) + (1-t)\Theta_i(w_t, x^*) + t\varphi_i(v) + (1-t)\varphi_i(x^*) - \varphi_i(w_t) \\ &\leq t\big[\Theta_i(w_t, v) + \varphi_i(v) - \varphi_i(w_t)\big] + (1-t)\langle w_t - x^*, B_iw_t\rangle \\ &= t\big[\Theta_i(w_t, v) + \varphi_i(v) - \varphi_i(w_t)\big] + (1-t)t\langle v - x^*, B_iw_t\rangle, \end{aligned}$$

and hence

$$0 \leq \Theta_i(w_t, v) + \varphi_i(v) - \varphi_i(w_t) + (1-t)\langle v - x^*, B_iw_t\rangle.$$

Letting $t \to 0$, we have, for each $v \in C$,

$$0 \leq \Theta_i(x^*, v) + \varphi_i(v) - \varphi_i(x^*) + \langle v - x^*, B_ix^*\rangle.$$

This implies that $x^* \in \mathrm{GMEP}(\Theta_i, \varphi_i, B_i)$ and hence $x^* \in \bigcap_{i=1}^N \mathrm{GMEP}(\Theta_i, \varphi_i, B_i)$. This together with Steps 2 and 3 attains $x^* \in \Omega$.

Finally, we show that x^* is a solution of VI (3.2). In fact, putting x_{t_n} in place of x_t in (3.26) and taking the limit as $t_n \to 0$, we obtain

$$\|x^* - p\|^2 \leq \frac{1}{\bar{\gamma} - 1}\langle (I-A)p, x^* - p\rangle, \quad \forall p \in \Omega.$$

In particular, x^* solves the following VI:

$$x^* \in \Omega, \quad \langle (A-I)p, x^* - p\rangle \leq 0, \quad \forall p \in \Omega,$$

or the equivalent dual variational inequality

$$x^* \in \Omega, \quad \langle (A-I)x^*, x^* - p\rangle \leq 0, \quad \forall p \in \Omega.$$

That is, $x^* \in \Omega$ is a solution of VI (3.2). Hence $x^* = \widetilde{x}$ by uniqueness. In a summary, we have proven that each cluster point of $\{x_t\}$ (as $t \to 0$) equals $\widetilde{x}$. Therefore $x_t \to \widetilde{x}$ as $t \to 0$. VI (3.2) can be rewritten as

$$\langle (2I - A)\widetilde{x} - \widetilde{x}, \widetilde{x} - p\rangle \geq 0, \quad \forall p \in \Omega.$$

So, in terms of (2.1), this is equivalent to the fixed point equation

$$P_\Omega(2I - A)\widetilde{x} = \widetilde{x}.$$

This completes the proof. □

Taking $T \equiv I$, $G \equiv I$, $\mu = 1$, and $\gamma = 1$ in Theorem 3.2, we have the following corollary.

Corollary 3.1 *Let* $\{x_t\}$ *be defined by*

$$x_t = P_C\big[(I - \theta_t A)\Delta_t^N R_t x_t + \theta_t\big(tVx_t + (1-t)\Delta_t^N R_t x_t\big)\big].$$

If $\lim_{t\to 0}\theta_t = 0$, *then* x_t *converges strongly as* $t \to 0$ *to* $\widetilde{x} \in \Omega := \bigcap_{i=1}^N \mathrm{GMEP}(\Theta_i, \varphi_i, B_i) \cap \mathrm{GSVI}(C, B_1, B_2)$, *which is the unique solution of the VI*

$$\big\langle (A - I)\widetilde{x}, \widetilde{x} - p\big\rangle \le 0, \quad \forall p \in \Omega. \tag{3.38}$$

Proof If $T \equiv I$, then T_r in Lemma 2.8 is the identity mapping. Thus the result follows from Theorem 3.2. □

We are now in a position to prove the strong convergence of the sequence $\{x_n\}$ generated by the general explicit iterative scheme (3.3) to $\widetilde{x} \in \Omega$, which is the unique solution to VI (3.2).

Theorem 3.3 *Let* $\{x_n\}$ *be the sequence generated by the explicit algorithm* (3.3). *Let* $\{\alpha_n\}$, $\{\beta_n\}$, $\{r_n\}$, $\{\lambda_n\}$, $\{\nu_n\}$, *and* $\{r_{i,n}\}_{i=1}^N$ *satisfy the following conditions*:

(C1) $\{\alpha_n\} \subset [0,1]$ *and* $\{\beta_n\} \subset (0,1]$, $\alpha_n \to 0$ *and* $\beta_n \to 0$ *as* $n \to \infty$;

(C2) $\sum_{n=0}^\infty \beta_n = \infty$;

(C3) $\sum_{n=0}^\infty |\alpha_{n+1} - \alpha_n| < \infty$, *and* $|\beta_{n+1} - \beta_n| \le o(\beta_{n+1}) + \sigma_n$, $\sum_{n=0}^\infty \sigma_n < \infty$ (*the perturbed control condition*);

(C4) $\{r_n\} \subset (0,\infty)$, $\liminf_{n\to\infty} r_n > 0$, *and* $\sum_{n=0}^\infty |r_{n+1} - r_n| < \infty$;

(C5) $\{\lambda_n\} \subset (0,\infty)$, $\lim_{n\to\infty}\lambda_n = \lambda > 0$, *and* $\sum_{n=0}^\infty |\lambda_{n+1} - \lambda_n| < \infty$;

(C6) $\{\nu_n\} \subset (0,\infty)$, $\lim_{n\to\infty}\nu_n = \nu > 0$, *and* $\sum_{n=0}^\infty |\nu_{n+1} - \nu_n| < \infty$;

(C7) $\{r_{i,n}\} \subset [c_i, d_i] \subset (0, 2\mu_i)$ $\forall i \in \{1,2,\dots,N\}$, *and* $\sum_{n=0}^\infty (\sum_{i=1}^N |r_{i,n+1} - r_{i,n}|) < \infty$.

Then $\{x_n\}$ *converges strongly to* $\widetilde{x} \in \Omega := \bigcap_{i=1}^N \mathrm{GMEP}(\Theta_i, \varphi_i, B_i) \cap \mathrm{GSVI}(C, F_1, F_2) \cap \mathrm{Fix}(T)$, *which is the unique solution of VI* (3.2).

Proof First, note that from condition (C1), without loss of generality, we assume that $\alpha_n\tau < 1$, $\beta_n\bar{\gamma} < 1$ and $\frac{2\beta_n(\bar{\gamma}-1)}{1-\beta_n} < 1$ for all $n \ge 0$. Let $\widetilde{x} \in \Omega$ be the unique solution of VI (3.2). (The existence of $\widetilde{x}$ follows from Theorem 3.2.)

