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Hermite–Hadamard type inequalities for fractional integrals via Green’s function

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Abstract

In the article, we establish the left Riemann–Liouville fractional Hermite–Hadamard type inequalities and the generalized Hermite–Hadamard type inequalities by using Green’s function and Jensen’s inequality, and present several new Hermite–Hadamard type inequalities for a class of convex as well as monotone functions.

MSC: 26D15

Keywords: Hermite–Hadamard inequality; convex function; Green’s function

1 Introduction

Convexity plays an important role in all the fields of pure and applied mathematics [1–12]. Many remarkable inequalities have been obtained in the literature by using convexity [13–22]. Among the inequalities, the most extensively and intensively attractive inequality in the last decades is the well-known Hermite–Hadamard inequality. This interesting result was obtained by Hermite and Hadamard independently, and it provides an equivalence with the convexity property. This inequality reads as follows: if the function $\psi : [\alpha_1, \alpha_2] \rightarrow \mathbb{R}$ is convex on $[\alpha_1, \alpha_2]$, then

$$\psi\left(\frac{\alpha_1 + \alpha_2}{2}\right) \leq \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \psi(x) dx \leq \frac{\psi(\alpha_1) + \psi(\alpha_2)}{2}. \quad (1.1)$$

If ψ is a concave function, then the inequalities in (1.1) will hold in reverse directions. The Hermite–Hadamard inequality gives an upper as well as lower estimations for the integral mean of any convex function defined on a closed and bounded interval which involves the endpoints and midpoint of the domain of the function. Also (1.1) provides the necessary and sufficient condition for the function to be convex. There are several applications of this inequality in the geometry of Banach spaces and nonlinear analysis [23, 24]. Some peculiar convex functions can be used in (1.1) to obtain classical inequalities for means. For some comprehensive surveys on various generalizations and developments of (1.1), we recommend [25]. Due to the great importance of this inequality, in the recent years many remarkable varieties of generalizations, refinements, extensions and different versions of Hermite–Hadamard inequality for different classes of convexity, such as preinvex, s -convex, harmonic convex, $\alpha(x)$ -convex, superquadratic, and co-ordinate convex functions, have been studied in the literature. Also there have been a large number of research

papers published on this subject, for interested readers we recommend to read the papers [26–37] and some of the references therein.

The following definitions for the left and right side Riemann–Liouville fractional integrals are well known in the literature.

Let $b_1, b_2 \in \mathbb{R}$ with $b_1 < b_2$ and $\psi \in L[b_1, b_2]$. Then the left and right Riemann–Liouville fractional integrals $J_{b_1+}^\alpha \psi$ and $J_{b_2-}^\alpha \psi$ of order $\alpha > 0$ are defined by

$$J_{b_1+}^\alpha \psi(x) = \frac{1}{\Gamma(\alpha)} \int_{b_1}^x (x-t)^{\alpha-1} \psi(t) dt, \quad x > b_1,$$

and

$$J_{b_2-}^\alpha \psi(x) = \frac{1}{\Gamma(\alpha)} \int_x^{b_2} (t-x)^{\alpha-1} \psi(t) dt, \quad x < b_2,$$

respectively, where $\Gamma(\alpha)$ is the gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$.

In [38], Sarikaya et al. established the Hermite–Hadamard type inequality for fractional integral as follows.

Theorem 1.1 *Let $\psi : [b_1, b_2] \rightarrow \mathbb{R}$ be a positive function with $0 \leq b_1 < b_2$, $\alpha > 0$, and $\psi \in L[b_1, b_2]$. If ψ is convex on $[b_1, b_2]$, then one has*

$$\psi\left(\frac{b_1 + b_2}{2}\right) \leq \frac{\Gamma(\alpha + 1)}{2(b_2 - b_1)^\alpha} [J_{b_1+}^\alpha \psi(b_2)] + J_{b_2-}^\alpha \psi(b_1) \leq \frac{\psi(b_1) + \psi(b_2)}{2}.$$

Remark 1.2 In Theorem 1.1, it is not necessary to suppose that ψ is a positive function and b_1, b_2 are positive real numbers. From the definition of left and right Riemann–Liouville fractional integrals, we clearly see that b_1 and b_2 can be any real numbers such that $b_1 < b_2$.

The main purpose of this paper is to give a new method to derive the left Riemann–Liouville fractional Hermite–Hadamard type inequalities as given in [39]. In this method we use Green’s function and obtain identities for the difference of the left Riemann–Liouville fractional Hermite–Hadamard inequality, and then we prove that these identities are non-negative. As a consequence, these inequalities provide the generalized Hermite–Hadamard inequality. Also, by using these identities for the class of convex, concave, and monotone functions, we obtain new Hermite–Hadamard type inequalities.

2 Main results

Let $b_1 < b_2$. Then the following four new Green’s functions $G_i : [b_1, b_2] \times [b_1, b_2] \mapsto \mathbb{R}$ ($i = 1, 2, 3, 4$) are defined by Mehmood et al. in [40]:

$$G_1(\lambda, \mu) = \begin{cases} b_1 - \mu, & b_1 \leq \mu \leq \lambda, \\ b_1 - \lambda, & \lambda \leq \mu \leq b_2, \end{cases} \tag{2.1}$$

$$G_2(\lambda, \mu) = \begin{cases} \lambda - b_2, & b_1 \leq \mu \leq \lambda, \\ \mu - b_2, & \lambda \leq \mu \leq b_2, \end{cases} \tag{2.2}$$

$$G_3(\lambda, \mu) = \begin{cases} \lambda - b_1, & b_1 \leq \mu \leq \lambda, \\ \mu - b_1, & \lambda \leq \mu \leq b_2, \end{cases} \tag{2.3}$$

$$G_4(\lambda, \mu) = \begin{cases} b_2 - \mu, & b_1 \leq \mu \leq \lambda, \\ b_2 - \lambda, & \lambda \leq \mu \leq b_2. \end{cases} \tag{2.4}$$

In [40], the authors established the following Lemma 2.1, which will be used to establish our main results.

