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Globally proper efficiency of set-valued optimization and vector variational inequality involving the generalized contingent epiderivative

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Abstract

In this paper, firstly, a new property of the cone subpreinvex set-valued map involving the generalized contingent epiderivative is obtained. As an application of this property, a sufficient optimality condition for constrained set-valued optimization problem in the sense of globally proper efficiency is derived. Finally, we establish the relations between the globally proper efficiency of the set-valued optimization problem and the globally proper efficiency of the vector variational inequality.

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1 Introduction

It is well known that convexity plays a crucial role in set-valued optimization. To generalize convexity of set-valued maps, some scholars introduced different kinds of generalized convex set-valued maps. Bhatia and Mehra [1] introduced the cone preinvex set-valued function which is a generalization of the cone convex set-valued map and derived a Lagrangian type duality for a fractional programming problem involving the cone preinvex set-valued map. Jia [2] defined the cone subpreinvex set-valued map and discussed the optimality condition and duality of set-valued optimization problems. Qiu [3] proposed the generalized cone preinvex set-valued map and used the cone-directed contingent derivative given by [4] to obtain necessary and sufficient optimality conditions of set-valued optimization problems in the sense of weak efficiency and strong efficiency, respectively. Zhou et al. [5], in real normed spaces, introduced a new concept of generalized cone convex set-valued map and established optimality conditions of set-valued optimization problem in the sense of Henig proper efficiency by applying the contingent epiderivative proposed by [6] and the generalized cone convexity of set-valued maps. Other generalized convexity of set-valued maps and their applications can be found in [7–10] and references therein.

On the other hand, Giannessi [11] initially introduced the notion of vector variational inequality in the finite dimensional Euclidean space. Since then, different types of extensions and generalizations of variational inequalities have been proposed in different settings, see

[6, 12–27] and references therein. In [6], optimality conditions are given by a certain kind of vector variational inequality in real normed spaces. We notice that such vector variational inequality is characterized by the contingent epiderivative of a set-valued map. Later, by virtue of the contingent epiderivative of set-valued maps, Liu and Gong [13] investigated the relations between the proper efficiency of set-valued optimization problem (for short, (SVOP)) and the proper efficiency of vector variational inequality based on the convexity assumption. By considering the generalized cone preinvexity and the contingent epiderivative of set-valued maps, Yu [20] disclosed the relations between Henig global efficiency of (SVOP) and Henig global efficiency of a kind of vector variational inequality. Yu and Kong [21] discussed the relations between a weakly approximate minimizer of a cone subinvex set-valued optimization problem and a weakly approximate solution of a kind of vector variational inequality characterized by the contingent epiderivative. We remark that the contingent epiderivative and the cone convexity (or the generalized cone convexity) of set-valued maps are used to deal with the relations between proper efficiency of (SVOP) and proper efficiency of vector variational inequality. However, it is worth noting that Chen and Jahn [12] pointed out that the existence of the contingent epiderivative of a set-valued map in a general setting is still an open question. To overcome this difficulty, Chen and Jahn [12] introduced a generalized contingent epiderivative of set-valued maps and derived the existence of the generalized contingent epiderivative under some standard assumptions. Therefore, it is interesting to study the optimality conditions of (SVOP) in the sense of globally proper efficiency via the generalized contingent epiderivative.

Motivated by the works [2, 5, 6, 13, 20, 21, 28], in this paper, we first propose a property of the cone subpreinvex set-valued map involving the generalized contingent epiderivative of set-valued maps which improves and generalizes the corresponding results in [5, 12]. A example is presented to illustrate this property. Second, we introduce a new type of generalized vector variational inequality problem (shortly, (GVVIP)) by virtue of the generalized contingent epiderivative of set-valued maps, propose the notion of globally proper efficiency of (GVVIP) and obtain the optimality conditions of the globally proper efficiency of (GVVIP).

This paper is organized as follows. In Sect. 2, we recall some definitions including the generalized contingent epiderivative of set-valued maps, the globally proper efficient point of a set and some generalized cone convexity of set-valued maps. In Sect. 3, a new property of the cone subpreinvex set-valued map is obtained and a sufficient optimality condition of constrained set-valued optimization problem in the sense of globally proper efficiency is derived. In Sect. 4, the relations between the globally proper efficiency of (SVOP) and the globally proper efficiency of (GVVIP) are disclosed.

