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General fractional integral inequalities for convex and m-convex functions via an extended generalized Mittag-Leffler function

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Abstract

In this paper some new general fractional integral inequalities for convex and m-convex functions by involving an extended Mittag-Leffler function are presented. These results produce inequalities for several kinds of fractional integral operators. Some interesting special cases of our main results are also pointed out.

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1 Introduction, definitions, and preliminaries

Convex functions are very important in the field of integral inequalities. A lot of fractional integral inequalities and novel results have been established due to convex functions (for more details, see [1, 8, 13, 14]).

Definition 1 A function $f : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$, is said to be a convex function if

$$f\big(tx + (1-t)y\big) \le tf(x) + (1-t)f(y) \tag{1}$$

holds for $t \in [0,1]$ and $x, y \in I$.

A convex function $f : I \to \mathbb{R}$ is also equivalently defined by the Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2},$$

where $a, b \in I$, $a < b$.

The concept of m-convexity was introduced in [17] and since then many properties, especially inequalities, have been obtained for this class of functions (see [3, 6, 7, 18]).

Definition 2 A function $f : [0, b] \to \mathbb{R}$, $b > 0$ is called m-convex, where $m \in [0, 1]$, if for every $x, y \in [0, b]$ and $t \in [0, 1]$, we have

$$f\bigl(tx + m(1-t)y\bigr) \leq tf(x) + m(1-t)f(y).$$

For $m = 1$, we recapture the definition of convex functions, and for $m = 0$, the definition of star-shaped functions defined on $[0, b]$. We recall that a function $f : [0, b] \to \mathbb{R}$ is called *star-shaped* if

$$f(tx) \leq tf(x) \quad \text{for all } t \in [0,1] \text{ and } x \in [0,b].$$

If we denote by $K_m(b)$ the set of m-convex functions defined on $[0, b]$ for which $f(0) < 0$, then

$$K_1(b) \subset K_m(b) \subset K_0(b),$$

whenever $m \in (0, 1)$. Note that in the class $K_1(b)$ there are only convex functions $f : [0, b] \to \mathbb{R}$ for which $f(0) \leq 0$ (see [4]), while $k_0(b)$ contains *star-shaped* functions.

Example 1.1 ([6]) The function $f : [0, \infty) \to \mathbb{R}$, given by

$$f(x) = \frac{1}{12}\bigl(x^4 - 5x^3 + 9x^2 - 5x\bigr),$$

is a $\frac{16}{17}$-convex function but it is not m-convex for any $m \in (\frac{16}{17}, 1]$.

For more results and inequalities related to m-convex functions, one can consult, for example, [3, 6, 7] along with the references therein.

Recently in [2] Andrić et al. defined an extended generalized Mittag-Leffler function $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\cdot;p)$ as follows.

Definition 3 ([2]) Let $\mu, \alpha, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\alpha), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$, and $0 < k \leq \delta + \Re(\mu)$. Then the extended generalized Mittag-Leffler function $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(t;p)$ is defined by

$$E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(t;p) = \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c-\gamma)} \frac{(c)_{nk}}{\Gamma(\mu n + \alpha)} \frac{t^n}{(l)_{n\delta}}, \tag{2}$$

where β_p is the generalized beta function defined by

$$\beta_p(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{-\frac{p}{t(1-t)}}\, dt$$

and $(c)_{nk}$ is the Pochhammer symbol defined as $(c)_{nk} = \frac{\Gamma(c+nk)}{\Gamma(c)}$.

In [2] properties of the generalized Mittag-Leffler function are discussed, and it is given that $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(t;p)$ is absolutely convergent for $k < \delta + \Re(\mu)$. Let S be the sum of series of absolute terms of the Mittag-Leffler function $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(t;p)$, then we have $|E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(t;p)| \leq S$. We use this property of Mittag-Leffler function in our results where we need.

The corresponding left and right sided extended generalized fractional integral operators are defined as follows.