Lemma 2.1 (see [40, Lemma 1]) *Let $b_1 < b_2$ and $G = G_1$ be the Green's function defined by (2.1). Then*

$$\psi(x) = \psi(b_1) + (x - b_1)\psi'(b_2) + \int_{b_1}^{b_2} G(x, \mu)\psi''(\mu) d\mu \tag{2.5}$$

if $\psi \in C^2([b_1, b_2])$.

Theorem 2.2 *Let $\psi \in C^2([b_1, b_2])$ be a convex function. Then the double inequality*

$$\psi\left(\frac{\alpha b_1 + b_2}{\alpha + 1}\right) \leq \frac{\Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} J_{b_1^+}^\alpha \psi(b_2) \leq \frac{\alpha\psi(b_1) + \psi(b_2)}{\alpha + 1} \tag{2.6}$$

holds for any $\alpha > 0$.

Proof Substituting $x = \frac{\alpha b_1 + b_2}{\alpha + 1}$ in (2.5), we get

$$\begin{aligned} \psi\left(\frac{\alpha b_1 + b_2}{\alpha + 1}\right) &= \psi(b_1) + \left(\frac{\alpha b_1 + b_2}{\alpha + 1} - b_1\right)\psi'(b_2) \\ &\quad + \int_{b_1}^{b_2} G\left(\frac{\alpha b_1 + b_2}{\alpha + 1}, \mu\right)\psi''(\mu) d\mu, \\ \psi\left(\frac{\alpha b_1 + b_2}{\alpha + 1}\right) &= \psi(b_1) + \frac{(b_2 - b_1)\psi'(b_2)}{\alpha + 1} + \int_{b_1}^{b_2} G\left(\frac{\alpha b_1 + b_2}{\alpha + 1}, \mu\right)\psi''(\mu) d\mu. \end{aligned} \tag{2.7}$$

Now, multiplying both sides of (2.5) by $\frac{\alpha(b_2-x)^{\alpha-1}}{(b_2-b_1)^\alpha}$ and integrating, we get

$$\begin{aligned} &\frac{\alpha}{(b_2 - b_1)^\alpha} \int_{b_1}^{b_2} (b_2 - x)^{\alpha-1} \psi(x) dx \\ &= \frac{\alpha}{(b_2 - b_1)^\alpha} \left[\int_{b_1}^{b_2} (b_2 - x)^{\alpha-1} \psi(b_1) dx + \int_{b_1}^{b_2} (b_2 - x)^{\alpha-1} (x - b_1)\psi'(b_2) dx \right. \\ &\quad \left. + \int_{b_1}^{b_2} \int_{b_1}^{b_2} (b_2 - x)^{\alpha-1} G(x, \mu)\psi''(\mu) d\mu dx \right] \\ &= \frac{\alpha}{(b_2 - b_1)^\alpha} \left[\psi(b_1) \frac{(b_2 - b_1)^\alpha}{\alpha} + \frac{\psi'(b_2)(b_2 - b_1)^{\alpha+1}}{\alpha(\alpha + 1)} \right. \\ &\quad \left. + \int_{b_1}^{b_2} \int_{b_1}^{b_2} (b_2 - x)^{\alpha-1} G(x, \mu)\psi''(\mu) d\mu dx \right], \\ &\frac{\Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} J_{b_1^+}^\alpha \psi(b_2) = \psi(b_1) + \frac{(b_2 - b_1)\psi'(b_2)}{\alpha + 1} \\ &\quad + \frac{\alpha}{(b_2 - b_1)^\alpha} \int_{b_1}^{b_2} \int_{b_1}^{b_2} (b_2 - x)^{\alpha-1} G(x, \mu)\psi''(\mu) d\mu dx. \end{aligned} \tag{2.8}$$

Subtracting (2.8) from (2.7), we obtain

$$\begin{aligned} & \psi\left(\frac{\alpha b_1 + b_2}{\alpha + 1}\right) - \frac{\Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} J_{b_1^+}^\alpha \psi(b_2) \\ &= \int_{b_1}^{b_2} \left[G\left(\frac{\alpha b_1 + b_2}{\alpha + 1}, \mu\right) - \frac{\alpha}{(b_2 - b_1)^\alpha} \int_{b_1}^{b_2} (b_2 - x)^{\alpha-1} G(x, \mu) dx \right] \psi''(\mu) d\mu. \end{aligned} \tag{2.9}$$

Clearly

$$\int_{b_1}^{b_2} (b_2 - x)^{\alpha-1} G(x, \mu) dx = \frac{1}{\alpha(\alpha + 1)} [(b_2 - \mu)^{\alpha+1} - (b_2 - b_1)^{\alpha+1}]. \tag{2.10}$$

Also, since

$$G\left(\frac{\alpha b_1 + b_2}{\alpha + 1}, \mu\right) = \begin{cases} b_1 - \mu, & b_1 \leq \mu \leq \frac{\alpha b_1 + b_2}{\alpha + 1}, \\ \frac{b_1 - b_2}{\alpha + 1}, & \frac{\alpha b_1 + b_2}{\alpha + 1} \leq \mu \leq b_2. \end{cases} \tag{2.11}$$

Therefore, if $b_1 \leq \mu \leq \frac{\alpha b_1 + b_2}{\alpha + 1}$, then from (2.10) and (2.11) we have

$$\begin{aligned} & G\left(\frac{\alpha b_1 + b_2}{\alpha + 1}, \mu\right) - \frac{\alpha}{(b_2 - b_1)^\alpha} \int_{b_1}^{b_2} (b_2 - x)^{\alpha-1} G(x, \mu) dx \\ &= b_1 - \mu - \frac{(b_2 - \mu)^{\alpha+1} - (b_2 - b_1)^{\alpha+1}}{(b_2 - b_1)^\alpha (\alpha + 1)}. \end{aligned}$$