2 Preliminaries

Throughout the paper, $\mathbb{R}^m_+$ represents the nonnegative orthant of $\mathbb{R}^m$ and $\mathbb{R}_+ := \mathbb{R}^1_+$, where $\mathbb{R}^m$ represents the m-dimensional Euclidean space. Let X be a linear space, Y and Z be two real normed spaces, respectively. For a set $K \subset Y$, $\operatorname{int} K$ denotes the interior of K. Let 0 denote the zero element for every space.

Let $C \subset Y$ and $D \subset Z$ be two closed pointed convex cones with $\operatorname{int} C \neq \emptyset$ and $\operatorname{int} D \neq \emptyset$. The cones C and D induce partial ordering of Y and Z, respectively. We denote by Y^* and Z^* the topological dual spaces of Y and Z, respectively. The topological dual cone C^+ of

C is defined as

$$C^+ = \{\mu \in Y^* | \mu(y) \geq 0, \forall y \in C\},$$

and the quasi-interior C^{+i} of C is defined as

$$C^{+i} = \{\mu \in Y^* | \mu(y) > 0, \forall y \in C \backslash \{0\}\},$$

where $\mu(y)$ denotes the value of the linear continuous functional μ at y. The meaning of D^+ is similar.

Let Γ be a nonempty subset in X and $F : \Gamma \rightrightarrows 2^Y$ be a set-valued map. Let

$$F(\Gamma) := \bigcup_{x\in\Gamma} F(x), \qquad \mu(F(x)) := \{\mu(y) | y \in F(x), x \in \Gamma\}, \qquad \mu(F(\Gamma)) := \bigcup_{x\in\Gamma} \mu(F(x)).$$

The graph and epigraph of F are, respectively, defined as

$$\mathrm{gr}(F) := \{(x, y) \in X \times Y | x \in \Gamma, y \in F(x)\}$$

and

$$\mathrm{epi}(F) := \{(x, y) \in X \times Y | x \in \Gamma, y \in F(x) + C\}.$$

Now, we recall some basic definitions which will be used in the sequel.

Definition 2.1 ([29]) Let Ω be a nonempty subset in X, $\bar{z} \in \mathrm{cl}\,\Omega$ and $h \in X$. The contingent cone of Ω at $\bar{z}$ is $T(\Omega, \bar{z}) := \{h \in X | \exists t_n \downarrow 0, \exists h_n \to h$, such that $\bar{z} + t_n h_n \in \Omega, \forall n \in \mathbb{N}\}$, or equivalently, $T(\Omega, \bar{z}) := \{h \in X | \exists \lambda_n \to +\infty, \exists z_n \in \Omega$, such that $\lim_{n\to\infty} z_n = \bar{z}$ and $\lim_{n\to\infty} \lambda_n(z_n - \bar{z}) = h\}$.

Definition 2.2 ([30, 31]) Let S be a nonempty subset in Y.

(i) A point $\bar{y} \in S$ is called a minimal point of S iff $(\bar{y} - C) \cap S = \{\bar{y}\}$. The set of all minimal points of S with respect to C is denoted by $\mathrm{Min}\, S$.

(ii) A point $\bar{y} \in S$ is called a globally proper efficient point of S iff there exists a pointed convex cone $C' \subset Y$ with $C \backslash \{0\} \subset \mathrm{int}\, C'$ such that $(S - \bar{y}) \cap (-C' \backslash \{0\}) = \emptyset$. The set of all globally proper efficient points of S with respect to C is denoted by $\mathrm{GPE}(S, C)$.

Remark 2.1 Notice that Yu and Liu [32] presented an equivalent characterization of the globally proper efficient point: For a nonempty subset S in Y, a point $\bar{y} \in S$ is called a globally proper efficient point of S iff there exists a pointed convex cone $C' \subset Y$ with $C \backslash \{0\} \subset \mathrm{int}\, C'$ such that $(S - \bar{y}) \cap (-\mathrm{int}\, C') = \emptyset$.

Definition 2.3 ([30])

(a) The cone C is called Daniell iff any decreasing sequence in Y having a lower bound converges to its infimum.

(b) A subset A of Y is said to be minorized iff there exists $y \in Y$ such that $A \subset \{y\} + C$.