Definition 4 ([2]) Let $\omega, \mu, \alpha, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\alpha), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$ and $0 < k \leq \delta + \Re(\mu)$. Let $f \in L_1[a,b]$ and $x \in [a,b]$. Then the extended generalized fractional integral operators $\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}f$ and $\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b^-}f$ are defined by

$$\big(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}f\big)(x;p) = \int_a^x (x-t)^{\alpha-1} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\big(\omega(x-t)^{\mu};p\big) f(t)\,dt \tag{3}$$

and

$$\big(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b^-}f\big)(x;p) = \int_x^b (t-x)^{\alpha-1} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\big(\omega(t-x)^{\mu};p\big) f(t)\,dt. \tag{4}$$

From extended generalized fractional integral operators, we have

$$\begin{aligned}
&(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}1)(x;p) \\
&\quad= \int_a^x (x-t)^{\alpha-1} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(w(x-t)^{\mu};p)\,dt \\
&\quad= \int_a^x (x-t)^{\alpha-1} \sum_{n=0}^{\infty} \frac{\mathrm{B}_p(\gamma+nk, c-\gamma)}{\mathrm{B}(\gamma, c-\gamma)} \frac{(c)_{nk}}{\Gamma(\mu n+\alpha)} \frac{w^n (x-t)^{\mu n}}{(l)_{n\delta}}\,dt \\
&\quad= \sum_{n=0}^{\infty} \frac{\mathrm{B}_p(\gamma+nk, c-\gamma)}{\mathrm{B}(\gamma, c-\gamma)} \frac{(c)_{nk}}{\Gamma(\mu n+\alpha)} \frac{w^n}{(l)_{n\delta}} \int_a^x (x-t)^{\mu n+\alpha-1}\,dt \\
&\quad= (x-a)^{\alpha} \sum_{n=0}^{\infty} \frac{\mathrm{B}_p(\gamma+nk, c-\gamma)}{\mathrm{B}(\gamma, c-\gamma)} \frac{(c)_{nk}}{\Gamma(\mu n+\alpha)} \frac{w^n}{(l)_{n\delta}} (x-a)^{\mu n} \frac{1}{\mu n+\alpha}.
\end{aligned}$$

Hence

$$\big(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}1\big)(x;p) = (x-a)^{\alpha} E^{\gamma,\delta,k,c}_{\mu,\alpha+1,l}\big(w(x-a)^{\mu};p\big),$$

and similarly

$$\big(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b^-}1\big)(x;p) = (b-x)^{\alpha} E^{\gamma,\delta,k,c}_{\mu,\alpha+1,l}\big(w(b-x)^{\mu};p\big).$$

We use the following notations in our results:

$$C_{\alpha,a^+}(x;p) = \big(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}1\big)(x;p) \tag{5}$$

and

$$C_{\alpha,b^-}(x;p) = \big(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b^-}1\big)(x;p). \tag{6}$$

For more information related to Mittag-Leffler functions and corresponding fractional integral operators, the readers are referred to [9–12, 15, 16, 19].

In this paper we give general fractional integral inequalities for convex and m-convex functions by involving an extended Mittag-Leffler function and deduce some results already published in [1, 5, 6, 8, 13]. Also we give a Hadamard type inequality for convex and m-convex functions by involving an extended Mittag-Leffler function.

2 Main results

Here we give some fractional integral inequalities for convex and m-convex functions via an extended generalized Mittag-Leffler function and corresponding fractional integral operators given in (3) and (4). The following lemma is useful to establish the results.

Lemma 2.1 *Let $f:[a,mb]\to\mathbb{R}$ be a differentiable function such that $f'\in L_1[a,mb]$ with $0\le a<mb$. Also let $g:[a,mb]\to\mathbb{R}$ be a continuous function on $[a,mb]$, then the following identity for extended generalized fractional integral operators holds*:

$$\begin{aligned}
&\left(\int_a^{mb} g(s)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega s^{\mu};p)\,ds\right)^{\alpha}\big[f(a)+f(mb)\big]\\
&\quad-\alpha\int_a^{mb}\left(\int_a^{t} g(s)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega s^{\mu};p)\,ds\right)^{\alpha-1} g(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^{\mu};p)f(t)\,dt\\
&\quad-\alpha\int_a^{mb}\left(\int_t^{mb} g(s)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega s^{\mu};p)\,ds\right)^{\alpha-1} g(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^{\mu};p)f(t)\,dt\\
&=\int_a^{mb}\left(\int_a^{t} g(s)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega s^{\mu};p)\,ds\right)^{\alpha} f'(t)\,dt\\
&\quad-\int_a^{mb}\left(\int_t^{mb} g(s)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega s^{\mu};p)\,ds\right)^{\alpha} f'(t)\,dt.
\end{aligned}\tag{7}$$

