Yang and Lu *Journal of Inequalities and Applications* (2018) 2018:183
https://doi.org/10.1186/s13660-018-1786-8

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH | Open Access

Some generalizations of inequalities for sector matrices

Chaojun Yang[1*] and Fangyan Lu[1]

*Correspondence:
cjyangmath@163.com
[1]Department of Mathematics, Soochow University, Suzhou, P.R. China

Abstract

In this paper, we generalize some Schatten p-norm inequalities for accretive-dissipative matrices obtained by Kittaneh and Sakkijha. Moreover, we present some inequalities for sector matrices.

MSC: Primary 15A45; secondary 15A15

Keywords: Sector matrix; Schatten p-norm; Accretive-dissipative matrix; Inequality

1 Introduction

Throughout this paper, let $\mathbb{M}_n$ be the set of all $n \times n$ complex matrices. We denote by I_n the identity matrix in $\mathbb{M}_n$. For two Hermitian matrices $A, B \in \mathbb{M}_n$, we use $A \geq B$ ($B \leq A$) to mean that $A - B$ is a positive semidefinite matrix. A matrix $A \in \mathbb{M}_n$ is called accretive-dissipative if in its Cartesian (or Toeptliz) decomposition, $A = \Re(A) + i\Im(A)$, the matrices $\Re(A)$ and $\Im(A)$ are positive semidefinite, where $\Re(A) = \frac{A+A^*}{2}$, $\Im(A) = \frac{A-A^*}{2i}$.

Let $|||\cdot|||$ denote any unitarily invariant norm on $\mathbb{M}_n$. Note that tr is the usual trace functional. For $p > 0$ and $A \in \mathbb{M}_n$, let $\|A\|_p = (\sum_{j=1}^n s_j^p(A))^{\frac{1}{p}}$, where $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$ are the singular values of A. Thus, $\|A\|_p = (\operatorname{tr}|A|^p)^{\frac{1}{p}}$. For $p \geq 1$, this is the Schatten p-norm of A. For more information about the Schatten p-norms, see [1, p. 92].

A real-valued continuous function f on an interval I is called matrix concave of order n if $f(\alpha A + (1-\alpha)B) \geq \alpha f(A) + (1-\alpha)f(B)$ for any two Hermitian matrices $A, B \in \mathbb{M}_n$ with spectrum in I and all $\alpha \in [0,1]$. Furthermore, f is called operator concave if f is matrix concave for all n.

The numerical range of $A \in \mathbb{M}_n$ is defined by

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}.$$

For $\alpha \in [0, \frac{\pi}{2})$, S_α denotes the sector in the complex plane as follows:

$$S_\alpha = \{z \in \mathbb{C} : \Re z \geq 0, |\Im z| \leq (\Re z)\tan\alpha\}.$$

Clearly, A is positive semidefinite if and only if $W(A) \subset S_0$, and if $W(A), W(B) \subset S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$, then $W(A+B) \subset S_\alpha$. As $0 \notin S_\alpha$, if $W(A) \subset S_\alpha$, then A is nonsingular.

In [7], Kittaneh and Sakkijha gave the following Schatten-p norm inequalities involving sums of accretive-dissipative matrices.

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Theorem 1.1 *Let $S, T \in \mathbb{M}_n$ be accretive-dissipative. Then*

$$2^{\frac{-p}{2}}\left(\|S\|_p^p + \|T\|_p^p\right) \leq \|S+T\|_p^p \leq 2^{\frac{3}{2}p-1}\left(\|S\|_p^p + \|T\|_p^p\right) \quad \text{for } p \geq 1.$$

In [5], Garg and Aujla showed the following inequalities:

$$\prod_{j=1}^{k} s_j\left(|A+B|^r\right) \leq \prod_{j=1}^{k} s_j\left(I_n + |A|^r\right) \prod_{j=1}^{k} s_j\left(I_n + |B|^r\right) \quad \text{for } 1 \leq k \leq n, 1 \leq r \leq 2; \tag{1}$$

and

$$\prod_{j=1}^{k} s_j\left(I_n + f\left(|A+B|\right)\right) \leq \prod_{j=1}^{k} s_j\left(I_n + f\left(|A|\right)\right) \prod_{j=1}^{k} s_j\left(I_n + f\left(|B|\right)\right) \quad \text{for } 1 \leq k \leq n, \tag{2}$$

where $A, B \in \mathbb{M}_n$ and $f : [0,\infty) \to [0,\infty)$ is an operator concave function.