(c) The domination property holds for a subset A of Y iff $A \subset \mathrm{Min}\, A + C$.

Definition 2.4 ([12]) Let Γ be a nonempty subset in X, $F : \Gamma \rightrightarrows 2^Y$ be a set-valued map and $(\bar{x}, \bar{y}) \in \mathrm{gr}(F)$. A set-valued map $D_g F(\bar{x}, \bar{y}) : \Gamma \rightrightarrows 2^Y$, defined by

$$D_g F(\bar{x}, \bar{y})(x) := \mathrm{Min}\bigl\{y \in Y \mid (x, y) \in T\bigl(\mathrm{epi}(F), (\bar{x}, \bar{y})\bigr)\bigr\}, \quad \forall x \in \Gamma,$$

is called the generalized contingent epiderivative of F at $(\bar{x}, \bar{y})$.

Definition 2.5 ([33]) A subset $\Gamma \subset X$ is called an invex set with respect to $\eta : \Gamma \times \Gamma \to X$ iff for each $x, y \in \Gamma$ and $\lambda \in [0, 1]$, $y + \lambda\eta(x, y) \in \Gamma$.

Remark 2.2 Obviously, a convex set is an invex set by taking $\eta(x, y) = x - y$ in Definition 2.5. In general, the converse is not true (see Example 2.1 in [34]).

Definition 2.6 ([1]) Let Γ be a nonempty invex subset in X with respect to η. The set-valued map $F : \Gamma \rightrightarrows 2^Y$ is called C-preinvex on Γ with respect to η iff

$$\lambda F(x) + (1 - \lambda)F(y) \subset F\bigl(y + \lambda\eta(x, y)\bigr) + C, \quad \forall \lambda \in [0, 1], \forall x, y \in \Gamma.$$

Remark 2.3 It is clear that the C-convexity of the set-valued map F is the C-preinvexity of the set-valued map F. However, the converse is not necessarily true (see Example 2.1 in [1]).

Now, we will give the concept of cone subpreinvex set-valued maps, which will be needed in the sequel.

Definition 2.7 ([2]) Let Γ be a nonempty invex subset in X with respect to η. The set-valued map $F : \Gamma \rightrightarrows 2^Y$ is called C-subpreinvex on Γ with respect to η iff, $\exists \theta \in \mathrm{int}\, C$ such that, $\forall x, y \in \Gamma$, $\forall \lambda \in [0, 1]$, $\forall \varepsilon > 0$,

$$\varepsilon\theta + \lambda F(x) + (1 - \lambda)F(y) \subset F\bigl(y + \lambda\eta(x, y)\bigr) + C.$$

Remark 2.4 Clearly, the C-preinvex set-valued map on Γ is a C-subpreinvex set-valued map on Γ. However, Example 2.1 shows that the converse is not necessarily true. Therefore, the C-subpreinvex set-valued map is a proper generalization of the C-preinvex set-valued map.

Example 2.1 Let $X = Y = \mathbb{R}$, $C = \mathbb{R}_+$ and $\Gamma = \{x \in \mathbb{R} \mid x \in [-2, 2]\backslash\{0\}\}$. The set-valued map $F : \Gamma \rightrightarrows 2^Y$ and the map $\eta : \Gamma \times \Gamma \to X$ are, respectively, defined as follows:

$$F(x) = \begin{cases} [0, \frac{1}{2}), & \text{if } x = -2, \\ [0, |x|[, & \text{if } x \in]-2, 0[\cup]0, 1]\backslash\{\frac{5}{6}\}, \\ ]0, \frac{1}{2}[, & \text{if } x = \frac{5}{6}, \\ ]0, x - 1[, & \text{if } x \in]1, 2] \end{cases}$$

and

$$\eta(x, y) = \begin{cases} x - y, & \text{if } xy > 0, \\ -\frac{y}{2}, & \text{if } xy < 0. \end{cases}$$