Proof On integrating by parts one can have

$$\begin{aligned}
&\int_a^{mb}\left(\int_a^{t} g(s)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega s^{\mu};p)\,ds\right)^{\alpha} f'(t)\,dt\\
&=\left(\int_a^{mb} g(s)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega s^{\mu};p)\,ds\right)^{\alpha} f(mb)\\
&\quad-\alpha\int_a^{mb}\left(\int_a^{t} g(s)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega s^{\mu};p)\,ds\right)^{\alpha-1} g(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^{\mu};p)f(t)\,dt
\end{aligned}\tag{8}$$

and

$$\begin{aligned}
&\int_a^{mb}\left(\int_t^{mb} g(s)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega s^{\mu};p)\,ds\right)^{\alpha} f'(t)\,dt\\
&=-\left(\int_a^{mb} g(s)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega s^{\mu};p)\,ds\right)^{\alpha} f(a)\\
&\quad+\alpha\int_a^{mb}\left(\int_t^{mb} g(s)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega s^{\mu};p)\,ds\right)^{\alpha-1} g(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^{\mu};p)f(t)\,dt.
\end{aligned}\tag{9}$$

Subtracting (9) from (8), we get (7) which is the required identity. □

If we take $m=1$ in (7), then we get the following identity for a convex function.

Corollary 2.2 *Let $f : [a,b] \subseteq [0,\infty) \to \mathbb{R}$ be a differentiable function such that $f' \in L_1[a,b]$ with $a < b$. Also let $g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, then the following identity for extended generalized fractional integral operators holds*:

$$\begin{aligned}
&\left(\int_a^b g(s)E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega s^\mu;p)\,ds\right)^\alpha \big[f(a)+f(b)\big] \\
&\quad - \alpha\int_a^b\left(\int_a^t g(s)E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega s^\mu;p)\,ds\right)^{\alpha-1} g(t)E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega t^\mu;p)f(t)\,dt \\
&\quad - \alpha\int_a^b\left(\int_t^b g(s)E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega s^\mu;p)\,ds\right)^{\alpha-1} g(t)E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega t^\mu;p)f(t)\,dt \\
&\quad = \int_a^b\left(\int_a^t g(s)E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega s^\mu;p)\,ds\right)^{\alpha} f'(t)\,dt \\
&\quad - \int_a^b\left(\int_t^b g(s)E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega s^\mu;p)\,ds\right)^{\alpha} f'(t)\,dt. \qquad (10)
\end{aligned}$$

We use identity (7) to establish the following fractional integral inequality.

Theorem 2.3 *Let $f : [a,mb] \to \mathbb{R}$ be a differentiable function such that $f' \in L_1[a,mb]$ with $0 \le a < mb$. Also let $g : [a,mb] \to \mathbb{R}$ be a continuous function on $[a,mb]$. If $|f'|$ is an m-convex function on $[a,mb]$, then the following inequality for extended generalized fractional integral operators holds*:

$$\begin{aligned}
&\Bigg|\left(\int_a^{mb} g(s)E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega s^\mu;p)\,ds\right)^\alpha \big(f(a)+f(mb)\big) \\
&\quad - \alpha\int_a^{mb}\left(\int_a^t g(s)E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega s^\mu;p)\,ds\right)^{\alpha-1} g(t)E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega t^\mu;p)f(t)\,dt \\
&\quad - \alpha\int_a^{mb}\left(\int_t^{mb} g(s)E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega s^\mu;p)\,ds\right)^{\alpha-1} g(t)E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega t^\mu;p)f(t)\,dt\Bigg| \\
&\quad \le \frac{(mb-a)^{\alpha+1}\|g\|_\infty^\alpha S^\alpha}{(\alpha+1)}\big(|f'(a)|+m|f'(b)|\big) \qquad (11)
\end{aligned}$$

for $k < \delta + \Re(\mu)$ and $\|g\|_\infty = \sup_{t\in[a,mb]}|g(t)|$.