By letting $A, B \geq 0$, $r = 1$ and $f(X) = X$ for any $X \in \mathbb{M}_n$ in (1) and (2), we have

$$\prod_{j=1}^{k} s_j(A+B) \leq \prod_{j=1}^{k} s_j(I_n + A) \prod_{j=1}^{k} s_j(I_n + B) \quad \text{for } 1 \leq k \leq n; \tag{3}$$

and

$$\prod_{j=1}^{k} s_j(I_n + A + B) \leq \prod_{j=1}^{k} s_j(I_n + A) \prod_{j=1}^{k} s_j(I_n + B) \quad \text{for } 1 \leq k \leq n. \tag{4}$$

In this paper, we give a generalization of Theorem 1.1. Moreover, we present some inequalities for sector matrices based on (3) and (4) which remove the absolute values in (1) and (2) from the right-hand side.

2 Main results

Before we give the main results, let us present the following lemmas that will be useful later.

Lemma 2.1 ([2, 11]) *Let $A_1, \ldots, A_n \in \mathbb{M}_n$ be positive semidefinite. Then*

$$\sum_{j=1}^{n} \|A_j\|_p^p \leq \left\|\sum_{j=1}^{n} A_j\right\|_p^p \leq n^{p-1} \sum_{j=1}^{n} \|A_j\|_p^p \quad \text{for } p \geq 1.$$

Lemma 2.2 ([3]) *Let $A, B \in \mathbb{M}_n$ be positive semidefinite. Then*

$$\|A + iB\|_p \leq \|A + B\|_p \leq \sqrt{2}\|A + iB\|_p \quad \text{for } p \geq 1.$$

Our first main result is a generalization of Theorem 1.1.

Theorem 2.3 *Let $A_1, \ldots, A_n \in \mathbb{M}_n$ be accretive-dissipative. Then*

$$2^{\frac{-p}{2}} \sum_{j=1}^{n} \|A_j\|_p^p \leq \left\|\sum_{j=1}^{n} A_j\right\|_p^p \leq \frac{(2n^2)^{\frac{p}{2}}}{n} \sum_{j=1}^{n} \|A_j\|_p^p \quad \text{for } p \geq 1.$$

Proof Let $A_j = B_j + iC_j$ be the Cartesian decompositions of A_j, $j = 1, \dots, n$. Then we have

$$\begin{aligned}
\left\| \sum_{j=1}^{n} A_j \right\|_p^p &= \left\| \sum_{j=1}^{n} (B_j + iC_j) \right\|_p^p \\
&= \left\| \sum_{j=1}^{n} B_j + i \sum_{j=1}^{n} C_j \right\|_p^p \\
&\geq 2^{\frac{-p}{2}} \left\| \sum_{j=1}^{n} B_j + \sum_{j=1}^{n} C_j \right\|_p^p \quad \text{(by Lemma 2.2)} \\
&= 2^{\frac{-p}{2}} \left\| \sum_{j=1}^{n} (B_j + C_j) \right\|_p^p \\
&\geq 2^{\frac{-p}{2}} \sum_{j=1}^{n} \| B_j + C_j \|_p^p \quad \text{(by Lemma 2.1)} \\
&\geq 2^{\frac{-p}{2}} \sum_{j=1}^{n} \| B_j + iC_j \|_p^p \quad \text{(by Lemma 2.2)} \\
&= 2^{\frac{-p}{2}} \sum_{j=1}^{n} \| A_j \|_p^p,
\end{aligned}$$

which proves the first inequality.