From now, we put $z_n = R_n x_n$, $u_n = \Delta_n^N z_n$, and $v_n = T_{r_n} u_n$. Take $p \in \Omega$. Then $p = T_{r_n} p$ by Lemma 2.8(iii), $p = \Delta_n^i p$ $(= T_{r_{i,n}}^{(\Theta_i,\varphi_i)}(I - r_{i,n}B_i)p)$ by Proposition 2.1(iii), and $p = R_n p$ by Lemma 1.1.

We divide the proof into several steps as follows.

Step 1. We show that $\{x_n\}$ is bounded. Indeed, utilizing Proposition 2.1(ii) and Proposition 2.2, we have

$$\begin{aligned} \|u_n - p\| &= \big\| T_{r_{N,n}}^{(\Theta_N,\varphi_N)}(I - r_{N,n}B_N)\Delta_n^{N-1} z_n - T_{r_{N,n}}^{(\Theta_N,\varphi_N)}(I - r_{N,n}B_N)\Delta_n^{N-1} p\big\| \\ &\le \big\|(I - r_{N,n}B_N)\Delta_n^{N-1} z_n - (I - r_{N,n}B_N)\Delta_n^{N-1} p\big\| \\ &\le \big\|\Delta_n^{N-1} z_n - \Delta_n^{N-1} p\big\| \\ &\le \cdots \\ &\le \big\|\Delta_n^0 z_n - \Delta_n^0 p\big\| = \|z_n - p\|. \end{aligned} \tag{3.39}$$

It is easy from the nonexpansion of R_n to see that

$$\|z_n - p\| = \|R_n x_n - R_n p\| \le \|x_n - p\|,$$

which together with the nonexpansion of T_{r_n} and (3.39) implies that

$$\|v_n - p\| = \|T_{r_n} u_n - T_{r_n} p\| \le \|u_n - p\| \le \|z_n - p\| \le \|x_n - p\|. \tag{3.40}$$

From (3.3) and (3.40), we get

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&\quad \le \bigl\|(I - \beta_n A)v_n + \beta_n\bigl(\alpha_n \gamma V x_n + (I - \alpha_n \mu G)v_n\bigr) - p\bigr\| \\
&\quad = \bigl\|(I - \beta_n A)v_n - (I - \beta_n A)p + \beta_n\bigl(\alpha_n \gamma V x_n + (I - \alpha_n \mu G)v_n - p\bigr) + \beta_n (I - A)p\bigr\| \\
&\quad \le \bigl\|(I - \beta_n A)v_n - (I - \beta_n A)p\bigr\| + \beta_n \bigl\|\alpha_n \gamma V x_n + (I - \alpha_n \mu G)v_n - p\bigr\| + \beta_n \bigl\|(I - A)p\bigr\| \\
&\quad = \bigl\|(I - \beta_n A)v_n - (I - \beta_n A)p\bigr\| \\
&\qquad + \beta_n \bigl\|(I - \alpha_n \mu G)v_n - (I - \alpha_n \mu G)p + \alpha_n(\gamma V x_n - \mu G p)\bigr\| + \beta_n \bigl\|(I - A)p\bigr\| \\
&\quad \le (1 - \beta_n \bar{\gamma})\|v_n - p\| + \beta_n \bigl[\bigl\|(I - \alpha_n \mu G)v_n - (I - \alpha_n \mu G)p\bigr\| \\
&\qquad + \alpha_n\bigl(\gamma \|V x_n - V p\| + \|\gamma V p - \mu G p\|\bigr)\bigr] + \beta_n \bigl\|(I - A)p\bigr\| \\
&\quad \le (1 - \beta_n \bar{\gamma})\|x_n - p\| + \beta_n \bigl[(1 - \alpha_n \tau)\|x_n - p\| + \alpha_n\bigl(\gamma l \|x_n - p\| + \bigl\|(\gamma V - \mu G)p\bigr\|\bigr)\bigr] \\
&\qquad + \beta_n \|I - A\| \|p\| \\
&\quad = \bigl[1 - \beta_n\bigl(\bar{\gamma} - 1 + \alpha_n(\tau - \gamma l)\bigr)\bigr]\|x_n - p\| + \beta_n \bigl[\|I - A\| \|p\| + \alpha_n \bigl\|(\gamma V - \mu G)p\bigr\|\bigr] \\
&\quad \le \bigl[1 - \beta_n(\bar{\gamma} - 1)\bigr]\|x_n - p\| + \beta_n \bigl[\|I - A\| \|p\| + \bigl\|(\gamma V - \mu G)p\bigr\|\bigr] \\
&\quad = \bigl[1 - \beta_n(\bar{\gamma} - 1)\bigr]\|x_n - p\| + \beta_n(\bar{\gamma} - 1)\frac{\|I - A\| \|p\| + \|(\gamma V - \mu G)p\|}{\bar{\gamma} - 1} \\
&\quad \le \max\left\{\|x_n - p\|, \frac{\|I - A\| \|p\| + \|(\gamma V - \mu G)p\|}{\bar{\gamma} - 1}\right\}.
\end{aligned}$$

By induction, we derive

$$\|x_n - p\| \le \max\left\{\|x_0 - p\|, \frac{\|I - A\| \|p\| + \|(\gamma V - \mu G)p\|}{\bar{\gamma} - 1}\right\}, \quad \forall n \ge 0.$$

This implies that $\{x_n\}$ is bounded and so are $\{Vx_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{z_n\}$, and $\{Gv_n\}$. As a consequence, with the control condition (C1), we get

$$\|x_{n+1} - v_n\| \le \beta_n \|w_n - A v_n\| \to 0 \quad (n \to \infty). \tag{3.41}$$

Step 2. We show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. To this end, let $y_n = F_{2,\nu_n} x_n$, $y_{n-1} = F_{2,\nu_{n-1}} x_{n-1}$, $z_n = F_{1,\lambda_n} y_n$, and $z_{n-1} = F_{1,\lambda_{n-1}} y_{n-1}$. Then we derive

$$\langle y - y_{n-1}, F_2 y_{n-1}\rangle + \frac{1}{\nu_{n-1}}\langle y - y_{n-1}, y_{n-1} - x_{n-1}\rangle \ge 0, \quad \forall y \in C, \tag{3.42}$$

and

$$\langle y-y_n, F_2 y_n\rangle + \frac{1}{\nu_n}\langle y-y_n, y_n-x_n\rangle \geq 0, \quad \forall y \in C. \tag{3.43}$$