Proof From Lemma 2.1, we have

$$\begin{aligned}
&\Bigg|\left(\int_a^{mb} g(s)E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega s^\mu;p)\,ds\right)^\alpha \big(f(a)+f(mb)\big) \\
&\quad - \alpha\int_a^{mb}\left(\int_a^t g(s)E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega s^\mu;p)\,ds\right)^{\alpha-1} g(t)E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega t^\mu;p)f(t)\,dt \\
&\quad - \alpha\int_a^{mb}\left(\int_t^{mb} g(s)E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega s^\mu;p)\,ds\right)^{\alpha-1} g(t)E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega t^\mu;p)f(t)\,dt\Bigg| \\
&\quad \le \int_a^{mb}\left|\int_a^t g(s)E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega s^\mu;p)\,ds\right|^\alpha |f'(t)|\,dt \\
&\quad + \int_a^{mb}\left|\int_t^{mb} g(s)E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega s^\mu;p)\,ds\right|^\alpha |f'(t)|\,dt. \qquad (12)
\end{aligned}$$

Using absolute convergence of the Mittag-Leffler function and $\|g\|_\infty = \sup_{t\in[a,b]} |g(t)|$, we have

$$\begin{aligned}&\bigg|\bigg(\int_a^{mb} g(s)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega s^\mu;p)\,ds\bigg)^\alpha \big(f(a)+f(mb)\big)\\&\quad -\alpha\int_a^{mb}\bigg(\int_a^t g(s)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega s^\mu;p)\,ds\bigg)^{\alpha-1} g(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^\mu;p)f(t)\,dt\\&\quad -\alpha\int_a^{mb}\bigg(\int_t^{mb} g(s)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega s^\mu;p)\,ds\bigg)^{\alpha-1} g(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^\mu;p)f(t)\,dt\bigg|\\&\quad\le \|g\|_\infty^\alpha S^\alpha\bigg(\int_a^{mb}(t-a)^\alpha\big|f'(t)\big|\,dt+\int_a^{mb}(mb-t)^\alpha\big|f'(t)\big|\,dt\bigg).\end{aligned}\tag{13}$$

Since $|f'|$ is an m-convex function, we have

$$\big|f'(t)\big|\le\frac{mb-t}{mb-a}\big|f'(a)\big|+m\frac{t-a}{mb-a}\big|f'(b)\big|\tag{14}$$

for $t\in[a,mb]$.

Using (14) in (13), we have

$$\begin{aligned}&\bigg|\bigg(\int_a^{mb} g(s)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega s^\mu;p)\,ds\bigg)^\alpha \big(f(a)+f(mb)\big)\\&\quad -\alpha\int_a^{mb}\bigg(\int_a^t g(s)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega s^\mu;p)\,ds\bigg)^{\alpha-1} g(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^\mu;p)f(t)\,dt\\&\quad -\alpha\int_a^{mb}\bigg(\int_t^{mb} g(s)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega s^\mu;p)\,ds\bigg)^{\alpha-1} g(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^\mu;p)f(t)\,dt\bigg|\\&\quad\le \|g\|_\infty^\alpha S^\alpha\bigg(\int_a^{mb}(t-a)^\alpha\bigg(\frac{mb-t}{mb-a}\big|f'(a)\big|+m\frac{t-a}{mb-a}\big|f'(b)\big|\bigg)\,dt\\&\quad\quad+\int_a^{mb}(mb-t)^\alpha\bigg(\frac{mb-t}{mb-a}\big|f'(a)\big|+m\frac{t-a}{mb-a}\big|f'(b)\big|\bigg)\,dt\bigg).\end{aligned}\tag{15}$$

After simple calculation of the above inequality, we get (11) which is required. □

If we take $m = 1$ in (11), then we get the following result for a convex function.

Corollary 2.4 *Let $f:[a,b]\subseteq[0,\infty)\to\mathbb{R}$ be a differentiable function such that $f'\in L_1[a,b]$ with $a<b$. Also let $g:[a,b]\to\mathbb{R}$ be a continuous function on $[a,b]$. If $|f'|$ is a convex function on $[a,b]$, then the following inequality for extended generalized fractional integral operators holds*:

$$\begin{aligned}&\bigg|\bigg(\int_a^{b} g(s)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega s^\mu;p)\,ds\bigg)^\alpha \big[f(a)+f(b)\big]\\&\quad -\alpha\int_a^{b}\bigg(\int_a^t g(s)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega s^\mu;p)\,ds\bigg)^{\alpha-1} g(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^\mu;p)f(t)\,dt\end{aligned}$$

$$-\alpha\int_a^b\left(\int_t^b g(s)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega s^{\mu};p)\,ds\right)^{\alpha-1}g(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^{\mu};p)f(t)\,dt\Bigg|$$
$$\leq\frac{(b-a)^{\alpha+1}\|g\|_{\infty}^{\alpha}S^{\alpha}}{(\alpha+1)}\left[\left|f'(a)\right|+\left|f'(b)\right|\right] \tag{16}$$

for $k<\delta+\Re(\mu)$ *and* $\|g\|_{\infty}=\sup_{t\in[a,b]}|g(t)|$.