To prove the second inequality, compute

$$\begin{aligned}
\left\| \sum_{j=1}^{n} A_j \right\|_p^p &= \left\| \sum_{j=1}^{n} (B_j + iC_j) \right\|_p^p \\
&= \left\| \sum_{j=1}^{n} B_j + i \sum_{j=1}^{n} C_j \right\|_p^p \\
&\leq \left\| \sum_{j=1}^{n} B_j + \sum_{j=1}^{n} C_j \right\|_p^p \quad \text{(by Lemma 2.2)} \\
&= \left\| \sum_{j=1}^{n} (B_j + C_j) \right\|_p^p \\
&\leq n^{p-1} \sum_{j=1}^{n} \| B_j + C_j \|_p^p \quad \text{(by Lemma 2.1)} \\
&\leq n^{p-1} 2^{\frac{p}{2}} \sum_{j=1}^{n} \| B_j + iC_j \|_p^p \quad \text{(by Lemma 2.2)} \\
&= \frac{(2n^2)^{\frac{p}{2}}}{n} \sum_{j=1}^{n} \| A_j \|_p^p,
\end{aligned}$$

which completes the proof. □

Remark 2.4 By letting $n = 2$ in Theorem 2.3, we thus get Theorem 1.1.

The following lemma is the well-known Fan–Hoffman inequality.

Lemma 2.5 ([12, p. 63]) *Let $A \in \mathbb{M}_n$. Then*

$$\lambda_j(\Re A) \le s_j(A),$$

where $\lambda_j(\cdot)$ denotes the jth largest eigenvalue.

In [4], Drury and Lin presented a reverse version of Lemma 2.5 as follows.

Lemma 2.6 *Let $A \in \mathbb{M}_n$ be such that $W(A) \subset S_\alpha$. Then*

$$s_j(A) \le \sec^2(\alpha)\lambda_j(\Re A),$$

where $\lambda_j(\cdot)$ denotes the jth largest eigenvalue.

Theorem 2.7 *Let $A, B \in \mathbb{M}_n$ be such that $W(A), W(B) \subset S_\alpha$. Then*

$$\prod_{j=1}^{k} s_j(A+B) \le \sec^{2k}(\alpha) \prod_{j=1}^{k} s_j(I_n + A) \prod_{j=1}^{k} s_j(I_n + B) \quad \text{for } 1 \le k \le n; \tag{5}$$

and

$$\prod_{j=1}^{k} s_j(I_n + A + B) \le \sec^{2k}(\alpha) \prod_{j=1}^{k} s_j(I_n + A) \prod_{j=1}^{k} s_j(I_n + B) \quad \text{for } 1 \le k \le n. \tag{6}$$

Proof We have

$$\begin{aligned}
\prod_{j=1}^{k} s_j(A+B) &\le \sec^{2k}(\alpha) \prod_{j=1}^{k} s_j\big(\Re(A+B)\big) \quad (\text{by Lemma 2.6}) \\
&= \sec^{2k}(\alpha) \prod_{j=1}^{k} s_j\big(\Re(A) + \Re(B)\big) \\
&\le \sec^{2k}(\alpha) \prod_{j=1}^{k} s_j\big(I_n + \Re(A)\big) \prod_{j=1}^{k} s_j\big(I_n + \Re(B)\big) \quad \big(\text{by (3)}\big) \\
&= \sec^{2k}(\alpha) \prod_{j=1}^{k} s_j\big(\Re(I_n + A)\big) \prod_{j=1}^{k} s_j\big(\Re(I_n + B)\big) \\
&\le \sec^{2k}(\alpha) \prod_{j=1}^{k} s_j(I_n + A) \prod_{j=1}^{k} s_j(I_n + B) \quad (\text{by Lemma 2.5})
\end{aligned}$$

which proves (5).

To prove (6), compute

$$\prod_{j=1}^{k} s_j(I_n + A + B) \le \sec^{2k}(\alpha) \prod_{j=1}^{k} s_j\big(\Re(I_n + A + B)\big) \quad (\text{by Lemma 2.6})$$

$$
\begin{aligned}
&= \sec^{2k}(\alpha)\prod_{j=1}^{k} s_j\big(I_n + \Re(A) + \Re(B)\big) \\
&\le \sec^{2k}(\alpha)\prod_{j=1}^{k} s_j\big(I_n + \Re(A)\big)\prod_{j=1}^{k} s_j\big(I_n + \Re(B)\big) \quad \big(\text{by } (4)\big) \\
&= \sec^{2k}(\alpha)\prod_{j=1}^{k} s_j\big(\Re(I_n + A)\big)\prod_{j=1}^{k} s_j\big(\Re(I_n + B)\big) \\
&\le \sec^{2k}(\alpha)\prod_{j=1}^{k} s_j(I_n + A)\prod_{j=1}^{k} s_j(I_n + B), \quad (\text{by Lemma 2.5})
\end{aligned}
$$