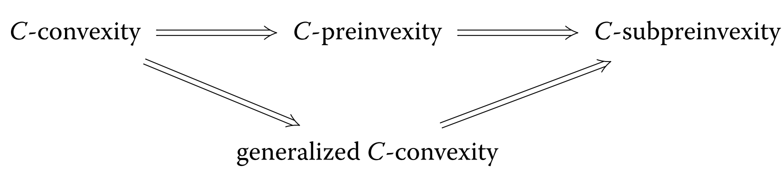
It is easy to check that Γ is an invex set with respect to η and there exists $\theta = 1 \in \operatorname{int}\mathbb{R}_+$, such that $\forall x, y \in \Gamma$, $\forall\lambda \in [0,1]$, $\forall\varepsilon > 0$,

$$\varepsilon\theta + \lambda F(x) + (1-\lambda)F(y) \subset F\bigl(y + \lambda\eta(x,y)\bigr) + C.$$

Therefore, F is C-subpreinvex on Γ with respect to η. On the other hand, if we take $\hat{x} = -2$, $\hat{y} = 1$ and $\hat{\lambda} = \frac{1}{3}$, then $0 \in [0, \frac{5}{6}[= \hat{\lambda}F(\hat{x}) + (1-\hat{\lambda})F(\hat{y})$. However, $0 \notin]0, \frac{1}{2}[+ C = F(\hat{y} + \hat{\lambda}\eta(\hat{x},\hat{y})) + C$. Thus, F is not C-preinvex on Γ with respect to η.

Remark 2.5 Zhou et al. [5] introduced the concept of generalized C-convex set-valued functions (see Defintion 2.4 in [5]) and pointed out that the generalized C-convex set-valued map is a proper generalization of the C-convex set-valued map (see Remark 2.1 in [5]). Clearly, if $\eta(x,y) = x - y$, that is, the invex set Γ is a convex set, then Definition 2.7 reduces to Definition 2.4 in [5]. However, if the invex set Γ is not a convex set in X, then the C-subpreinvexity of set-valued maps cannot imply the generalized C-convexity of set-valued maps. Therefore, the set-valued map F in Example 2.1 is not the generalized C-convex on Γ.

A chain of inclusion relations can be established now:



3 Optimality condition

In this section, firstly, we will use the generalized contingent epiderivative of set-valued maps to present a property of the C-subpreinvex set-valued map. Secondly, as an application of this property, we will give a sufficient optimality condition of constrained set-valued optimization problem in the sense of globally proper efficiency.

Theorem 3.1 *Let C be a closed pointed convex cone being Daniell, Γ be a nonempty invex subset in X with respect to η and the set-valued map $F : \Gamma \rightrightarrows 2^Y$ be C-subpreinvex on Γ with respect to η. Let $(\bar{x},\bar{y}) \in \operatorname{gr}(F)$. For any $x \in \Gamma$, write $\Phi(\eta(x,\bar{x})) := \{y \in Y | (\eta(x,\bar{x}), y) \in T(\operatorname{epi}(F), (\bar{x},\bar{y}))\}$. If $\Phi(\eta(x,\bar{x}))$ is minorized and fulfills the domination property for any $x \in \Gamma$, then*

$$F(x) - \{\bar{y}\} \subset D_g F(\bar{x},\bar{y})\bigl(\eta(x,\bar{x})\bigr) + C, \quad \forall x \in \Gamma.$$

Proof Take any $x \in \Gamma$ and $y \in F(x)$. Let $\{t_n\}$ be a sequence in $\mathbb{R}$ such that $t_n \in (0,1)$ with $\lim_{n\to\infty} t_n = 0$. Since F is a C-subpreinvex set-valued map on Γ with respect to η, there exists $\theta \in \operatorname{int}(C)$, for $x, \bar{x} \in \Gamma$, $y \in F(x)$ and $\bar{y} \in F(\bar{x})$,

$$\begin{aligned} \bar{y} + t_n(y - \bar{y} + t_n\theta) &= t_n^2\theta + (1-t_n)\bar{y} + t_n y \\ &\in t_n^2\theta + (1-t_n)F(\bar{x}) + t_n F(x) \\ &\subset F\bigl(\bar{x} + t_n\eta(x,\bar{x})\bigr) + C. \end{aligned} \tag{3.1}$$

Now, we define two sequences $\{x_n\}$ and $\{y_n\}$ as follows:

$$x_n := \bar{x} + t_n\eta(x,\bar{x}), \qquad y_n := \bar{y} + t_n(y - \bar{y} + t_n\theta). \tag{3.2}$$