Remark 2.5 In Theorem 2.3.

(i) If we put $p=0$, then we get [6, Theorem 3.2].
(ii) If we put $\omega=p=0$ and $m=1$, then we get [13, Theorem 6].
(iii) If we take $\omega=p=0$, $m=1$, $\alpha=\frac{\mu}{k}$, and $g(s)=1$, then we get [8, Corollary 2.3].
(iv) For $g(s)=1$ along with $\omega=p=0$, $m=1$, and $\alpha=\mu$, we get [13, Corollary 2].

Remark 2.6 In Corollary 2.4.

(i) If we put $p=0$, then we get [1, Theorem 3.2].
(ii) If we put $\omega=p=0$, then we get [13, Theorem 6].
(iii) For $\omega=p=0$, $\alpha=\frac{\mu}{k}$, and $g(s)=1$, we get [8, Corollary 2.3].
(iv) For $g(s)=1$ along with $\omega=p=0$, we get [13, Corollary 2].

Next we give the following fractional integral inequality.

Theorem 2.7 *Let* $f:[a,mb]\to\mathbb{R}$ *be a differentiable function such that* $f\in L_1[a,mb]$ *with* $0\leq a<mb$. *Also let* $g:[a,mb]\to\mathbb{R}$ *be a continuous function on* $[a,mb]$. *If* $|f'|^q$ *is a convex function on* $[a,mb]$, *then for* $q>0$ *the following inequality for extended generalized fractional integral operators holds*:

$$\Bigg|\left(\int_a^{mb} g(s)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega s^{\mu};p)\,ds\right)^{\alpha}\big(f(a)+f(mb)\big)$$
$$-\alpha\int_a^{mb}\left(\int_a^{t} g(s)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega s^{\mu};p)\,ds\right)^{\alpha-1}g(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^{\mu};p)f(t)\,dt$$
$$-\alpha\int_a^{mb}\left(\int_t^{mb} g(s)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega s^{\mu};p)\,ds\right)^{\alpha-1}g(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^{\mu};p)f(t)\,dt\Bigg|$$
$$\leq\frac{2(mb-a)^{\alpha+1}\|g\|_{\infty}^{\alpha}S^{\alpha}}{(\alpha p+1)^{\frac{1}{q}}}\left(\frac{|f'(a)|^q+m|f'(b)|^q}{2}\right)^{\frac{1}{q}} \tag{17}$$

for $k<\delta+\Re(\mu)$ *and* $\|g\|_{\infty}=\sup_{t\in[a,mb]}|g(t)|$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

Proof From Lemma 2.1 and by using Hölder's inequality, we have

$$\Bigg|\left(\int_a^{mb} g(s)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega s^{\mu};p)\,ds\right)^{\alpha}\big(f(a)+f(mb)\big)$$
$$-\alpha\int_a^{mb}\left(\int_a^{t} g(s)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega s^{\mu};p)\,ds\right)^{\alpha-1}g(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^{\mu};p)f(t)\,dt$$
$$-\alpha\int_a^{mb}\left(\int_t^{mb} g(s)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega s^{\mu};p)\,ds\right)^{\alpha-1}g(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^{\mu};p)f(t)\,dt\Bigg|$$

$$\leq \left(\int_a^{mb}\left|\int_a^t g(s)E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega s^\mu;p)\,ds\right|^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_a^{mb}\left|f'(t)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\left(\int_a^{mb}\left|\int_t^{mb} g(s)E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega s^\mu;p)\,ds\right|^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_a^{mb}\left|f'(t)\right|^q dt\right)^{\frac{1}{q}}. \tag{18}$$