which completes the proof. □

Corollary 2.8 *Let $A, B \in \mathbb{M}_n$ be such that $W(A), W(B) \subset S_\alpha$. Then, for all unitarily invariant norms $|||\cdot|||$ on $\mathbb{M}_n$,*

$$|||A + B||| \le \sec^2(\alpha)|||I_n + A|||\,|||I_n + B|||;$$

and

$$|||I_n + A + B||| \le \sec^2(\alpha)|||I_n + A|||\,|||I_n + B|||.$$

Proof From (5) and (6), we obtain

$$\prod_{j=1}^{k} s_j(A + B) \le \prod_{j=1}^{k} s_j\big(\sec(\alpha)(I_n + A)\big)s_j\big(\sec(\alpha)(I_n + B)\big) \quad \text{for } 1 \le k \le n;$$

and

$$\prod_{j=1}^{k} s_j(I_n + A + B) \le \prod_{j=1}^{k} s_j\big(\sec(\alpha)(I_n + A)\big)s_j\big(\sec(\alpha)(I_n + B)\big) \quad \text{for } 1 \le k \le n,$$

which is equivalent to the following inequalities:

$$\prod_{j=1}^{k} s_j^{\frac{1}{2}}(A + B) \le \prod_{j=1}^{k} s_j^{\frac{1}{2}}\big(\sec(\alpha)(I_n + A)\big)s_j^{\frac{1}{2}}\big(\sec(\alpha)(I_n + B)\big) \quad \text{for } 1 \le k \le n;$$

and

$$\prod_{j=1}^{k} s_j^{\frac{1}{2}}(I_n + A + B) \le \prod_{j=1}^{k} s_j^{\frac{1}{2}}\big(\sec(\alpha)(I_n + A)\big)s_j^{\frac{1}{2}}\big(\sec(\alpha)(I_n + B)\big) \quad \text{for } 1 \le k \le n.$$

By the property of majorization [1, p. 42], we have

$$\sum_{j=1}^{k} s_j^{\frac{1}{2}}(A + B) \le \sum_{j=1}^{k} s_j^{\frac{1}{2}}\big(\sec(\alpha)(I_n + A)\big)s_j^{\frac{1}{2}}\big(\sec(\alpha)(I_n + B)\big) \quad \text{for } 1 \le k \le n;$$

and

$$\sum_{j=1}^{k} s_j^{\frac{1}{2}}(I_n + A + B) \le \sum_{j=1}^{k} s_j^{\frac{1}{2}}\big(\sec(\alpha)(I_n + A)\big) s_j^{\frac{1}{2}}\big(\sec(\alpha)(I_n + B)\big) \quad \text{for } 1 \le k \le n.$$

Now, by the Cauchy–Schwarz inequality, we obtain

$$\sum_{j=1}^{k} s_j^{\frac{1}{2}}(A + B) \le \left(\sum_{j=1}^{k} s_j\big(\sec(\alpha)(I_n + A)\big)\right)^{\frac{1}{2}} \left(\sum_{j=1}^{k} s_j\big(\sec(\alpha)(I_n + B)\big)\right)^{\frac{1}{2}} \quad \text{for } 1 \le k \le n;$$

and

$$\sum_{j=1}^{k} s_j^{\frac{1}{2}}(I_n + A + B) \le \left(\sum_{j=1}^{k} s_j\big(\sec(\alpha)(I_n + A)\big)\right)^{\frac{1}{2}} \left(\sum_{j=1}^{k} s_j\big(\sec(\alpha)(I_n + B)\big)\right)^{\frac{1}{2}}$$
$$\text{for } 1 \le k \le n,$$

which is equivalent to the following inequalities:

$$\big\| |A + B|^{\frac{1}{2}} \big\|_k^2 \le \big\|\sec(\alpha)(I_n + A)\big\|_k \big\|\sec(\alpha)(I_n + B)\big\|_k;$$

and

$$\big\| |I_n + A + B|^{\frac{1}{2}} \big\|_k^2 \le \big\|\sec(\alpha)(I_n + A)\big\|_k \big\|\sec(\alpha)(I_n + B)\big\|_k.$$