Now, let

$$f(\mu) = b_1 - \mu - \frac{(b_2 - \mu)^{\alpha+1} - (b_2 - b_1)^{\alpha+1}}{(b_2 - b_1)^\alpha (\alpha + 1)}.$$

Then

$$f'(\mu) = -1 + \frac{(b_2 - \mu)^\alpha}{(b_2 - b_1)^\alpha} \leq 0,$$

which shows that f is decreasing and $f(b_1) \leq 0$, hence $f(\mu) \leq 0$ for all $\mu \in [b_1, \frac{\alpha b_1 + b_2}{\alpha + 1}]$, i.e.,

$$G\left(\frac{\alpha b_1 + b_2}{\alpha + 1}, \mu\right) - \frac{\alpha}{(b_2 - b_1)^\alpha} \int_{b_1}^{b_2} (b_2 - x)^{\alpha-1} G(x, \mu) dx \leq 0. \tag{2.12}$$

If $\frac{\alpha b_1 + b_2}{\alpha + 1} \leq \mu \leq b_2$, then making use of (2.10) and (2.11) we get

$$\begin{aligned}
 & G\left(\frac{\alpha b_1 + b_2}{\alpha + 1}, \mu\right) - \frac{\alpha}{(b_2 - b_1)^\alpha} \int_{b_1}^{b_2} (b_2 - x)^{\alpha-1} G(x, \mu) dx \\
 &= \frac{b_1 - b_2}{\alpha + 1} - \frac{\alpha}{(b_2 - b_1)^\alpha} \left[\frac{(b_2 - \mu)^{\alpha+1} - (b_2 - b_1)^{\alpha+1}}{\alpha(\alpha + 1)} \right] \\
 &= -\frac{(b_2 - b_1)}{(\alpha + 1)} - \frac{(b_2 - \mu)^{\alpha+1} - (b_2 - b_1)^{\alpha+1}}{(b_2 - b_1)^\alpha (\alpha + 1)} \\
 &= \frac{-(b_2 - b_1)^{\alpha+1} - (b_2 - \mu)^{\alpha+1} + (b_2 - b_1)^{\alpha+1}}{(\alpha + 1)(b_2 - b_1)^\alpha} \\
 &= \frac{-(b_2 - \mu)^{\alpha+1}}{(\alpha + 1)(b_2 - b_1)^\alpha} \leq 0.
 \end{aligned} \tag{2.13}$$

Since ψ is convex, therefore, $\psi''(\mu) \geq 0$ and so by using (2.12) and (2.13) in (2.9), we deduce

$$\psi\left(\frac{\alpha b_1 + b_2}{\alpha + 1}\right) \leq \frac{\Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} J_{b_1^+}^\alpha \psi(b_2),$$

which is the first inequality of (2.6).

Next, we prove the second inequality of (2.6). Let $x = b_2$ in (2.5), then we have

$$\psi(b_2) = \psi(b_1) + (b_2 - b_1)\psi'(b_2) + \int_{b_1}^{b_2} G(b_2, \mu)\psi''(\mu) d\mu.$$

Adding $\alpha\psi(b_1)$ on both sides and then dividing by $(\alpha + 1)$, we get

$$\frac{\alpha\psi(b_1) + \psi(b_2)}{\alpha + 1} = \psi(b_1) + \frac{(b_2 - b_1)\psi'(b_2)}{\alpha + 1} + \frac{1}{\alpha + 1} \int_{b_1}^{b_2} G(b_2, \mu)\psi''(\mu) d\mu. \tag{2.14}$$

Subtracting (2.8) from (2.14), we obtain

$$\begin{aligned}
 & \frac{\alpha\psi(b_1) + \psi(b_2)}{\alpha + 1} - \frac{\Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} J_{b_1^+}^\alpha \psi(b_2) \\
 &= \int_{b_1}^{b_2} \left[\frac{G(b_2, \mu)}{\alpha + 1} - \frac{\alpha}{(b_2 - b_1)^\alpha} \int_{b_1}^{b_2} (b_2 - x)^{\alpha-1} G(x, \mu) dx \right] \psi''(\mu) d\mu.
 \end{aligned} \tag{2.15}$$

Using the Green’s function and (2.10), we obtain

$$\begin{aligned}
 & \frac{G(b_2, \mu)}{\alpha + 1} - \frac{\alpha}{(b_2 - b_1)^\alpha} \left[\frac{(b_2 - \mu)^{\alpha+1} - (b_2 - b_1)^{\alpha+1}}{\alpha(\alpha + 1)} \right] \\
 &= \frac{b_1 - \mu}{\alpha + 1} - \frac{(b_2 - \mu)^{\alpha+1} - (b_2 - b_1)^{\alpha+1}}{(\alpha + 1)(b_2 - b_1)^\alpha} \\
 &= \frac{(b_1 - \mu)(b_2 - b_1)^\alpha + (b_2 - b_1)^{\alpha+1} - (b_2 - \mu)^{\alpha+1}}{(\alpha + 1)(b_2 - b_1)^\alpha} \\
 &= \frac{(b_2 - b_1)^\alpha (b_2 - \mu) - (b_2 - \mu)^{\alpha+1}}{(\alpha + 1)(b_2 - b_1)^\alpha} \\
 &= \frac{(b_2 - \mu)[(b_2 - b_1)^\alpha - (b_2 - \mu)^\alpha]}{(\alpha + 1)(b_2 - b_1)^\alpha} \geq 0
 \end{aligned} \tag{2.16}$$

for all $b_1 \leq \mu \leq b_2$.