Putting $y = y_n$ in (3.42) and $y = y_{n-1}$ in (3.43), we obtain

$$\langle y_n-y_{n-1}, F_2 y_{n-1}\rangle + \frac{1}{\nu_{n-1}}\langle y_n-y_{n-1}, y_{n-1}-x_{n-1}\rangle \geq 0 \tag{3.44}$$

and

$$\langle y_{n-1}-y_n, F_2 y_n\rangle + \frac{1}{\nu_n}\langle y_{n-1}-y_n, y_n-x_n\rangle \geq 0. \tag{3.45}$$

Adding up (3.44) and (3.45), we have

$$\langle y_n-y_{n-1}, F_2 y_{n-1}-F_2 y_n\rangle + \left\langle y_n-y_{n-1}, \frac{y_{n-1}-x_{n-1}}{\nu_{n-1}} - \frac{y_n-x_n}{\nu_n}\right\rangle \geq 0,$$

which together with the monotonicity of F_2 implies that

$$\left\langle y_n-y_{n-1}, \frac{y_{n-1}-x_{n-1}}{\nu_{n-1}} - \frac{y_n-x_n}{\nu_n}\right\rangle \geq 0,$$

and hence

$$\left\langle y_n-y_{n-1}, y_{n-1}-y_n+y_n-x_{n-1} - \frac{\nu_{n-1}}{\nu_n}(y_n-x_n)\right\rangle \geq 0.$$

It follows that

$$\begin{aligned}
\|y_n-y_{n-1}\|^2 &\leq \left\langle y_n-y_{n-1}, x_n-x_{n-1} + \left(1-\frac{\nu_{n-1}}{\nu_n}\right)(y_n-x_n)\right\rangle \\
&\leq \|y_n-y_{n-1}\|\left(\|x_n-x_{n-1}\| + \frac{1}{\nu_n}|\nu_n-\nu_{n-1}|\|y_n-x_n\|\right),
\end{aligned}$$

which immediately yields

$$\|y_n-y_{n-1}\| \leq \|x_n-x_{n-1}\| + \frac{1}{\nu_n}|\nu_n-\nu_{n-1}|\|y_n-x_n\|. \tag{3.46}$$

By using arguments similar to those of (3.46), we get

$$\|z_n-z_{n-1}\| \leq \|y_n-y_{n-1}\| + \frac{1}{\lambda_n}|\lambda_n-\lambda_{n-1}|\|z_n-y_n\|. \tag{3.47}$$

Substituting (3.46) for (3.47), we have

$$\begin{aligned}
\|z_n-z_{n-1}\| &\leq \|y_n-y_{n-1}\| + \frac{1}{\lambda_n}|\lambda_n-\lambda_{n-1}|\|z_n-y_n\| \\
&\leq \|x_n-x_{n-1}\| + \frac{1}{\nu_n}|\nu_n-\nu_{n-1}|\|y_n-x_n\| + \frac{1}{\lambda_n}|\lambda_n-\lambda_{n-1}|\|z_n-y_n\|.
\end{aligned} \tag{3.48}$$

Note that $v_n = T_{r_n}u_n$ and $v_{n-1} = T_{r_{n-1}}u_{n-1}$. By using arguments similar to those of (3.46), we obtain

$$\|v_n - v_{n-1}\| \le \|u_n - u_{n-1}\| + \frac{1}{r_n}|r_n - r_{n-1}|\|v_n - u_n\|. \tag{3.49}$$

Also, utilizing arguments similar to those of (3.25) in the proof of Theorem 3.1, we have

$$\begin{aligned}
\|u_n - u_{n-1}\| &= \bigl\|\Delta_n^N z_n - \Delta_{n-1}^N z_{n-1}\bigr\| \\
&\le |r_{N,n} - r_{N,n-1}|\biggl[\bigl\|B_N\Delta_n^{N-1}z_n\bigr\| + \frac{1}{r_{N,n}}\bigl\|T_{r_{N,n}}^{(\Theta_N,\varphi_N)}(I - r_{N,n}B_N)\Delta_n^{N-1}z_n \\
&\quad - (I - r_{N,n}B_N)\Delta_n^{N-1}z_n\bigr\|\biggr] + \cdots + |r_{1,n} - r_{1,n-1}|\biggl[\bigl\|B_1\Delta_n^0 z_n\bigr\| \\
&\quad + \frac{1}{r_{1,n}}\bigl\|T_{r_{1,n}}^{(\Theta_1,\varphi_1)}(I - r_{1,n}B_1)\Delta_n^0 z_n - (I - r_{1,n}B_1)\Delta_n^0 z_n\bigr\|\biggr] \\
&\quad + \bigl\|\Delta_n^0 z_n - \Delta_{n-1}^0 z_{n-1}\bigr\| \\
&\le \widetilde{M}_1\sum_{i=1}^N |r_{i,n} - r_{i,n-1}| + \|z_n - z_{n-1}\|,
\end{aligned} \tag{3.50}$$

where $\widetilde{M}_1 > 0$ is a constant such that, for each $n \ge 0$,

$$\sum_{i=1}^N\biggl[\bigl\|B_i\Delta_n^{i-1}z_n\bigr\| + \frac{1}{r_{i,n}}\bigl\|T_{r_{i,n}}^{(\Theta_i,\varphi_i)}(I - r_{i,n}B_i)\Delta_n^{i-1}z_n - (I - r_{i,n}B_i)\Delta_n^{i-1}z_n\bigr\|\biggr]\} \le \widetilde{M}_1.$$

So it follows from (3.48), (3.49), and (3.50) that

$$\begin{aligned}
\|v_n - v_{n-1}\| &\le \|u_n - u_{n-1}\| + \frac{1}{r_n}|r_n - r_{n-1}|\|v_n - u_n\| \\
&\le \widetilde{M}_1\sum_{i=1}^N |r_{i,n} - r_{i,n-1}| + \|z_n - z_{n-1}\| + \frac{1}{r_n}|r_n - r_{n-1}|\|v_n - u_n\| \\
&\le \widetilde{M}_1\sum_{i=1}^N |r_{i,n} - r_{i,n-1}| + \|x_n - x_{n-1}\| + \frac{1}{\nu_n}|\nu_n - \nu_{n-1}|\|y_n - x_n\| \\
&\quad + \frac{1}{\lambda_n}|\lambda_n - \lambda_{n-1}|\|z_n - y_n\| + \frac{1}{r_n}|r_n - r_{n-1}|\|v_n - u_n\|.
\end{aligned} \tag{3.51}$$