By (3.1) and (3.2), we have $(x_n, y_n) \in \operatorname{epi}(F)$ and $\lim_{n\to\infty}(x_n, y_n) = (\bar{x}, \bar{y})$. It follows from the definitions of $\{x_n\}$ and $\{y_n\}$ that

$$\lim_{n\to\infty}\frac{1}{t_n}(x_n - \bar{x}) = \eta(x,\bar{x}) \quad \text{and} \quad \lim_{n\to\infty}\frac{1}{t_n}(y_n - \bar{y}) = y - \bar{y}.$$

Therefore,

$$\bigl(\eta(x,\bar{x}), y - \bar{y}\bigr) \in T\bigl(\operatorname{epi}(F), (\bar{x}, \bar{y})\bigr),$$

i.e.,

$$y - \bar{y} \in \Phi\bigl(\eta(x,\bar{x})\bigr) = \bigl\{y \,|\, \bigl(\eta(x,\bar{x}), y\bigr) \in T\bigl(\operatorname{epi}(F), (\bar{x}, \bar{y})\bigr)\bigr\}.$$

Since $\Phi(\eta(x,\bar{x}))$ satisfies the domination property, we obtain

$$\Phi\bigl(\eta(x,\bar{x})\bigr) \subset D_gF(\bar{x},\bar{y})\bigl(\eta(x,\bar{x})\bigr) + C.$$

Thus,

$$F(x) - \{\bar{y}\} \subset D_gF(\bar{x},\bar{y})\bigl(\eta(x,\bar{x})\bigr) + C, \quad \forall x \in \Gamma.$$

□

Remark 3.1 In Theorem 3.1, the conditions that C is Daniell and $\Phi(\eta(x,\bar{x}))$ is minorized guarantee the existence of $D_gF(\bar{x},\bar{y})(\eta(x,\bar{x}))$.

Remark 3.2 Notice that Theorem 3.1 generalizes and improves Lemma 1 in [12] and Theorem 3.1 in [5] in the following aspects: (i) The convex set and the C-convexity of Lemma 1 in [12] are extended to the invex set and the C-subpreinvexity of Theorem 3.1, respectively. (ii) The convex set, the C-convexity and the contingent epiderivative of Theorem 3.1 in [5] are replaced by the invex set, the C-subpreinvexity and the generalized contingent epiderivative of Theorem 3.1, respectively.

The following example is used to illustrate Theorem 3.1.

Example 3.1 Let $X = Y = \mathbb{R}$, $C = \mathbb{R}_+$ and $\Gamma = \{x \in \mathbb{R} | x \in [-2,2]\backslash\{0\}\}$. The set-valued map $F : \Gamma \rightrightarrows 2^Y$ and the map $\eta : \Gamma \times \Gamma \to X$ are, respectively, defined as follows:

$$F(x) = \begin{cases} [0,|x|[, & \text{if } x \in [-2,0[, \\ [0,x+1[, & \text{if } x \in]0,1], \\ [0,\frac{1}{2}[, & \text{if } x \in]1,2]\backslash\{\frac{3}{2}\}, \\ ]1,2[, & \text{if } x = \frac{3}{2} \end{cases}$$

and

$$\eta(x,y)=\begin{cases} x-y, & \text{if } x,y\in[-2,0[\text{ or } x,y\in]0,2],\\ -\frac{y}{3}, & \text{if otherwise.}\end{cases}$$

It is easy to check that Γ is an invex set with respect to η and F is C-subpreinvex on Γ with respect to η. Therefore, Lemma 1 in [12] and Theorem 3.1 in [5] cannot be applied. On the other hand, if we take $(\bar{x},\bar{y})=(\frac{1}{3},0)$, then $T(\text{epi}(F),(\bar{x},\bar{y}))=\mathbb{R}\times\mathbb{R}_+$. Moreover, for any $x\in\Gamma$, we have $\Phi(\eta(x,\bar{x}))=\mathbb{R}_+$. Clearly, $\Phi(\eta(x,\bar{x}))$ is minorized and fulfills the domination property. It follows from Definition 2.4 that $D_gF(\bar{x},\bar{y})(\eta(x,\bar{x}))=0$. Thus,

$$F(x)-\{\bar{y}\}\subset D_gF(\bar{x},\bar{y})\bigl(\eta(x,\bar{x})\bigr)+C,\quad \forall x\in\Gamma.$$

Now we consider the following constrained set-valued optimization problem:

$$(\text{SVOP})\quad \min F(x)\quad \text{s.t. } x\in E:=\bigl\{x\in\Gamma \mid G(x)\cap(-D)\neq\emptyset\bigr\},$$

where $F:\Gamma\rightrightarrows 2^Y$ and $G:\Gamma\rightrightarrows 2^Z$ are two set-valued maps with nonempty value. We always assume that the feasible set E is nonempty.