Using absolute convergence of the Mittag-Leffler function and $\|g\|_\infty = \sup_{t\in[a,b]}|g(t)|$, we have

$$\left|\left(\int_a^{mb} g(s)E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega s^\mu;p)\,ds\right)^\alpha\big(f(a)+f(mb)\big)\right.$$
$$-\alpha\int_a^{mb}\left(\int_a^t g(s)E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega s^\mu;p)\,ds\right)^{\alpha-1}g(t)E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega t^\mu;p)f(t)\,dt$$
$$\left.-\alpha\int_a^{mb}\left(\int_t^{mb} g(s)E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega s^\mu;p)\,ds\right)^{\alpha-1}g(t)E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega t^\mu;p)f(t)\,dt\right|$$
$$\leq \|g\|_\infty^\alpha S^\alpha\left(\left(\int_a^{mb}|t-a|^{\alpha p}\,dt\right)^{\frac{1}{p}}\right.$$
$$\left.+\left(\int_a^{mb}|mb-t|^{\alpha p}\,dt\right)^{\frac{1}{p}}\right)\left(\int_a^{mb}\left|f'(t)\right|^q dt\right)^{\frac{1}{q}}. \tag{19}$$

Since $|f'(t)|^q$ is an m-convex function, we have

$$\left|f'(t)\right|^q \leq \frac{mb-t}{mb-a}\left|f'(a)\right|^q + m\frac{t-a}{mb-a}\left|f'(b)\right|^q. \tag{20}$$

Using (20) in (19), we have

$$\left|\left(\int_a^{mb} g(s)E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega s^\mu;p)\,ds\right)^\alpha\big(f(a)+f(mb)\big)\right.$$
$$-\alpha\int_a^{mb}\left(\int_a^t g(s)E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega s^\mu;p)\,ds\right)^{\alpha-1}g(t)E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega t^\mu;p)f(t)\,dt$$
$$\left.-\alpha\int_a^{mb}\left(\int_t^{mb} g(s)E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega s^\mu;p)\,ds\right)^{\alpha-1}g(t)E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega t^\mu;p)f(t)\,dt\right|$$
$$\leq \|g\|_\infty^\alpha S^\alpha\left(\left(\int_a^{mb}|t-a|^{\alpha p}\,dt\right)^{\frac{1}{p}}+\left(\int_a^{mb}|mb-t|^{\alpha p}\,dt\right)^{\frac{1}{p}}\right)$$
$$\times\left(\int_a^{mb}\frac{mb-t}{mb-a}\left|f'(a)\right|^q + m\frac{t-a}{mb-a}\left|f'(b)\right|^q\right)^{\frac{1}{q}}. \tag{21}$$

After simple calculation of the above inequality, we get (17) which is required. □

If we take $m = 1$ in (17), then we get the following result for a convex function.

Corollary 2.8 *Let $f:[a,b]\subseteq[0,\infty)\to\mathbb{R}$ be a differentiable function such that $f'\in L_1[a,b]$ with $a<b$. Also let $g:[a,b]\to\mathbb{R}$ be a continuous function on $[a,b]$. If $|f'|^q$ is a convex function on $[a,b]$, then for $q>0$ the following inequality for extended generalized fractional* *integral operators holds*:

$$\begin{aligned}&\left|\left(\int_a^b g(s)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega s^\mu;p)\,ds\right)^\alpha\left[f(a)+f(b)\right]\right.\\&\quad-\alpha\int_a^b\left(\int_a^t g(s)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega s^\mu;p)\,ds\right)^{\alpha-1}g(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^\mu;p)f(t)\,dt\\&\quad\left.-\alpha\int_a^b\left(\int_t^b g(s)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega s^\mu;p)\,ds\right)^{\alpha-1}g(t)E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^\mu;p)f(t)\,dt\right|\\&\quad\le\frac{2(b-a)^{\alpha+1}\|g\|_\infty^\alpha S^\alpha}{(\alpha p+1)^{\frac{1}{q}}}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}}\end{aligned}\tag{22}$$

for $k<\delta+\Re(\mu)$ *and* $\|g\|_\infty=\sup_{t\in[a,b]}|g(t)|$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

Remark 2.9 In Theorem 2.7.

(i) If we put $p=0$, then we get [6, Theorem 3.6].
(ii) If we put $\omega=p=0$ and $m=1$, then we get [13, Theorem 7].
(iii) If we take $\omega=p=0$, $m=1$ along with $\alpha=\frac{\mu}{k}$, then we get [8, Theorem 2.5].
(iv) If we take $g(s)=1$, $m=1$, and $\omega=p=0$, then we get [5, Theorem 2.3].
(v) If we put $\omega=p=0$, $m=1$, and $\alpha=1$, then we get [5, Corollary 3].

Remark 2.10 In Corollary 2.8.