Since $\liminf_{n\to\infty} r_n > 0$, $\lim_{n\to\infty}\lambda_n = \lambda > 0$, and $\lim_{n\to\infty}\nu_n = \nu > 0$, it is easy to see from (3.51) that, for each $n \ge 0$,

$$\begin{aligned}
\|v_n - v_{n-1}\| &\le \|x_n - x_{n-1}\| + \widetilde{M}\Biggl[\sum_{i=1}^N |r_{i,n} - r_{i,n-1}| + |\nu_n - \nu_{n-1}| \\
&\quad + |\lambda_n - \lambda_{n-1}| + |r_n - r_{n-1}|\Biggr],
\end{aligned} \tag{3.52}$$

where $\widetilde{M} > 0$ is a constant such that

$$\sup_{n\geq 0}\left\{\widetilde{M}_1 + \frac{1}{\nu_n}\|y_n - x_n\| + \frac{1}{\lambda_n}\|z_n - y_n\| + \frac{1}{r_n}\|v_n - u_n\|\right\} \leq \widetilde{M}.$$

Now, simple calculations yield that

$$\begin{aligned} w_n - w_{n-1} &= \alpha_n\gamma Vx_n + (I - \alpha_n\mu G)v_n - \alpha_{n-1}\gamma Vx_{n-1} - (I - \alpha_{n-1}\mu G)v_{n-1} \\ &= (\alpha_n - \alpha_{n-1})(\gamma Vx_{n-1} - \mu Gv_{n-1}) + \alpha_n\gamma(Vx_n - Vx_{n-1}) \\ &\quad + (I - \alpha_n\mu G)v_n - (I - \alpha_n\mu G)v_{n-1}. \end{aligned}$$

In terms of (3.52) and Lemma 2.6, we obtain

$$\begin{aligned} \|w_n - w_{n-1}\| &\leq |\alpha_n - \alpha_{n-1}|\big(\gamma\|Vx_{n-1}\| + \mu\|Gv_{n-1}\|\big) + \alpha_n\gamma l\|x_n - x_{n-1}\| \\ &\quad + (1 - \tau\alpha_n)\|v_n - v_{n-1}\| \\ &\leq |\alpha_n - \alpha_{n-1}|\big(\gamma\|Vx_{n-1}\| + \mu\|Gv_{n-1}\|\big) + \alpha_n\gamma l\|x_n - x_{n-1}\| \\ &\quad + (1 - \tau\alpha_n)\|x_n - x_{n-1}\| + \widetilde{M}\Bigg[\sum_{i=1}^{N}|r_{i,n} - r_{i,n-1}| \\ &\quad + |\nu_n - \nu_{n-1}| + |\lambda_n - \lambda_{n-1}| + |r_n - r_{n-1}|\Bigg] \\ &= |\alpha_n - \alpha_{n-1}|\big(\gamma\|Vx_{n-1}\| + \mu\|Gv_{n-1}\|\big) + \big(1 - \alpha_n(\tau - \gamma l)\big)\|x_n - x_{n-1}\| \\ &\quad + \widetilde{M}\Bigg[\sum_{i=1}^{N}|r_{i,n} - r_{i,n-1}| + |\nu_n - \nu_{n-1}| + |\lambda_n - \lambda_{n-1}| + |r_n - r_{n-1}|\Bigg] \\ &\leq \|x_n - x_{n-1}\| + \widetilde{M}_2\Bigg[\sum_{i=1}^{N}|r_{i,n} - r_{i,n-1}| + |\alpha_n - \alpha_{n-1}| \\ &\quad + |\nu_n - \nu_{n-1}| + |\lambda_n - \lambda_{n-1}| + |r_n - r_{n-1}|\Bigg], \end{aligned} \tag{3.53}$$

where $\widetilde{M}_2 = \sup_{n\geq 0}\{\gamma\|Vx_n\| + \mu\|Gv_n\| + \widetilde{M}\}$. By (3.53) and Lemma 2.5, we derive

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \big\|(I - \beta_n A)v_n + \beta_n w_n - (I - \beta_{n-1}A)v_{n-1} - \beta_{n-1}w_{n-1}\big\| \\ &\leq \big\|(I - \beta_n A)(v_n - v_{n-1})\big\| + |\beta_n - \beta_{n-1}|\|A\|\|v_{n-1}\| \\ &\quad + \beta_n\|w_n - w_{n-1}\| + |\beta_n - \beta_{n-1}|\|w_{n-1}\| \\ &\leq (1 - \beta_n\bar{\gamma})\|v_n - v_{n-1}\| + \beta_n\Bigg[\|x_n - x_{n-1}\| + \widetilde{M}_2\Bigg(\sum_{i=1}^{N}|r_{i,n} - r_{i,n-1}| \\ &\quad + |\alpha_n - \alpha_{n-1}| + |\nu_n - \nu_{n-1}| + |\lambda_n - \lambda_{n-1}| + |r_n - r_{n-1}|\Bigg)\Bigg] + |\beta_n - \beta_{n-1}|\widetilde{M}_3 \\ &\leq (1 - \beta_n\bar{\gamma})\Bigg[\|x_n - x_{n-1}\| + \widetilde{M}\Bigg(\sum_{i=1}^{N}|r_{i,n} - r_{i,n-1}| + |\nu_n - \nu_{n-1}| + |\lambda_n - \lambda_{n-1}| \end{aligned}$$

$$\begin{aligned}
&+|r_n-r_{n-1}|\Bigg)\Bigg]+\beta_n\Bigg[\|x_n-x_{n-1}\|+\widetilde{M}_2\Bigg(\sum_{i=1}^{N}|r_{i,n}-r_{i,n-1}|+|\alpha_n-\alpha_{n-1}| \\
&+|\nu_n-\nu_{n-1}|+|\lambda_n-\lambda_{n-1}|+|r_n-r_{n-1}|\Bigg)\Bigg]+|\beta_n-\beta_{n-1}|\widetilde{M}_3 \\
\le{}&\big(1-\beta_n(\bar{\gamma}-1)\big)\|x_n-x_{n-1}\|+\big(1-\beta_n(\bar{\gamma}-1)\big)\widetilde{M}_2\Bigg(\sum_{i=1}^{N}|r_{i,n}-r_{i,n-1}| \\
&+|\alpha_n-\alpha_{n-1}|+|\nu_n-\nu_{n-1}|+|\lambda_n-\lambda_{n-1}|+|r_n-r_{n-1}|\Bigg)+|\beta_n-\beta_{n-1}|\widetilde{M}_3 \\
\le{}&\big(1-\beta_n(\bar{\gamma}-1)\big)\|x_n-x_{n-1}\|+\widetilde{M}_2\Bigg(\sum_{i=1}^{N}|r_{i,n}-r_{i,n-1}|+|\alpha_n-\alpha_{n-1}| \\
&+|\nu_n-\nu_{n-1}|+|\lambda_n-\lambda_{n-1}|+|r_n-r_{n-1}|\Bigg)+|\beta_n-\beta_{n-1}|\widetilde{M}_3 \\
\le{}&\big(1-\beta_n(\bar{\gamma}-1)\big)\|x_n-x_{n-1}\|+\widetilde{M}_2\Bigg(\sum_{i=1}^{N}|r_{i,n}-r_{i,n-1}|+|\alpha_n-\alpha_{n-1}| \\
&+|\nu_n-\nu_{n-1}|+|\lambda_n-\lambda_{n-1}|+|r_n-r_{n-1}|\Bigg)+\big(o(\beta_n)+\sigma_{n-1}\big)\widetilde{M}_3, \qquad (3.54)
\end{aligned}$$