Definition 3.1 ([31]) $\bar{x}\in E$ is called a globally proper efficient solution of (SVOP) iff there exists $\bar{y}\in F(\bar{x})$ such that $\bar{y}\in \text{GPE}(F(E),C)$. The pair $(\bar{x},\bar{y})$ is called a globally proper efficient element of (SVOP).

Definition 3.2 ([35]) Let $\bar{x}\in E$ and $\bar{y}\in F(\bar{x})$. $(\bar{x},\bar{y})$ is called a positive properly efficient element of (SVOP) iff there exists $\mu\in C^{+i}$ such that $\mu(F(x)-\bar{y})\geq 0$ for all $x\in E$.

Lemma 3.1 ([35]) *A positive properly efficient element of* (SVOP) *must be a globally proper efficient element of* (SVOP).

By applying Lemma 3.1, Gong et al. [28] obtained a sufficient condition involving multiplier functionals for a globally proper efficient solution of (SVOP) (see Theorem 3.4 in [28]). Next, we will use Lemma 3.1 to establish a sufficient optimality condition characterized by the generalized contingent epiderivative of set-valued maps in the sense of globally proper efficiency.

Theorem 3.2 *Let C and D be two closed pointed convex cones being Daniell, and let Γ be a nonempty invex subset in X with respect to η. Let $\bar{x}\in E$, $\bar{y}\in F(\bar{x})$ and $\bar{z}\in G(\bar{x})\cap(-D)$. For any $x\in\Gamma$, write $\Psi(\eta(x,\bar{x})):=\{(y,z)\in Y\times Z \mid (\eta(x,\bar{x}),y,z)\in T(\text{epi}(F,G),(\bar{x},\bar{y},\bar{z}))\}$. The set $\Psi(\eta(x,\bar{x}))$ is minorized and fulfills the domination property for any $x\in\Gamma$. Suppose that the following conditions are satisfied*:

(i) *The set-valued map (F,G) is $C\times D$-subpreinvex on Γ with respect to η*;

(ii) *There exists $(\mu,\nu)\in C^{+i}\times D^+$ such that*

$$(\mu,\nu)\bigl(D_g(F,G)(\bar{x},\bar{y},\bar{z})\bigl(\eta(x,\bar{x})\bigr)\bigr)\geq 0,\quad \forall x\in\Gamma \tag{3.3}$$

and

$$\nu(\bar{z}) = 0. \tag{3.4}$$

Then $(\bar{x}, \bar{y})$ *is a globally proper efficient element of* (SVOP).

Proof From Theorem 3.1, we have

$$(F,G)(x) - \{(\bar{y},\bar{z})\} \subset D_g(F,G)(\bar{x},\bar{y},\bar{z})\big(\eta(x,\bar{x})\big) + C \times D, \quad \forall x \in \Gamma. \tag{3.5}$$

It follows from (3.5) that for any $x \in \Gamma$ and $(y,z) \in (F,G)(x)$, there exist $c \in C$ and $d \in D$ such that

$$(y - \bar{y} - c, z - \bar{z} - d) \in D_g(F,G)(\bar{x},\bar{y},\bar{z})\big(\eta(x,\bar{x})\big). \tag{3.6}$$

By (3.3) and (3.6), we get

$$\mu(y - \bar{y} - c) + \nu(z - \bar{z} - d) \geq 0,$$

i.e.,

$$\mu(y - \bar{y}) + \nu(z) \geq \mu(c) + \nu(\bar{z}) + \nu(d). \tag{3.7}$$

Since $(\mu,\nu) \in C^{+i} \times D^{+}$, we have $\mu(c) + \nu(d) \geq 0$, which together with (3.4) and (3.7) yields

$$\mu(y - \bar{y}) + \nu(z) \geq 0, \quad \forall x \in \Gamma, \forall (y,z) \in (F,G)(x). \tag{3.8}$$