Now, using the convexity of ψ and (2.16) in (2.15), we get

$$\frac{\alpha\psi(b_1) + \psi(b_2)}{\alpha + 1} \geq \frac{\Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} J_{b_1^+}^\alpha \psi(b_2),$$

which is the second inequality of (2.6). □

Next, we present new Hermite–Hadamard type inequalities for the class of monotone and convex functions.

Theorem 2.3 *Let $\psi \in C^2([b_1, b_2])$ and $\alpha > 0$. Then the following statements are true:*

(i) *If $|\psi''|$ is an increasing function, then*

$$\left| \frac{\alpha\psi(b_1) + \psi(b_2)}{\alpha + 1} - \frac{\Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} J_{b_1^+}^\alpha \psi(b_2) \right| \leq \frac{|\psi''(b_2)|\alpha(b_2 - b_1)^2}{2(\alpha + 1)(\alpha + 2)}. \tag{2.17}$$

(ii) *If $|\psi''|$ is a decreasing function, then*

$$\left| \frac{\alpha\psi(b_1) + \psi(b_2)}{\alpha + 1} - \frac{\Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} J_{b_1^+}^\alpha \psi(b_2) \right| \leq \frac{|\psi''(b_1)|\alpha(b_2 - b_1)^2}{2(\alpha + 1)(\alpha + 2)}.$$

(iii) *If $|\psi''|$ is a convex function, then*

$$\left| \frac{\alpha\psi(b_1) + \psi(b_2)}{\alpha + 1} - \frac{\Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} J_{b_1^+}^\alpha \psi(b_2) \right| \leq \frac{\max\{|\psi''(b_1)|, |\psi''(b_2)|\}\alpha(b_2 - b_1)^2}{2(\alpha + 1)(\alpha + 2)}.$$

Proof (i) It follows from (2.15) that

$$\begin{aligned} & \left| \frac{\alpha\psi(b_1) + \psi(b_2)}{\alpha + 1} - \frac{\Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} J_{b_1^+}^\alpha \psi(b_2) \right| \\ & \leq \frac{1}{(\alpha + 1)(b_2 - b_1)^\alpha} \int_{b_1}^{b_2} [(b_2 - b_1)^\alpha(b_2 - \mu) - (b_2 - \mu)^{\alpha+1}] |\psi''(\mu)| d\mu. \end{aligned} \tag{2.18}$$

Since $(b_2 - b_1)^\alpha(b_2 - \mu) - (b_2 - \mu)^{\alpha+1} \geq 0$ and $|\psi''|$ is an increasing function, therefore we have

$$\begin{aligned} & \left| \frac{\alpha\psi(b_1) + \psi(b_2)}{\alpha + 1} - \frac{\Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} J_{b_1^+}^\alpha \psi(b_2) \right| \\ & \leq \frac{|\psi''(b_2)|}{(\alpha + 1)(b_2 - b_1)^\alpha} \int_{b_1}^{b_2} [(b_2 - b_1)^\alpha(b_2 - \mu) - (b_2 - \mu)^{\alpha+1}] d\mu \\ & = \frac{|\psi''(b_2)|}{(\alpha + 1)(b_2 - b_1)^\alpha} \left[(b_2 - b_1)^\alpha \frac{(b_2 - b_1)^2}{2} - \frac{(b_2 - b_1)^{\alpha+2}}{\alpha + 2} \right] \\ & = \frac{|\psi''(b_2)|\alpha(b_2 - b_1)^2}{2(\alpha + 1)(\alpha + 2)}, \end{aligned}$$

which is inequality (2.17).

Part (ii) can be proved in a similar way, we omit the details.

For part (iii), making use of (2.18) and the fact that every convex function ψ defined on the interval $[b_1, b_2]$ is bounded above by $\max\{\psi(b_1), \psi(b_2)\}$, we get

$$\begin{aligned} & \left| \frac{\alpha\psi(b_1) + \psi(b_2)}{\alpha + 1} - \frac{\Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} J_{b_1^+}^\alpha \psi(b_2) \right| \\ & \leq \frac{\max\{|\psi''(b_1)|, |\psi''(b_2)|\}}{(\alpha + 1)(b_2 - b_1)^\alpha} \int_{b_1}^{b_2} [(b_2 - b_1)^\alpha (b_2 - \mu) - (b_2 - \mu)^{\alpha+1}] d\mu \\ & = \frac{\max\{|\psi''(b_1)|, |\psi''(b_2)|\} \alpha (b_2 - b_1)^2}{2(\alpha + 1)(\alpha + 2)}. \end{aligned} \quad \square$$

Remark 2.4 Let $\alpha = 1$. Then Theorem 2.3 leads to

$$\begin{aligned} & \left| \frac{\psi(b_1) + \psi(b_2)}{2} - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \psi(t) dt \right| \leq \frac{|\psi''(b_2)|(b_2 - b_1)^2}{12}, \\ & \left| \frac{\psi(b_1) + \psi(b_2)}{2} - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \psi(t) dt \right| \leq \frac{|\psi''(b_1)|(b_2 - b_1)^2}{12}, \\ & \left| \frac{\psi(b_1) + \psi(b_2)}{2} - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \psi(t) dt \right| \leq \frac{\max\{|\psi''(b_1)|, |\psi''(b_2)|\} (b_2 - b_1)^2}{12}. \end{aligned}$$

Theorem 2.5 Let $\psi \in C^2([b_1, b_2])$ and $\alpha > 0$. Then the following statements are true:

(i) If $|\psi''|$ is an increasing function, then

$$\begin{aligned} & \left| \psi\left(\frac{\alpha b_1 + b_2}{\alpha + 1}\right) - \frac{\Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} J_{b_1^+}^\alpha \psi(b_2) \right| \\ & \leq \frac{(b_2 - b_1)^2}{(\alpha + 1)^{\alpha+3}(\alpha + 2)} \\ & \quad \times \left[\left| \psi''\left(\frac{\alpha b_1 + b_2}{\alpha + 1}\right) \right| \left(\frac{\alpha[(\alpha + 1)^{\alpha+1} - 2\alpha^{\alpha+1}]}{2} \right) + |\psi''(b_2)| \alpha^{\alpha+2} \right]. \end{aligned}$$