By the generalization of Fan dominance theorem [8], we have

$$\big|\big|\big| |A + B|^{\frac{1}{2}} \big|\big|\big|^2 \le \big|\big|\big|\sec(\alpha)(I_n + A)\big|\big|\big| \big|\big|\big|\sec(\alpha)(I_n + B)\big|\big|\big|; \tag{7}$$

and

$$\big|\big|\big| |I_n + A + B|^{\frac{1}{2}} \big|\big|\big|^2 \le \big|\big|\big|\sec(\alpha)(I_n + A)\big|\big|\big| \big|\big|\big|\sec(\alpha)(I_n + B)\big|\big|\big|. \tag{8}$$

Let $A + B = U|A + B|$, $I_n + A + B = V|I_n + A + B|$ be the polar decomposition of $A + B$ and $I_n + A + B$, respectively, where U and V are unitary matrices. Thus, by (7), we have

$$\begin{aligned} |||A + B||| &= \big|\big|\big| U|A + B| \big|\big|\big| \\ &= \big|\big|\big| \big(|A + B|^{\frac{1}{2}}\big)^2 \big|\big|\big| \\ &\le \big|\big|\big| |A + B|^{\frac{1}{2}} \big|\big|\big|^2 \\ &\le \big|\big|\big|\sec(\alpha)(I_n + A)\big|\big|\big| \big|\big|\big|\sec(\alpha)(I_n + B)\big|\big|\big| \\ &= \sec(\alpha)^2 \big|\big|\big|(I_n + A)\big|\big|\big| \big|\big|\big|(I_n + B)\big|\big|\big|. \end{aligned}$$

Similarly, by (8) we have

$$|||I_n + A + B||| \le \sec^2(\alpha) |||I_n + A||| \, |||I_n + B|||,$$

which completes the proof. □

Taking $k = n$ in Theorem 2.7, we obtain the following corollary.

Corollary 2.9 *Let $A, B \in \mathbb{M}_n$ be such that $W(A), W(B) \subset S_\alpha$. Then*

$$\left|\det(A + B)\right| \le \sec^{2n}(\alpha)\left|\det(I_n + A)\right|\left|\det(I_n + B)\right|;$$

and

$$\left|\det(I_n + A + B)\right| \le \sec^{2n}(\alpha)\left|\det(I_n + A)\right|\left|\det(I_n + B)\right|.$$

Lemma 2.10 ([13]) *Let $A \in \mathbb{M}_n$ be such that $W(A) \subset S_\alpha$. Then, for all unitarily invariant norms $|||\cdot|||$ on $\mathbb{M}_n$,*

$$|||A||| \le \sec(\alpha)\left|\left|\left|\Re(A)\right|\right|\right|.$$

Next we give an improvement of Corollary 2.8.

Theorem 2.11 *Let $A, B \in \mathbb{M}_n$ be such that $W(A), W(B) \subset S_\alpha$. Then, for all unitarily invariant norms $|||\cdot|||$ on $\mathbb{M}_n$,*

$$|||A + B||| \le \sec(\alpha)|||I_n + A|||\,|||I_n + B|||; \tag{9}$$

and

$$|||I_n + A + B||| \le \sec(\alpha)|||I_n + A|||\,|||I_n + B|||. \tag{10}$$

Proof By (3), (4), and the proof of Corollary 2.8, we obtain

$$\left|\left|\left|\Re(A) + \Re(B)\right|\right|\right| \le \left|\left|\left|I_n + \Re(A)\right|\right|\right|\,\left|\left|\left|I_n + \Re(B)\right|\right|\right|; \tag{11}$$

and

$$\left|\left|\left|I_n + \Re(A) + \Re(B)\right|\right|\right| \le \left|\left|\left|I_n + \Re(A)\right|\right|\right|\,\left|\left|\left|I_n + \Re(B)\right|\right|\right|. \tag{12}$$

Hence

$$\begin{aligned} |||A + B||| &\le \sec(\alpha)\left|\left|\left|\Re(A + B)\right|\right|\right| \quad \text{(by Lemma 2.10)} \\ &= \sec(\alpha)\left|\left|\left|\Re(A) + \Re(B)\right|\right|\right| \\ &\le \sec(\alpha)\left|\left|\left|I_n + \Re(A)\right|\right|\right|\,\left|\left|\left|I_n + \Re(B)\right|\right|\right| \quad \big(\text{by (11)}\big) \\ &= \sec(\alpha)\left|\left|\left|\Re(I_n + A)\right|\right|\right|\,\left|\left|\left|\Re(I_n + B)\right|\right|\right| \\ &\le \sec(\alpha)|||I_n + A|||\,|||I_n + B|||, \end{aligned}$$

which proves (9).