(i) If we put $p=0$, then we get [1, Theorem 3.5].
(ii) If we put $\omega=p=0$, then we get [13, Theorem 7].
(iii) If we put $\omega=p=0$, $\alpha=1$, then we get [13, Corollary 3].
(iv) If we take $\omega=p=0$ along with $\alpha=\frac{\mu}{k}$, then we get [8, Theorem 2.5].
(v) If we take $g(s)=1$ and $\omega=p=0$, then we get [5, Theorem 2.3].

In the next result we give Hadamard type inequalities for m-convex functions via an extended Mittag-Leffler function.

Theorem 2.11 *Let* $f:[a,mb]\to\mathbb{R}$ *be a function such that* $f\in L_1[a,mb]$ *with* $0\le a<mb$. *If* f *is* m*-convex on* $[a,mb]$, *then the following inequalities for extended generalized fractional integral operators hold*:

$$\begin{aligned}&2f\left(\frac{a+mb}{2}\right)C_{\alpha,(\frac{a+mb}{2})^+}(mb;p)\\&\quad\le\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega',(\frac{a+mb}{2})^+}f\right)(mb;p)+m^{\alpha+1}\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,m^\mu\omega',(\frac{a+mb}{2m})^-}f\right)\left(\frac{a}{m};p\right)\\&\quad\le\frac{1}{mb-a}\left(f(a)-m^2f\left(\frac{a}{m^2}\right)\right)C_{\alpha+1,(\frac{a+mb}{2})^+}(mb;p)\\&\qquad+m^{\alpha+1}\left(f(b)+mf\left(\frac{a}{m^2}\right)\right)C_{\alpha,(\frac{a+mb}{2m})^-}\left(\frac{a}{m};p\right),\end{aligned}\tag{23}$$

where $\omega'=\frac{2^\mu\omega}{(mb-a)^\mu}$.

Proof Since f is an m-convex function, we have

$$2f\left(\frac{a+mb}{2}\right) \leq f\left(\frac{t}{2}a + \frac{2-t}{2}mb\right) + mf\left(\frac{2-t}{2m}a + \frac{t}{2}b\right). \tag{24}$$

Also from m-convexity of f, we have

$$\begin{aligned} &f\left(\frac{t}{2}a + m\frac{2-t}{2}b\right) + mf\left(\frac{2-t}{2m}a + \frac{t}{2}b\right) \\ &\quad \leq \frac{t}{2}\left(f(a) - m^2 f\left(\frac{a}{m^2}\right)\right) + m\left(f(b) + mf\left(\frac{a}{m^2}\right)\right). \end{aligned} \tag{25}$$

Multiplying (24) by $t^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega t^{\mu};p)$ on both sides and then integrating over $[0,1]$, we have

$$\begin{aligned} &2f\left(\frac{a+mb}{2}\right)\int_0^1 t^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega t^{\mu};p)\,dt \\ &\quad \leq \int_0^1 t^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega t^{\mu};p) f\left(\frac{t}{2}a + \frac{2-t}{2}mb\right)dt \\ &\qquad + m\int_0^1 t^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega t^{\mu};p) f\left(\frac{2-t}{2m}a + \frac{t}{2}b\right)dt. \end{aligned} \tag{26}$$

Putting $u = \frac{t}{2}a + \frac{2-t}{2}mb$ and $v = \frac{2-t}{2m}a + \frac{t}{2}b$ in (26), we have

$$\begin{aligned} &2f\left(\frac{a+mb}{2}\right)\int_{\frac{a+mb}{2}}^{mb} (mb-u)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}\left(\omega'(mb-u)^{\mu};p\right)du \\ &\quad \leq \int_{\frac{a+mb}{2}}^{mb} (mb-u)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}\left(\omega'(mb-u)^{\mu};p\right)f(u)\,du \\ &\qquad + m^{\alpha+1}\int_{\frac{a}{m}}^{\frac{a+mb}{2m}} \left(v-\frac{a}{m}\right)^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}\left(m^{\mu}\omega'\left(v-\frac{a}{m}\right)^{\mu}:p\right)f(v)\,dv. \end{aligned}$$

By using (3), (4), and (5) we get the first inequality of (23).