(ii) If $|\psi''|$ is a decreasing function, then

$$\begin{aligned} & \left| \psi\left(\frac{\alpha b_1 + b_2}{\alpha + 1}\right) - \frac{\Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} J_{b_1^+}^\alpha \psi(b_2) \right| \\ & \leq \frac{(b_2 - b_1)^2}{(\alpha + 1)^{\alpha+3}(\alpha + 2)} \\ & \quad \times \left[|\psi''(b_1)| \left(\frac{\alpha[(\alpha + 1)^{\alpha+1} - 2\alpha^{\alpha+1}]}{2} \right) + \left| \psi''\left(\frac{\alpha b_1 + b_2}{\alpha + 1}\right) \right| \alpha^{\alpha+2} \right]. \end{aligned}$$

(iii) If $|\psi''|$ is a convex function, then

$$\begin{aligned} & \left| \psi\left(\frac{\alpha b_1 + b_2}{\alpha + 1}\right) - \frac{\Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} J_{b_1^+}^\alpha \psi(b_2) \right| \\ & \leq \frac{(b_2 - b_1)^2}{(\alpha + 1)^{\alpha+3}(\alpha + 2)} \\ & \quad \times \left[\max\left\{ \left| \psi''(b_1) \right|, \left| \psi''\left(\frac{\alpha b_1 + b_2}{\alpha + 1}\right) \right| \right\} \left(\frac{\alpha[(\alpha + 1)^{\alpha+1} - 2\alpha^{\alpha+1}]}{2} \right) \right. \\ & \quad \left. + \max\left\{ \left| \psi''\left(\frac{\alpha b_1 + b_2}{\alpha + 1}\right) \right|, \left| \psi''(b_2) \right| \right\} \alpha^{\alpha+2} \right]. \end{aligned}$$

Proof (i) It follows from (2.9) that

$$\begin{aligned}
 & \psi\left(\frac{\alpha b_1 + b_2}{\alpha + 1}\right) - \frac{\Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} J_{b_1^+}^\alpha \psi(b_2) \\
 &= \int_{b_1}^{\frac{\alpha b_1 + b_2}{\alpha + 1}} \left[G\left(\frac{\alpha b_1 + b_2}{\alpha + 1}, \mu\right) - \frac{\alpha}{(b_2 - b_1)^\alpha} \int_{b_1}^{b_2} (b_2 - x)^{\alpha-1} G(x, \mu) dx \right] \psi''(\mu) d\mu \\
 & \quad + \int_{\frac{\alpha b_1 + b_2}{\alpha + 1}}^{b_2} \left[G\left(\frac{\alpha b_1 + b_2}{\alpha + 1}, \mu\right) - \frac{\alpha}{(b_2 - b_1)^\alpha} \int_{b_1}^{b_2} (b_2 - x)^{\alpha-1} G(x, \mu) dx \right] \psi''(\mu) d\mu \\
 &= -\frac{1}{(\alpha + 1)(b_2 - b_1)^\alpha} \left[\int_{b_1}^{\frac{\alpha b_1 + b_2}{\alpha + 1}} \{(b_2 - \mu)^{\alpha+1} - (b_2 - b_1)^{\alpha+1} \right. \\
 & \quad \left. - (b_1 - \mu)(b_2 - b_1)^\alpha(\alpha + 1)\} \psi''(\mu) d\mu + \int_{\frac{\alpha b_1 + b_2}{\alpha + 1}}^{b_2} (b_2 - \mu)^{\alpha+1} \psi''(\mu) d\mu \right]. \tag{2.19}
 \end{aligned}$$

Taking the absolute function and using the triangular inequality, we get

$$\begin{aligned}
 & \left| \psi\left(\frac{\alpha b_1 + b_2}{\alpha + 1}\right) - \frac{\Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} J_{b_1^+}^\alpha \psi(b_2) \right| \\
 & \leq \frac{1}{(\alpha + 1)(b_2 - b_1)^\alpha} \left[\int_{b_1}^{\frac{\alpha b_1 + b_2}{\alpha + 1}} \{(b_2 - \mu)^{\alpha+1} - (b_2 - b_1)^{\alpha+1} \right. \\
 & \quad \left. - (b_1 - \mu)(b_2 - b_1)^\alpha(\alpha + 1)\} |\psi''(\mu)| d\mu + \int_{\frac{\alpha b_1 + b_2}{\alpha + 1}}^{b_2} (b_2 - \mu)^{\alpha+1} |\psi''(\mu)| d\mu \right] \\
 & \leq \frac{1}{(\alpha + 1)(b_2 - b_1)^\alpha} \left[\left| \psi''\left(\frac{\alpha b_1 + b_2}{\alpha + 1}\right) \right| \int_{b_1}^{\frac{\alpha b_1 + b_2}{\alpha + 1}} \{(b_2 - \mu)^{\alpha+1} - (b_2 - b_1)^{\alpha+1} \right. \\
 & \quad \left. - (b_1 - \mu)(b_2 - b_1)^\alpha(\alpha + 1)\} d\mu + |\psi''(b_2)| \int_{\frac{\alpha b_1 + b_2}{\alpha + 1}}^{b_2} (b_2 - \mu)^{\alpha+1} d\mu \right] \\
 & = \frac{1}{(\alpha + 1)(b_2 - b_1)^\alpha} \left[\left| \psi''\left(\frac{\alpha b_1 + b_2}{\alpha + 1}\right) \right| (b_2 - b_1)^{\alpha+2} \left\{ \frac{-\alpha^{\alpha+2}}{(\alpha + 1)^{\alpha+2}(\alpha + 2)} \right. \right. \\
 & \quad \left. \left. + \frac{1}{(\alpha + 2)} - \frac{1}{(\alpha + 1)} + \frac{1}{2(\alpha + 1)} \right\} + |\psi''(b_2)| (b_2 - b_1)^{\alpha+2} \left\{ \frac{\alpha^{\alpha+2}}{(\alpha + 1)^{\alpha+2}(\alpha + 2)} \right\} \right] \\
 & = \frac{(b_2 - b_1)^2}{(\alpha + 1)^{\alpha+3}(\alpha + 2)} \left[\left| \psi''\left(\frac{\alpha b_1 + b_2}{\alpha + 1}\right) \right| \left\{ \frac{\alpha[(\alpha + 1)^{\alpha+1} - 2\alpha^{\alpha+1}]}{2} \right\} + |\psi''(b_2)| \alpha^{\alpha+2} \right].
 \end{aligned}$$