where $\widetilde{M}_3=\sup_{n\ge 0}\{\|A\|\|\nu_n\|+\|w_n\|\}$. By taking $s_{n+1}=\|x_{n+1}-x_n\|$, $\omega_n=\beta_n(\bar{\gamma}-1)$, $\omega_n\delta_n=\widetilde{M}_3 o(\beta_n)$, and

$$\gamma_n=\sigma_{n-1}\widetilde{M}_3+\widetilde{M}_2\Bigg(\sum_{i=1}^{N}|r_{i,n}-r_{i,n-1}|+|\alpha_n-\alpha_{n-1}|+|\nu_n-\nu_{n-1}|+|\lambda_n-\lambda_{n-1}|+|r_n-r_{n-1}|\Bigg),$$

we deduce from (3.54) that

$$s_{n+1}\le(1-\omega_n)s_n+\omega_n\delta_n+\gamma_n.$$

Hence, by conditions (C2)–(C7) and Lemma 2.2, we obtain

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.$$

Step 3. We show that $\lim_{n\to\infty}\|x_{n+1}-w_n\|=0$. Indeed, from (3.41) and condition (C1), we derive

$$\begin{aligned}
\|x_{n+1}-w_n\|&\le\|x_{n+1}-\nu_n\|+\|\nu_n-w_n\| \\
&\le\beta_n\|w_n-A\nu_n\|+\alpha_n\|\gamma Vx_n-\mu G\nu_n\|\to 0 \quad (n\to\infty).
\end{aligned}$$

Step 4. We show that $\lim_{n\to\infty}\|x_n-w_n\|=0$. In fact, by Step 2 and Step 3, we get

$$\|x_n-w_n\|\le\|x_n-x_{n+1}\|+\|x_{n+1}-w_n\|\to 0 \quad (n\to\infty).$$

Step 5. We show that $\lim_{n\to\infty}\|x_n-z_n\|=0$ and $\lim_{n\to\infty}\|x_n-Rx_n\|=0$. In fact, we first derive $\lim_{n\to\infty}\|x_n-z_n\|=0$ by using arguments similar to those of (3.9) in the proof of

Theorem 3.1, and then we obtain $\lim_{n\to\infty}\|x_n - Rx_n\| = 0$ by using arguments similar to those of (3.37) in the proof of Theorem 3.2.

Step 6. We show that $\lim_{n\to\infty}\|z_n - u_n\| = 0$ and $\lim_{n\to\infty}\|x_n - \Delta_n^N x_n\| = 0$. In fact, by using arguments similar to those of (3.12) and (3.13) in the proof of Theorem 3.1, we obtain the desired conclusions.

Step 7. We show that $\lim_{n\to\infty}\|u_n - v_n\| = 0$ and $\lim_{n\to\infty}\|x_n - T_{r_n}x_n\| = 0$. In fact, by using arguments similar to those of (3.14) and (3.15) in the proof of Theorem 3.1, we obtain the desired conclusions.

Step 8. We show that $\limsup_{n\to\infty}\langle (I-A)\widetilde{x}, x_n - \widetilde{x}\rangle \le 0$. To this end, take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\bigl\langle (I-A)\widetilde{x}, x_n - \widetilde{x}\bigr\rangle = \lim_{k\to\infty}\bigl\langle (I-A)\widetilde{x}, x_{n_k} - \widetilde{x}\bigr\rangle.$$

Without loss of generality, we may assume that $x_{n_k} \rightharpoonup \hat{x}$. Utilizing Steps 5, 6, and 7 and arguments similar to those of Steps 2, 3, and 4 in the proof of Theorem 3.2, we derive $\hat{x} \in \Omega$. Thus, from VI (3.2), we conclude

$$\limsup_{n\to\infty}\bigl\langle (I-A)\widetilde{x}, x_n - \widetilde{x}\bigr\rangle = \lim_{k\to\infty}\bigl\langle (I-A)\widetilde{x}, x_{n_k} - \widetilde{x}\bigr\rangle = \bigl\langle (I-A)\widetilde{x}, \hat{x} - \widetilde{x}\bigr\rangle \le 0.$$

Step 9. We show that $\lim_{n\to\infty}\|x_n - \widetilde{x}\| = 0$. Note that $\widetilde{x} \in \Omega$. From (3.3), $\widetilde{x} = R_n\widetilde{x}$, $\widetilde{x} = \Delta_n^N\widetilde{x}$, and $\widetilde{x} = T_{r_n}\widetilde{x}$, we obtain

$$w_n - \widetilde{x} = (I - \alpha_n\mu G)v_n - (I - \alpha_n\mu G)\widetilde{x} + \alpha_n(\gamma V x_n - \mu G\widetilde{x})$$

and

$$\begin{aligned} x_{n+1} - \widetilde{x} &= x_{n+1} - (I - \beta_n A)v_n - \beta_n w_n + (I - \beta_n A)v_n + \beta_n w_n - \widetilde{x} \\ &= x_{n+1} - (I - \beta_n A)v_n - \beta_n w_n + (I - \beta_n A)(v_n - \widetilde{x}) + \beta_n(w_n - \widetilde{x}) + \beta_n(I - A)\widetilde{x}. \end{aligned}$$