To prove (10), compute

$$\begin{aligned}|||I_n + A + B||| &\le \sec(\alpha)\left|\left|\left|\Re(I_n + A + B)\right|\right|\right| \quad \text{(by Lemma 2.10)}\\ &= \sec(\alpha)\left|\left|\left|I_n + \Re(A) + \Re(B)\right|\right|\right|\\ &\le \sec(\alpha)\left|\left|\left|I_n + \Re(A)\right|\right|\right|\left|\left|\left|I_n + \Re(B)\right|\right|\right| \quad \big(\text{by (12)}\big)\\ &= \sec(\alpha)\left|\left|\left|\Re(I_n + A)\right|\right|\right|\left|\left|\left|\Re(I_n + B)\right|\right|\right|\\ &\le \sec(\alpha)|||I_n + A|||\,|||I_n + B|||,\end{aligned}$$

which completes the proof. □

The following lemma can be obtained by Lemma 2.5.

Lemma 2.12 ([6, p. 510]) *If $A \in \mathbb{M}_n$ has positive definite real part, then*

$$\det(\Re A) \le |\det A|.$$

Lemma 2.13 ([10]) *Let $A \in \mathbb{M}_n$ be such that $W(A) \subset S_\alpha$. Then*

$$\sec^n(\alpha)\det(\Re A) \ge |\det A|.$$

Now we are ready to give an improvement of Corollary 2.9.

Theorem 2.14 *Let $A, B \in \mathbb{M}_n$ be such that $W(A), W(B) \subset S_\alpha$. Then*

$$\left|\det(A + B)\right| \le \sec^n(\alpha)\left|\det(I_n + A)\right|\left|\det(I_n + B)\right|; \tag{13}$$

and

$$\left|\det(I_n + A + B)\right| \le \sec^n(\alpha)\left|\det(I_n + A)\right|\left|\det(I_n + B)\right|. \tag{14}$$

Proof Letting $k = n$ in (3) and (4), we have

$$\det\big(\Re(A) + \Re(B)\big) \le \det\big(I_n + \Re(A)\big)\det\big(I_n + \Re(B)\big); \tag{15}$$

and

$$\det\big(I_n + \Re(A) + \Re(B)\big) \le \det\big(I_n + \Re(A)\big)\det\big(I_n + \Re(B)\big). \tag{16}$$

Thus

$$\begin{aligned}\left|\det(A + B)\right| &\le \sec^n(\alpha)\det\big(\Re(A + B)\big) \quad \text{(by Lemma 2.13)}\\ &= \sec^n(\alpha)\det\big(\Re(A) + \Re(B)\big)\\ &\le \sec^n(\alpha)\det\big(I_n + \Re(A)\big)\det\big(I_n + \Re(B)\big) \quad \big(\text{by (15)}\big)\end{aligned}$$

$$= \sec^n(\alpha)\det\big(\Re(I_n + A)\big)\det\big(\Re(I_n + B)\big)$$
$$\le \sec^n(\alpha)\big|\det(I_n + A)\big|\big|\det(I_n + B)\big| \quad \text{(by Lemma 2.12)}$$

which proves (13).

To prove (14), compute

$$\begin{aligned}\big|\det(I_n + A + B)\big| &\le \sec^n(\alpha)\det\big(\Re(I_n + A + B)\big) \quad \text{(by Lemma 2.13)}\\ &= \sec^n(\alpha)\det\big(I_n + \Re(A) + \Re(B)\big)\\ &\le \sec^n(\alpha)\det\big(I_n + \Re(A)\big)\det\big(I_n + \Re(B)\big) \quad \big(\text{by (16)}\big)\\ &= \sec^n(\alpha)\det\big(\Re(I_n + A)\big)\det\big(\Re(I_n + B)\big)\\ &\le \sec^n(\alpha)\big|\det(I_n + A)\big|\big|\det(I_n + B)\big| \quad \text{(by Lemma 2.12)}\end{aligned}$$

which completes the proof. □

Lemma 2.15 ([9]) *Let $A, B \in \mathbb{M}_n$ be positive semidefinite. Then*

$$\big|\det(A + iB)\big| \le \det(A + B) \le 2^{\frac{n}{2}}\big|\det(A + iB)\big|.$$