Part (ii) can be proved by using the same procedure.

Next, we prove part (iii). We clearly see that

$$\begin{aligned}
 & \left| \psi\left(\frac{\alpha b_1 + b_2}{\alpha + 1}\right) - \frac{\Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} J_{b_1^+}^\alpha \psi(b_2) \right| \\
 & \leq \frac{1}{(\alpha + 1)(b_2 - b_1)^\alpha} \left[\int_{b_1}^{\frac{\alpha b_1 + b_2}{\alpha + 1}} \{(b_2 - \mu)^{\alpha+1} - (b_2 - b_1)^{\alpha+1} \right. \\
 & \quad \left. - (b_1 - \mu)(b_2 - b_1)^\alpha(\alpha + 1)\} |\psi''(\mu)| d\mu + \int_{\frac{\alpha b_1 + b_2}{\alpha + 1}}^{b_2} (b_2 - \mu)^{\alpha+1} |\psi''(\mu)| d\mu \right].
 \end{aligned}$$

Since every convex function ψ defined on an interval $[b_1, b_2]$ is bounded above by $\max\{\psi(b_1), \psi(b_2)\}$. Therefore, we have

$$\begin{aligned} & \left| \psi\left(\frac{\alpha b_1 + b_2}{\alpha + 1}\right) - \frac{\Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} J_{b_1^+}^\alpha \psi(b_2) \right| \\ & \leq \frac{1}{(\alpha + 1)(b_2 - b_1)^\alpha} \left[\max\left\{ \left| \psi''(b_1) \right|, \left| \psi''\left(\frac{\alpha b_1 + b_2}{\alpha + 1}\right) \right| \right\}, \right. \\ & \int_{b_1}^{\frac{\alpha b_1 + b_2}{\alpha + 1}} \left\{ (b_2 - \mu)^{\alpha + 1} - (b_2 - b_1)^{\alpha + 1} - (b_1 - \mu)(b_2 - b_1)^\alpha (\alpha + 1) \right\} d\mu \\ & \quad \left. + \max\left\{ \left| \psi''\left(\frac{\alpha b_1 + b_2}{\alpha + 1}\right) \right|, \left| \psi''(b_2) \right| \right\} \int_{\frac{\alpha b_1 + b_2}{\alpha + 1}}^{b_2} (b_2 - \mu)^{\alpha + 1} d\mu \right] \\ & = \frac{(b_2 - b_1)^2}{(\alpha + 1)^{\alpha + 3} (\alpha + 2)} \left[\max\left\{ \left| \psi''(b_1) \right|, \left| \psi''\left(\frac{\alpha b_1 + b_2}{\alpha + 1}\right) \right| \right\} \left(\frac{\alpha [(\alpha + 1)^{\alpha + 1} - 2\alpha^{\alpha + 1}]}{2} \right) \right. \\ & \quad \left. + \max\left\{ \left| \psi''\left(\frac{\alpha b_1 + b_2}{\alpha + 1}\right) \right|, \left| \psi''(b_2) \right| \right\} (\alpha^{\alpha + 2}) \right], \end{aligned}$$

which is our required inequality. □

Remark 2.6 Let $\alpha = 1$. Then Theorem 2.5 leads to

$$\begin{aligned} & \left| \psi\left(\frac{b_1 + b_2}{2}\right) - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \psi(t) dt \right| \leq \frac{(b_2 - b_1)^2}{48} \left[\left| \psi''\left(\frac{b_1 + b_2}{2}\right) \right| + \left| \psi''(b_2) \right| \right], \\ & \left| \psi\left(\frac{b_1 + b_2}{2}\right) - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \psi(t) dt \right| \leq \frac{(b_2 - b_1)^2}{48} \left[\left| \psi''(b_1) \right| + \left| \psi''\left(\frac{b_1 + b_2}{2}\right) \right| \right], \\ & \left| \psi\left(\frac{b_1 + b_2}{2}\right) - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \psi(t) dt \right| \\ & \leq \frac{(b_2 - b_1)^2}{48} \left[\max\left\{ \left| \psi''(b_1) \right|, \left| \psi''\left(\frac{b_1 + b_2}{2}\right) \right| \right\} + \max\left\{ \left| \psi''\left(\frac{b_1 + b_2}{2}\right) \right|, \left| \psi''(b_2) \right| \right\} \right]. \end{aligned}$$

Theorem 2.7 Let $\psi \in C^2([b_1, b_2])$ and $|\psi''|$ be a convex function. Then the inequality

$$\begin{aligned} & \left| \psi\left(\frac{\alpha b_1 + b_2}{\alpha + 1}\right) - \frac{\Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} J_{b_1^+}^\alpha \psi(b_2) \right| \\ & \leq \frac{(b_2 - b_1)^2}{6(\alpha + 1)^3 (\alpha + 3)} \left[\left| \psi''(b_1) \right| (9\alpha^2 + 23\alpha + 12) + \frac{\left| \psi''(b_2) \right| (7\alpha^2 + 17\alpha + 12)}{\alpha + 2} \right] \end{aligned}$$

holds for any $\alpha > 0$.