Applying (2.1), (3.40) and Lemmas 2.1, 2.5, and 2.6, we deduce that

$$\begin{aligned} \|w_n - \widetilde{x}\|^2 &= \bigl\|(I - \alpha_n\mu G)v_n - (I - \alpha_n\mu G)\widetilde{x} + \alpha_n(\gamma V x_n - \mu G\widetilde{x})\bigr\|^2 \\ &\le \bigl\|(I - \alpha_n\mu G)v_n - (I - \alpha_n\mu G)\widetilde{x}\bigr\|^2 + 2\alpha_n\langle \gamma V x_n - \mu G\widetilde{x}, w_n - \widetilde{x}\rangle \\ &\le (1 - \alpha_n\tau)^2\|v_n - \widetilde{x}\|^2 + 2\alpha_n\|\gamma V x_n - \mu G\widetilde{x}\|\|w_n - \widetilde{x}\| \\ &\le \|x_n - \widetilde{x}\|^2 + 2\alpha_n\|\gamma V x_n - \mu G\widetilde{x}\|\|w_n - \widetilde{x}\|, \end{aligned}$$

and hence

$$\begin{aligned} &\|x_{n+1} - \widetilde{x}\|^2 \\ &\quad = \bigl\|(I - \beta_n A)(v_n - \widetilde{x}) + \beta_n(w_n - \widetilde{x}) + \beta_n(I - A)\widetilde{x} + x_{n+1} - (I - \beta_n A)v_n - \beta_n w_n\bigr\|^2 \\ &\quad \le \bigl\|(I - \beta_n A)(v_n - \widetilde{x})\bigr\|^2 + 2\beta_n\langle w_n - \widetilde{x}, x_{n+1} - \widetilde{x}\rangle \\ &\qquad + 2\beta_n\bigl\langle (I - A)\widetilde{x}, x_{n+1} - \widetilde{x}\bigr\rangle + 2\bigl\langle x_{n+1} - (I - \beta_n A)v_n - \beta_n w_n, x_{n+1} - \widetilde{x}\bigr\rangle \end{aligned}$$

$$\begin{aligned}
&\le \big\|(I-\beta_nA)(v_n-\widetilde{x})\big\|^2 + 2\beta_n\langle w_n-\widetilde{x}, x_{n+1}-\widetilde{x}\rangle + 2\beta_n\big\langle (I-A)\widetilde{x}, x_{n+1}-\widetilde{x}\big\rangle \\
&\le (1-\beta_n\bar{\gamma})^2\|v_n-\widetilde{x}\|^2 + 2\beta_n\|w_n-\widetilde{x}\|\|x_{n+1}-\widetilde{x}\| + 2\beta_n\big\langle (I-A)\widetilde{x}, x_{n+1}-\widetilde{x}\big\rangle \\
&\le (1-\beta_n\bar{\gamma})^2\|x_n-\widetilde{x}\|^2 + \beta_n\big(\|w_n-\widetilde{x}\|^2 + \|x_{n+1}-\widetilde{x}\|^2\big) + 2\beta_n\big\langle (I-A)\widetilde{x}, x_{n+1}-\widetilde{x}\big\rangle \\
&\le (1-\beta_n\bar{\gamma})^2\|x_n-\widetilde{x}\|^2 + \beta_n\big[\|x_n-\widetilde{x}\|^2 + 2\alpha_n\|\gamma Vx_n-\mu G\widetilde{x}\|\|w_n-\widetilde{x}\|\big] \\
&\quad + \beta_n\|x_{n+1}-\widetilde{x}\|^2 + 2\beta_n\big\langle (I-A)\widetilde{x}, x_{n+1}-\widetilde{x}\big\rangle \\
&= \big[(1-\beta_n\bar{\gamma})^2+\beta_n\big]\|x_n-\widetilde{x}\|^2 + 2\alpha_n\beta_n\|\gamma Vx_n-\mu G\widetilde{x}\|\|w_n-\widetilde{x}\| + \beta_n\|x_{n+1}-\widetilde{x}\|^2 \\
&\quad + 2\beta_n\big\langle (I-A)\widetilde{x}, x_{n+1}-\widetilde{x}\big\rangle.
\end{aligned} \tag{3.55}$$

It then follows from (3.55) that

$$\begin{aligned}
&\|x_{n+1}-\widetilde{x}\|^2 \\
&\quad\le \frac{(1-\beta_n\bar{\gamma})^2+\beta_n}{1-\beta_n}\|x_n-\widetilde{x}\|^2 + \frac{\beta_n}{1-\beta_n}\big[2\alpha_n\|\gamma Vx_n-\mu G\widetilde{x}\|\|w_n-\widetilde{x}\| \\
&\quad\quad + 2\big\langle (I-A)\widetilde{x}, x_{n+1}-\widetilde{x}\big\rangle\big] \\
&\quad= \left(1-\frac{2\beta_n(\bar{\gamma}-1)}{1-\beta_n}\right)\|x_n-\widetilde{x}\|^2 + \frac{2\beta_n(\bar{\gamma}-1)}{1-\beta_n}\cdot\frac{1}{2(\bar{\gamma}-1)}\big[2\alpha_n\|\gamma Vx_n-\mu G\widetilde{x}\|\|w_n-\widetilde{x}\| \\
&\quad\quad + \beta_n\bar{\gamma}^2\|x_n-\widetilde{x}\|^2 + 2\big\langle (I-A)\widetilde{x}, x_{n+1}-\widetilde{x}\big\rangle\big] \\
&\quad= (1-\xi_n)\|x_n-\widetilde{x}\|^2 + \xi_n\delta_n,
\end{aligned}$$

where $\xi_n = \frac{2\beta_n(\bar{\gamma}-1)}{1-\beta_n}$, $\delta_n = \frac{1}{2(\bar{\gamma}-1)}[2\alpha_n\|\gamma Vx_n - \mu G\widetilde{x}\|\|w_n-\widetilde{x}\| + \beta_n\bar{\gamma}^2\|x_n-\widetilde{x}\|^2 + 2\langle (I-A)\widetilde{x}, x_{n+1}-\widetilde{x}\rangle]$. It can be readily seen from Step 2 and conditions (C1) and (C2) that $\xi_n \to 0$, $\sum_{n=0}^{\infty}\xi_n = \infty$, and $\limsup_{n\to\infty}\delta_n \le 0$. By Lemma 2.2, we conclude that $\lim_{n\to\infty}\|x_n-\widetilde{x}\| = 0$. This completes the proof. □

Taking $T \equiv I$, $G \equiv I$, $\mu = 1$, and $\gamma = 1$ in Theorem 3.3, we have the following corollary.

Corollary 3.2 *Let $\{x_n\}$ be generated by the following iterative algorithm*:

$$\begin{cases} w_n = \alpha_n Vx_n + (1-\alpha_n)\Delta_n^N R_n x_n, \\ x_{n+1} = P_C[(I-\beta_nA)\Delta_n^N R_n x_n + \beta_n w_n], \quad \forall n \ge 1. \end{cases}$$

Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\lambda_n\}$, $\{\nu_n\}$, and $\{r_{i,n}\}_{i=1}^N$ satisfy conditions (C1)–(C3) *and* (C5)–(C7) *in Theorem* 3.3. *Then $\{x_n\}$ converges strongly to $\widetilde{x} \in \Omega := \bigcap_{i=1}^N \mathrm{GMEP}(\Theta_i, \varphi_i, B_i) \cap \mathrm{GSVI}(C, F_1, F_2)$, which is the unique solution of VI* (3.38).