Now, we prove $(\bar{x}, \bar{y})$ is a positive properly efficient element of (SVOP). Otherwise, for $\mu \in C^{+i}$, there exist $\hat{x} \in E$ and $\hat{y} \in F(\hat{x})$ such that

$$\mu(\hat{y} - \bar{y}) < 0. \tag{3.9}$$

Since $\hat{x} \in E$, there exists $\hat{z} \in G(\hat{x})$ such that $\hat{z} \in -D$. Therefore,

$$\nu(\hat{z}) \leq 0. \tag{3.10}$$

By (3.9) and (3.10), we get

$$\mu(\hat{y} - \bar{y}) + \nu(\hat{z}) < 0,$$

which contradicts (3.8). Hence, $(\bar{x}, \bar{y})$ is a positive properly efficient element of (SVOP). By Lemma 3.1, $(\bar{x}, \bar{y})$ is a globally proper efficient element of (SVOP). □

Remark 3.3 Compared with Theorem 4.1 in [36], Theorem 3.2 has some weaker conditions and a stronger conclusion. Therefore, Theorem 3.2 is a proper generalization of Theorem 4.1 in [36].

4 Generalized vector variational inequality

In this section, we will introduce a generalized vector variational inequality problem (GVVIP) and disclose the close relations between the globally proper efficiency of (SVOP) and the globally proper efficiency of (GVVIP).

Let $\bar{x} \in E$ and $\bar{y} \in F(\bar{x})$. We always assume that $\eta(x,\bar{x})$ belongs to the domain of $D_gF(\bar{x},\bar{y})$ for any $x \in E$. The generalized vector variational inequality problem is to find $\bar{x} \in E$ and $\bar{y} \in F(\bar{x})$ such that

$$\text{(GVVIP)} \quad D_gF(\bar{x},\bar{y})\bigl(\eta(x,\bar{x})\bigr) \cap (-C) = \emptyset, \quad \forall x \in E.$$

Definition 4.1 A pair $(\bar{x},\bar{y})$ is called a globally proper efficient element of (GVVIP) iff there exists a pointed convex cone $C' \subset Y$ with $C \backslash \{0\} \subset \operatorname{int} C'$ such that

$$D_gF(\bar{x},\bar{y})\bigl(\eta(x,\bar{x})\bigr) \cap \bigl(-\operatorname{int} C'\bigr) = \emptyset, \quad \forall x \in E.$$

Remark 4.1 When $\eta(x,\bar{x}) = x - \bar{x}$ and the generalized contingent epiderivative of the set-valued map becomes the contingent epiderivative of the set-valued map, Definition 4.1 reduces to Definition 18 in [13] and Definition 2.10 in [20].

We will use the standard assumptions: Let C be a closed pointed convex cone being Daniell and $\operatorname{int} C \neq \emptyset$, and suppose that $\Phi(\eta(x,\bar{x}))$ given by Theorem 3.1 is minorized and fulfills the domination property for any $x \in \Gamma$.

Theorem 4.1 *Let the standard assumptions hold. If* $(\bar{x},\bar{y})$ *is a globally proper efficient element of* (SVOP), *then* $(\bar{x},\bar{y})$ *is a globally proper efficient element of* (GVVIP).

Proof Suppose that $(\bar{x},\bar{y})$ is not a globally proper efficient element of (GVVIP), then for any pointed convex cone $\hat{C} \subset Y$ with $C \backslash \{0\} \subset \operatorname{int} \hat{C}$, there exist $x' \in E$ and $y' \in D_gF(\bar{x},\bar{y})(\eta(x',\bar{x}))$ such that

$$y' \in -\operatorname{int} \hat{C}. \tag{4.1}$$

It follows from Definition 2.4 that $(\eta(x',\bar{x}),y') \in T(\operatorname{epi}(F),(\bar{x},\bar{y}))$. Therefore, there exist a sequence $\{(x_n,y_n)\}_{n\in\mathbb{N}}$ in $\operatorname{epi}(F)$ and a sequence $\{t_n\}_{n\in\mathbb{N}}$ of positive real numbers such that $(\bar{x},\bar{y}) = \lim_{n\to\infty}(x_n,y_n)$ and

$$\bigl(\eta\bigl(x',\bar{x}\bigr),y'\bigr) = \lim_{n\to\infty} t_n\bigl((x_n,y_n) - (\bar{x},\bar{y})\bigr). \tag{4.2}$$