Proof It follows from (2.19) that

$$\begin{aligned} & \psi\left(\frac{\alpha b_1 + b_2}{\alpha + 1}\right) - \frac{\Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} J_{b_1^+}^\alpha \psi(b_2) \\ & = \frac{1}{(\alpha + 1)(b_2 - b_1)^\alpha} \left[\int_{b_1}^{\frac{\alpha b_1 + b_2}{\alpha + 1}} \left\{ (b_1 - \mu)(b_2 - b_1)^\alpha (\alpha + 1) \right. \right. \\ & \quad \left. \left. - (b_2 - \mu)^{\alpha + 1} + (b_2 - b_1)^{\alpha + 1} \right\} \psi''(\mu) d\mu - \int_{\frac{\alpha b_1 + b_2}{\alpha + 1}}^{b_2} (b_2 - \mu)^{\alpha + 1} \psi''(\mu) d\mu \right]. \end{aligned}$$

Let $t \in [0, 1]$ and $\mu = tb_1 + (1 - t)b_2$. Then

$$\begin{aligned}
 & \psi\left(\frac{\alpha b_1 + b_2}{\alpha + 1}\right) - \frac{\Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} J_{b_1^+}^\alpha \psi(b_2) \\
 &= \frac{1}{(\alpha + 1)(b_2 - b_1)^\alpha} \left[\int_1^{\frac{\alpha}{\alpha+1}} \{ [b_1 - tb_1 - (1 - t)b_2](b_2 - b_1)^\alpha (\alpha + 1) \right. \\
 &\quad \left. - [b_2 - tb_1 - (1 - t)b_2]^{\alpha+1} + (b_2 - b_1)^{\alpha+1} \} \psi''(tb_1 + (1 - t)b_2)(b_1 - b_2) dt \right. \\
 &\quad \left. - \int_{\frac{\alpha}{\alpha+1}}^0 [b_2 - tb_1 - (1 - t)b_2]^{\alpha+1} \psi''(tb_1 + (1 - t)b_2)(b_1 - b_2) dt \right] \\
 &= \frac{-(b_2 - b_1)^{\alpha+2}}{(\alpha + 1)(b_2 - b_1)^\alpha} \left[\int_1^{\frac{\alpha}{\alpha+1}} \{(-1 + t)(\alpha + 1) + 1\} \psi''(tb_1 + (1 - t)b_2) dt \right. \\
 &\quad \left. - \int_1^{\frac{\alpha}{\alpha+1}} t^{\alpha+1} \psi''(tb_1 + (1 - t)b_2) dt + \int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha+1} \psi''(tb_1 + (1 - t)b_2) dt \right] \\
 &= \frac{(b_2 - b_1)^2}{\alpha + 1} \left[\int_{\frac{\alpha}{\alpha+1}}^1 (\alpha t + t - \alpha) \psi''(tb_1 + (1 - t)b_2) dt \right. \\
 &\quad \left. - \int_0^1 t^{\alpha+1} \psi''(tb_1 + (1 - t)b_2) dt \right]. \tag{2.20}
 \end{aligned}$$

Taking absolute on both sides and using the convexity of $|\psi''|$, we get

$$\begin{aligned}
 & \left| \psi\left(\frac{\alpha b_1 + b_2}{\alpha + 1}\right) - \frac{\Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} J_{b_1^+}^\alpha \psi(b_2) \right| \\
 & \leq \frac{(b_2 - b_1)^2}{\alpha + 1} \left[\int_{\frac{\alpha}{\alpha+1}}^1 (\alpha t + t - \alpha) |\psi''(tb_1 + (1 - t)b_2)| dt \right. \\
 &\quad \left. + \int_0^1 t^{\alpha+1} |\psi''(tb_1 + (1 - t)b_2)| dt \right] \\
 & \leq \frac{(b_2 - b_1)^2}{\alpha + 1} \left[\int_{\frac{\alpha}{\alpha+1}}^1 (t(\alpha + 1) - \alpha) [t|\psi''(b_1)| + (1 - t)|\psi''(b_2)|] dt \right. \\
 &\quad \left. + \int_0^1 t^{\alpha+1} [t|\psi''(b_1)| + (1 - t)|\psi''(b_2)|] dt \right] \\
 & = \frac{(b_2 - b_1)^2}{\alpha + 1} \left[\frac{|\psi''(b_1)|(3\alpha + 2)}{6(\alpha + 1)^2} + \frac{|\psi''(b_2)|}{6(\alpha + 1)^2} + \frac{|\psi''(b_1)|}{\alpha + 3} + \frac{|\psi''(b_2)|}{(\alpha + 2)(\alpha + 3)} \right] \\
 & = \frac{(b_2 - b_1)^2}{6(\alpha + 1)^3(\alpha + 3)} \left[|\psi''(b_1)|(9\alpha^2 + 23\alpha + 12) + \frac{|\psi''(b_2)|(7\alpha^2 + 17\alpha + 12)}{\alpha + 2} \right],
 \end{aligned}$$

which completes the proof. □

Remark 2.8 In Theorem 2.7, if we take $\alpha = 1$, then we obtain

$$\left| \psi\left(\frac{b_1 + b_2}{2}\right) - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \psi(t) dt \right| \leq \frac{(b_2 - b_1)^2}{48} [11|\psi''(b_1)| + 3|\psi''(b_2)|].$$

Theorem 2.9 *Let $\psi \in C^2([b_1, b_2])$ and $|\psi''|$ be a convex function. Then the inequality*

$$\begin{aligned} & \left| \frac{\alpha\psi(b_1) + \psi(b_2)}{\alpha + 1} - \frac{\Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} J_{b_1^+}^\alpha \psi(b_2) \right| \\ & \leq \frac{\alpha(b_2 - b_1)^2}{3(\alpha + 1)(\alpha + 3)} \left[|\psi''(b_1)| + |\psi''(b_2)| \left(\frac{\alpha + 5}{2(\alpha + 2)} \right) \right] \end{aligned}$$

holds for any $\alpha > 0$.