Remark 3.1 Compared with Proposition 3.3, Theorem 3.4, and Theorem 3.7 in [11], respectively, our Theorems 3.1, 3.2, and 3.3 improve and develop them in the following aspects:

(i) GSVI (1.3) with solutions being also fixed points of a continuous pseudocontinuous mapping in [12, Proposition 3.3, Theorem 3.4, and Theorem 3.7] is extended to GSVI (1.3) with solutions being also common solutions of a finite family of generalized mixed equilibrium problems (GMEPs) and fixed points of a continuous pseudocontinuous mapping in our Theorems 3.1, 3.2, and 3.3;

(ii) in the argument process of our Theorems 3.1, 3.2, and 3.3, we use the variable parameters λ_t and ν_t (resp., λ_n and ν_n) in place of the fixed parameters λ and ν in the proof of [12, Proposition 3.3, Theorem 3.4, and Theorem 3.7], and additionally deal with a pool of variable parameters $\{r_{i,t}\}_{i=1}^N$ (resp., $\{r_{i,n}\}_{i=1}^N$) involving a finite family of GMEPs;

(iii) the iterative schemes in our Theorems 3.1, 3.2, and 3.3 are more advantageous and more flexible than the iterative schemes in [12, Proposition 3.3, Theorem 3.4, and Theorem 3.7], because they can be applied to solving three problems (i.e., GSVI (1.3), a finite family of GMEPs, and the fixed point problem of a continuous pseudocontractive mapping) and involve much more parameter sequences;

(iv) it is worth emphasizing that our general implicit iterative scheme (3.1) is very different from Jung's composite implicit iterative scheme in [12], because the term "$T_{r_t}Rx_t$" in Jung's implicit scheme is replaced by the term "$T_{r_t}\Delta_t^N R_t x_t$" in our implicit scheme (3.1). Moreover, the term "$T_{r_n}Rx_n$" in Jung's explicit scheme is replaced by the term "$T_{r_n}\Delta_n^N R_n x_n$" in our explicit scheme (3.3).

4 Numerical examples

The purpose of this section is to give two examples and numerical results to illustrate the applicability, effectiveness, and stability of our algorithm.

Example 4.1 (Example of Theorem 3.3) Let $H = \mathbf{R}$ and $C = [0, 100]$. Let the inner product $\langle\cdot,\cdot\rangle : \mathbf{R}\times\mathbf{R}\to\mathbf{R}$ be defined by $\langle x,y\rangle = xy$. Let $N = 2$, $Vx = 2x$, $Gx = \frac{1}{2}x$, $Tx = x$, $B_1x = \frac{1}{2}x$, $B_2x = \frac{1}{3}x$, $F_1x = \frac{1}{2}x$, $F_2x = x$, $\Theta_1(x,y) = y^2 - x^2$, $\Theta_2(x,y) = -3x^2 + xy + 2y^2$, $\varphi_1 x = x^2$, $\varphi_2 x = 0$, and $Ax = \frac{3}{2}x$. Let $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{1}{3(n+1)}$, $r_n = 1$, $r_{1,n} = \frac{1}{2}$, $r_{2,n} = 1$, $\lambda_n = 1$, $\nu_n = \frac{1}{2}$, $\gamma = \frac{1}{8}$, $\mu = \frac{2}{3}$. It is easy to calculate that $T_{r_{1,n}}^{(\Theta_1,\varphi_1)}x = \frac{1}{3}x$, $T_{r_{2,n}}^{(\Theta_2,\varphi_2)}x = \frac{1}{6}x$, $T_{r_n}x = x$, $F_{1,\lambda_n}x = \frac{1}{2}x$, and $F_{2,\nu_n}x = \frac{1}{2}x$. Choose an arbitrary initial guess $x_1 = 4$. We get the numerical results of Algorithm (3.3).

Table 1 shows the value of the sequence $\{x_n\}$.

Figure 1 shows the convergence of the iterative sequence of Algorithm (3.3).

Solution: We can see from both Table 1 and Fig. 1 that the sequence $\{x_n\}$ converges to 0, that is, 0 is the solution in Example 4.1. In addition, it is also easy to check from Example 4.1 that $\bigcap_{i=1}^2 \mathrm{GMEP}(\Theta_i,\varphi_i,B_i)\cap \mathrm{GSVI}(C,F_1,F_2)\cap \mathrm{Fix}(T) = \{0\}$. Therefore, the iterative algorithm of Theorem 3.3 is efficient.

Example 4.2 (Example of Theorem 3.7 in [12]) Let $H = \mathbf{R}$ and $C = [0, 100]$. Let the inner product $\langle\cdot,\cdot\rangle : \mathbf{R}\times\mathbf{R}\to\mathbf{R}$ be defined by $\langle x,y\rangle = xy$. Let $Vx = 2x$, $Gx = \frac{1}{2}x$, $Tx = x$, $F_1x = \frac{1}{2}x$, $F_2x = x$, and $Ax = \frac{3}{2}x$. Let $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{1}{3(n+1)}$, $r_n = 1$, $\lambda = 1$, $\nu = \frac{1}{2}$, $\gamma = \frac{1}{8}$, $\mu = \frac{2}{3}$. Choose an arbitrary initial guess $x_1 = 4$. We get the numerical results of Algorithm (1.5) (Algorithm (3.10) of [12]).

Table 1 The values of x_n

n	x_n	n	x_n
1	4.0000	7	1.5285×10^{-11}
2	1.8261×10^{-1}	8	9.1892×10^{-14}
3	3.3191×10^{-3}	9	5.2325×10^{-16}
4	3.7633×10^{-5}	10	2.8636×10^{-18}
5	3.2426×10^{-7}	11	1.5212×10^{-20}
6	2.3546×10^{-9}	12	7.8994×10^{-23}