Now multiplying (25) by $t^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega t^{\mu};p)$ on both sides and then integrating over $[0,1]$, we have

$$\begin{aligned} &\int_0^1 t^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega t^{\mu};p) f\left(\frac{t}{2}a + m\frac{2-t}{2}b\right)dt \\ &\qquad + m\int_0^1 t^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega t^{\mu};p) f\left(\frac{2-t}{2m}a + \frac{t}{2}b\right) \\ &\quad \leq \frac{1}{2}\left(f(a) - m^2 f\left(\frac{a}{m^2}\right)\right)\int_0^1 t^{\alpha}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega t^{\mu};p)\,dt \\ &\qquad + m\left(f(b) + mf\left(\frac{a}{m^2}\right)\right)\int_0^1 t^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega t^{\mu};p)\,dt. \end{aligned} \tag{27}$$

Putting $u = \frac{t}{2}a + m\frac{2-t}{2}b$ and $v = \frac{2-t}{2m}a + \frac{t}{2}b$ in (27), we have

$$\begin{aligned}
&\int_{\frac{a+mb}{2}}^{mb} (mb-u)^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}\big(\omega'(mb-u)^{\mu};p\big) f(u)\,du \\
&\quad + \int_{\frac{a}{m}}^{\frac{a+mb}{2m}} \left(v-\frac{a}{m}\right)^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}\left(m^{\mu}\omega'\left(v-\frac{a}{m}\right)^{\mu};p\right) f(v)\,dv \\
&\quad \le \frac{1}{2}\left(f(a) - m^2 f\left(\frac{a}{m^2}\right)\right) \int_{\frac{a+mb}{2}}^{mb} (mb-u)^{\alpha} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}\big(\omega'(mb-u)^{\mu};p\big)\,dt \\
&\quad + m^{\alpha+1}\left(f(b) + mf\left(\frac{a}{m^2}\right)\right) \\
&\quad \times \int_{\frac{a}{m}}^{\frac{a+mb}{2m}} \left(v-\frac{a}{m}\right)^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}\left(m^{\mu}\omega'\left(v-\frac{a}{m}\right)^{\mu};p\right)\,dt.
\end{aligned} \tag{28}$$

By using (3), (4), and (6), we get the second inequality of (23). □

If we take $m = 1$ in (23), then we get the following Hadamard type inequality for a convex function.

Corollary 2.12 *Let $f : [a,b] \subseteq [0,\infty) \to \mathbb{R}$ be a function such that $f \in L_1[a,b]$ with $a < b$. If f is convex on $[a,b]$, then the following inequalities for extended generalized fractional integral operators hold*:

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right) C_{\alpha,(\frac{a+b}{2})^+}(b;p) \\
&\quad \le \left[\left(\epsilon_{\mu,\alpha,l,\omega',(\frac{a+b}{2})^+}^{\gamma,\delta,k,c} f\right)(b;p) + \left(\epsilon_{\mu,\alpha,l,\omega',(\frac{a+b}{2})_-}^{\gamma,\delta,k,c} f\right)(a;p)\right] \\
&\quad \le \frac{f(a)+f(b)}{2} C_{\alpha,(\frac{a+b}{2})^-}(a;p),
\end{aligned} \tag{29}$$

where $\omega' = \frac{2^{\mu}\omega}{(b-a)^{\mu}}$.

Remark 2.13 In Theorem 2.11.

(i) If we put $p = 0$, then we get [6, Theorem 3.10].

(ii) If we put $\omega = p = 0$, $m = 1$, and $\alpha = 1$, then we get the classical Hadamard inequality.

Remark 2.14 In Corollary 2.12.

(i) If we put $p = 0$, then we get [1, Theorem 3.9].

(ii) If we put $\omega = p = 0$ and $\alpha = 1$, then we get the classical Hadamard inequality.

(iii) If we take $\omega = p = 0$, then we get [14, Theorem 4].

3 Concluding remarks

We have investigated more general fractional integral inequalities. By selecting specific values of parameters quite interesting results can be obtained. For example selecting $p = 0$, fractional integral inequalities for fractional integral operators defined by Salim and Faraj in [12], selecting $l = \delta = 1$, fractional integral inequalities for fractional integral operators defined by Rahman et al. in [11], selecting $p = 0$ and $l = \delta = 1$, fractional integral inequalities for fractional integral operators defined by Shukla and Prajapati in [15] (see also [16]), selecting $p = 0$ and $l = \delta = k = 1$, fractional integral inequalities for fractional integral operators defined by Prabhakar in [10], selecting $p = \omega = 0$, fractional integral inequalities for Riemann–Liouville fractional integral operators.

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