We remark that (2) extends the well-known Rotfel'd inequality:

$$\det\big(I_n + \mu|A + B|^p\big) \le \det\big(I_n + \mu|A|^p\big)\det\big(I_n + \mu|B|^p\big) \quad \text{for } \mu > 0, 0 \le p \le 1. \tag{17}$$

Finally, we present two inequalities for accretive-dissipative matrices.

Theorem 2.16 *Let $A, B \in \mathbb{M}_n$ be accretive-dissipative and $\mu > 0$. Then*

$$\big|\det(A + B)\big| \le 2^n\big|\det(I_n + A)\big|\big|\det(I_n + B)\big|; \tag{18}$$

and

$$\big|\det\big(I_n + \mu(A + B)\big)\big| \le 2^n\big|\det(I_n + \mu A)\big|\big|\det(I_n + \mu B)\big|. \tag{19}$$

In particular,

$$\big|\det(I_n + A + B)\big| \le 2^n\big|\det(I_n + A)\big|\big|\det(I_n + B)\big|.$$

Proof Let $A = A_1 + iA_2$ and $B = B_1 + iB_2$ be the Cartesian decompositions of A and B. By (3) and (17), we obtain

$$\det(A_1 + A_2 + B_1 + B_2) \le \det(I_n + A_1 + A_2)\det(I_n + B_1 + B_2); \tag{20}$$

and

$$\det\big(I_n + \mu(A_1 + A_2 + B_1 + B_2)\big) \le \det\big(I_n + \mu(A_1 + A_2)\big)\det\big(I_n + \mu(B_1 + B_2)\big). \tag{21}$$

Hence

$$\begin{aligned}
\left|\det(A+B)\right| &= \left|\det(A_1+iA_2+B_1+iB_2)\right| \\
&= \left|\det\big((A_1+B_1)+i(A_2+B_2)\big)\right| \\
&\le \det(A_1+B_1+A_2+B_2) \quad \text{(by Lemma 2.15)} \\
&= \det(A_1+A_2+B_1+B_2) \\
&\le \det(I_n+A_1+A_2)\det(I_n+B_1+B_2) \quad \big(\text{by (20)}\big) \\
&\le 2^n\left|\det(I_n+A_1+iA_2)\right|\left|\det(I_n+B_1+iB_2)\right| \quad \text{(by Lemma 2.15)} \\
&= 2^n\left|\det(I_n+A)\right|\left|\det(I_n+B)\right|,
\end{aligned}$$

which proves (18).

To prove (19), compute

$$\begin{aligned}
&\left|\det\big(I_n+\mu(A+B)\big)\right| \\
&\quad= \left|\det\big(I_n+\mu(A_1+iA_2+B_1+iB_2)\big)\right| \\
&\quad= \left|\det\big(I_n+\mu(A_1+B_1)+\mu i(A_2+B_2)\big)\right| \\
&\quad\le \det\big(I_n+\mu(A_1+B_1+A_2+B_2)\big) \quad \text{(by Lemma 2.15)} \\
&\quad= \det\big(I_n+\mu(A_1+A_2+B_1+B_2)\big) \\
&\quad\le \det\big(I_n+\mu(A_1+A_2)\big)\det\big(I_n+\mu(B_1+B_2)\big) \quad \big(\text{by (21)}\big) \\
&\quad\le 2^n\left|\det\big(I_n+\mu(A_1+iA_2)\big)\right|\left|\det\big(I_n+\mu(B_1+iB_2)\big)\right| \quad \text{(by Lemma 2.15)} \\
&\quad= 2^n\left|\det(I_n+\mu A)\right|\left|\det(I_n+\mu B)\right|,
\end{aligned}$$

which completes the proof. □

Funding
This research is supported by the National Natural Science Foundation of P.R. China (No. 11571247).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed almost the same amount of work to the manuscript. All authors read and approved the final manuscript.

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Received: 27 April 2018 Accepted: 18 July 2018 Published online: 23 July 2018

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