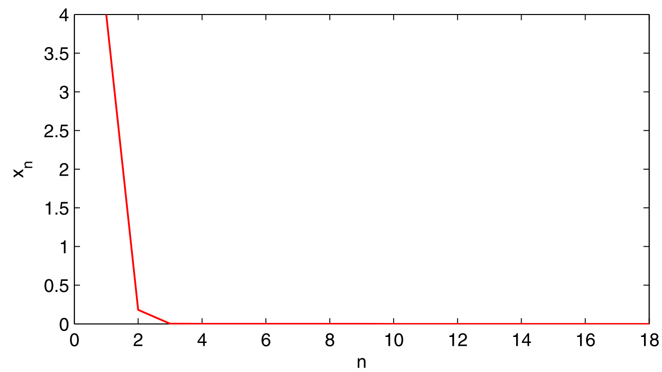

Figure 1 The convergence of $\{x_n\}$ with initial $x_1 = 4$

Table 2 The values of x_n

n	x_n	n	x_n
1	4.0000	7	9.0468×10^{-4}
2	1.0278	8	2.2236×10^{-4}
3	2.5219×10^{-1}	9	5.4731×10^{-5}
4	6.1587×10^{-2}	10	1.3488×10^{-5}
5	1.5055×10^{-2}	11	3.3278×10^{-6}
6	3.6870×10^{-3}	12	8.2181×10^{-7}

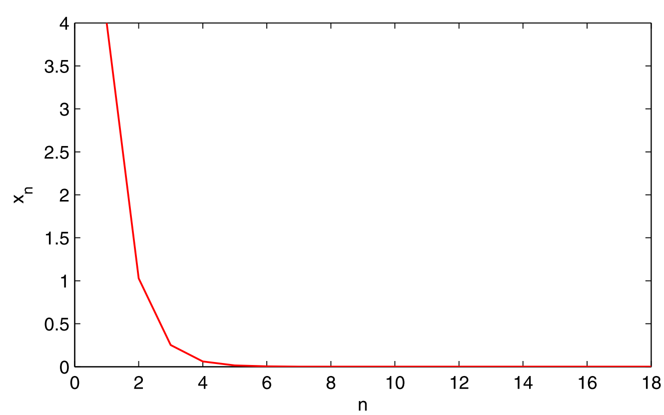

Figure 2 The convergence of $\{x_n\}$ with initial $x_1 = 4$

Table 2 shows the value of the sequence $\{x_n\}$.

The Fig. 2 shows the convergence of the iterative sequence of Algorithm (1.5).

Solution: We can see from both Table 2 and Fig. 2 that the sequence $\{x_n\}$ converges to 0, that is, 0 is the solution in Example 4.2. In addition, it is also easy to check from Example 4.2 that $\mathrm{GSVI}(C, F_1, F_2) \cap \mathrm{Fix}(T) = \{0\}$.

Remark 4.1 From Tables 1, 2 and Figs. 1, 2, it is readily seen that the convergence of $\{x_n\}$ to 0 in Example 4.1 is faster than the one of $\{x_n\}$ to 0 in Example 4.2. Therefore, our algorithm is more applicable, efficient, and stable than the algorithm in [12].

5 Application

In this section, applying our main result Theorem 3.3, we can prove strong convergence theorems for approximating the solution of the standard constrained convex optimization problem.

Let C be a closed convex subset of H. The standard constrained convex optimization problem is to find $x^* \in C$ such that

$$f(x^*) = \min_{x\in C} f(x), \tag{5.1}$$

where $f : C \to \mathbf{R}$ is a convex, Fréchet differentiable function. The set of the solutions of (5.1) is denoted by Φ_f.

Lemma 5.1 (Optimality condition, [25]) *A necessary condition of optimality for a point $x^* \in C$ to be a solution of the minimization problem (5.1) is that x^* solves the variational inequality*

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0 \tag{5.2}$$

for all $x \in C$. Equivalently, $x^* \in C$ solves the fixed point equation

$$x^* = P_C(I - \lambda \nabla f)x^*$$

for every $\lambda > 0$. If, in addition, f is convex, then the optimality condition (5.2) is also sufficient.

Theorem 5.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let f_i ($i = 1, 2, \ldots, N$): $C \to \mathbf{R}$ be a real-valued convex function with the gradient ∇f_i being $\frac{1}{L_{f_i}}$-inverse strongly monotone and continuous with $L_{f_i} > 0$. Let Θ_i, φ_i, A, V, G, F_1, F_2, R_n, F_{1,λ_n}, F_{2,ν_n}, T_{r_n}, and $T_{r_{i,n}}^{(\Theta_i,\varphi_i)}$ be defined as in Theorem 3.3. Given $x_1 \in C$ and let $\{x_n\}$ be the sequence generated by the following explicit algorithm*:

$$\begin{cases} w_n = \alpha_n \gamma V x_n + (I - \alpha_n \mu G) T_{r_n} \Lambda_n^N R_n x_n, \\ x_{n+1} = P_C[(I - \beta_n A) T_{r_n} \Lambda_n^N R_n x_n + \beta_n w_n], \quad \forall n \geq 1, \end{cases} \tag{5.3}$$

where $\Lambda_n^i = T_{r_{i,n}}^{(\Theta_i,\varphi_i)}(I - r_{i,n}\nabla f_i) T_{r_{i-1,n}}^{(\Theta_{i-1},\varphi_{i-1})}(I - r_{i-1,n}\nabla f_{i-1}) \cdots T_{r_{1,n}}^{(\Theta_1,\varphi_1)}(I - r_{1,n}\nabla f_1)$ and $\Lambda_n^0 = I$. Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{r_n\}$, $\{\lambda_n\}$, $\{\nu_n\}$, and $\{r_{i,n}\}_{i=1}^N$ satisfy conditions (C1)–(C7) *in Theorem 3.3. Then $\{x_n\}$ converges strongly to $\widetilde{x} \in \Omega := \bigcap_{i=1}^N \mathrm{MEP}(\Theta_i, \varphi_i) \cap \bigcap_{i=1}^N \Phi_{f_i} \cap \mathrm{GSVI}(C, F_1, F_2) \cap \mathrm{Fix}(T)$, which is the unique solution of VI* (3.2).

Proof By using Lemma 5.1 and Theorem 3.3, we obtain the desired conclusion directly. □

6 Conclusions

We introduced and analyzed one general implicit iterative scheme and another general explicit iterative scheme for finding a solution of a general system of variational inequalities (GSVI) with the constraints of finitely many generalized mixed equilibrium problems and a fixed point problem of a continuous pseudocontractive mapping in a Hilbert space.

Moreover, we established strong convergence of the proposed implicit and explicit iterative schemes to a solution of the GSVI, which is the unique solution of a certain variational inequality. Our Theorems 3.1–3.3 not only improve and develop the main results of [1] and [12] but also improve and develop Theorems 3.1 and 3.2 of [9], Theorems 3.1 and 3.2 of [10], and Proposition 3.1, Theorems 3.2 and 3.5 of [11].

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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Author details
[1]Department of Mathematics, Shanghai Normal University, Shanghai, China. [2]LAMPS and Department of Mathematics and Statistics, York University, Toronto, Canada.

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