According to (4.2), we get

$$y' = \lim_{n\to\infty} t_n(y_n - \bar{y})). \tag{4.3}$$

By (4.1) and (4.3), there exists $n_0 \in \mathbb{N}$ such that

$$t_n(y_n - \bar{y}) \in -\operatorname{int} \hat{C}, \quad \forall n \geq n_0,$$

which implies that

$$y_n \in \bar{y} - \operatorname{int} \hat{C}, \quad \forall n \geq n_0. \tag{4.4}$$

Since $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subset \operatorname{epi}(F)$, there exist $y_n^* \in F(x_n)$ and $c_n \in C$ such that $y_n = y_n^* + c_n$, which together with (4.4) yields

$$y_n^* = y_n - c_n \in \bar{y} - \operatorname{int} \hat{C} - C \subset \bar{y} - \operatorname{int} \hat{C}, \quad \forall n \geq n_0.$$

Therefore, $y_n^* - \bar{y} \in (F(E) - \bar{y}) \cap (-\operatorname{int} \hat{C})$, which contradicts the fact that $(\bar{x}, \bar{y})$ is a globally proper efficient element of (SVOP). Hence, $(\bar{x}, \bar{y})$ is a globally proper efficient element of (GVVIP). □

Remark 4.2 It is worth noting that the contingent epiderivative of Theorem 8 in [13] is replaced with the generalized contingent epiderivative. Therefore, Theorem 4.1 improves Theorem 8 in [13].

Theorem 4.2 *Let the standard assumptions hold. Let Γ be a nonempty invex subset in X with respect to η and the set-valued map $F : \Gamma \rightrightarrows 2^Y$ be C-subpreinvex on Γ with respect to η. If $(\bar{x}, \bar{y})$ is a globally proper efficient element of* (GVVIP), *then $(\bar{x}, \bar{y})$ is a globally proper efficient element of* (SVOP).

Proof Since $(\bar{x}, \bar{y})$ is a globally proper efficient element of (GVVIP), there exists a pointed convex cone $C' \subset Y$ with $C\backslash\{0\} \subset \operatorname{int} C'$ such that

$$D_g F(\bar{x}, \bar{y})\big(\eta(x, \bar{x})\big) \cap \big(-\operatorname{int} C'\big) = \emptyset, \quad \forall x \in E. \tag{4.5}$$

Suppose that $(\bar{x}, \bar{y})$ is not a globally proper efficient element of (SVOP), then for the pointed convex cone $C' \subset Y$ with $C\backslash\{0\} \subset \operatorname{int} C'$, there exist $\hat{x} \in E$ and $\hat{y} \in F(\hat{x})$ such that

$$\hat{y} - \bar{y} \in -\operatorname{int} C'. \tag{4.6}$$

It follows from Theorem 3.1 that there exist $\hat{a} \in D_g F(\bar{x}, \bar{y})(\eta(\hat{x}, \bar{x}))$ and $\hat{c} \in C$ such that

$$\hat{y} - \bar{y} = \hat{a} + \hat{c}. \tag{4.7}$$

By (4.6) and (4.7), we have

$$\hat{a} = \hat{y} - \bar{y} - \hat{c} \in -\operatorname{int} C' - C \subset -\operatorname{int} C',$$

which contradicts (4.5). Hence, $(\bar{x}, \bar{y})$ is a globally proper efficient element of (SVOP). □

Remark 4.3 Theorem 4.2 generalizes and improves Theorem 7 in [13] in the following three aspects: (i) The convex set becomes an invex set. (ii) The contingent epiderivative is generalized to the generalized contingent epiderivative. (iii) The C-convexity of F is extended to C-subpreinvexity of F.

5 Conclusions

In this paper, based on the generalized contingent epiderivative of set-valued maps, we obtained a new property of the cone subpreinvex set-valued map. By applying this property, we derived a sufficient optimality condition in the sense of globally proper efficiency in the constrained set-valued optimization problem. We also introduced a new kind of generalized variational inequality problem. Moreover, the relations between the globally proper efficiency of the set-valued optimization problem and the globally proper efficiency of the generalized variational inequality problem are disclosed. These results are new and are extensions of the corresponding ones in set-valued optimization.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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