Proof From (2.15), we have

$$\begin{aligned} & \frac{\alpha\psi(b_1) + \psi(b_2)}{\alpha + 1} - \frac{\Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} J_{b_1^+}^\alpha \psi(b_2) \\ & = \frac{1}{(\alpha + 1)(b_2 - b_1)^\alpha} \int_{b_1}^{b_2} [(b_2 - b_1)^\alpha (b_2 - \mu) - (b_2 - \mu)^{\alpha+1}] \psi''(\mu) d\mu. \end{aligned}$$

Let $t \in [0, 1]$ and $\mu = tb_1 + (1 - t)b_2$. Then we obtain

$$\begin{aligned} & \frac{\alpha\psi(b_1) + \psi(b_2)}{\alpha + 1} - \frac{\Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} J_{b_1^+}^\alpha \psi(b_2) \\ & = \frac{b_2 - b_1}{(\alpha + 1)(b_2 - b_1)^\alpha} \\ & \quad \times \int_0^1 \{ (b_2 - b_1)^\alpha [t(b_2 - b_1)] - [t(b_2 - b_1)]^{\alpha+1} \} \psi''(tb_1 + (1 - t)b_2) dt \\ & = \frac{(b_2 - b_1)^2}{\alpha + 1} \int_0^1 (t - t^{\alpha+1}) \psi''(tb_1 + (1 - t)b_2) dt. \end{aligned} \tag{2.21}$$

Taking absolute on both sides and using the convexity of $|\psi''|$, we get

$$\begin{aligned} & \left| \frac{\alpha\psi(b_1) + \psi(b_2)}{\alpha + 1} - \frac{\Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} J_{b_1^+}^\alpha \psi(b_2) \right| \\ & \leq \frac{(b_2 - b_1)^2}{\alpha + 1} \int_0^1 (t - t^{\alpha+1}) |\psi''(tb_1 + (1 - t)b_2)| dt \\ & \leq \frac{(b_2 - b_1)^2}{\alpha + 1} \int_0^1 (t - t^{\alpha+1}) [t|\psi''(b_1)| + (1 - t)|\psi''(b_2)|] dt \\ & = \frac{(b_2 - b_1)^2}{\alpha + 1} \left[|\psi''(b_1)| \left(\frac{\alpha}{3(\alpha + 3)} \right) + |\psi''(b_2)| \left(\frac{\alpha^2 + 5\alpha}{6(\alpha + 2)(\alpha + 3)} \right) \right] \\ & = \frac{\alpha(b_2 - b_1)^2}{3(\alpha + 1)(\alpha + 3)} \left[|\psi''(b_1)| + |\psi''(b_2)| \left(\frac{\alpha + 5}{2(\alpha + 2)} \right) \right], \end{aligned}$$

which is our required inequality. □

Remark 2.10 In Theorem 2.9, if we take $\alpha = 1$, then we obtain

$$\left| \frac{\psi(b_1) + \psi(b_2)}{2} - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \psi(t) dt \right| \leq \frac{(b_2 - b_1)^2}{24} [|\psi''(b_1)| + |\psi''(b_2)|].$$

Theorem 2.11 *Let $\psi \in C^2([b_1, b_2])$ and $|\psi''|$ be a concave function. Then*

$$\begin{aligned} & \left| \psi\left(\frac{\alpha b_1 + b_2}{\alpha + 1}\right) - \frac{\Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} J_{b_1^+}^\alpha \psi(b_2) \right| \\ & \leq \frac{(b_2 - b_1)^2}{\alpha + 1} \left[\frac{1}{2(\alpha + 1)} \left| \psi''\left(\frac{3\alpha b_1 + 2b_1 + b_2}{3(\alpha + 1)}\right) \right| + \frac{1}{\alpha + 2} \left| \psi''\left(\frac{\alpha b_1 + 2b_1 + b_2}{\alpha + 3}\right) \right| \right] \end{aligned}$$

for any $\alpha > 0$.

Proof It follows from (2.20) that

$$\begin{aligned} & \psi\left(\frac{\alpha b_1 + b_2}{\alpha + 1}\right) - \frac{\Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} J_{b_1^+}^\alpha \psi(b_2) \\ & = \frac{(b_2 - b_1)^2}{\alpha + 1} \\ & \quad \times \left[\int_{\frac{\alpha}{\alpha+1}}^1 \{t(\alpha + 1) - \alpha\} \psi''(tb_1 + (1 - t)b_2) dt - \int_0^1 t^{\alpha+1} \psi''(tb_1 + (1 - t)b_2) dt \right]. \end{aligned}$$

Taking absolute on both sides and using Jensen’s integral inequality, we get

$$\begin{aligned} & \left| \psi\left(\frac{\alpha b_1 + b_2}{\alpha + 1}\right) - \frac{\Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} J_{b_1^+}^\alpha \psi(b_2) \right| \\ & \leq \frac{(b_2 - b_1)^2}{\alpha + 1} \left[\int_{\frac{\alpha}{\alpha+1}}^1 \{t(\alpha + 1) - \alpha\} dt \left| \psi''\left(\frac{\int_{\frac{\alpha}{\alpha+1}}^1 \{t(\alpha + 1) - \alpha\} \{tb_1 + (1 - t)b_2\} dt}{\int_{\frac{\alpha}{\alpha+1}}^1 \{t(\alpha + 1) - \alpha\} dt}\right) \right| \right. \\ & \quad \left. + \int_0^1 t^{\alpha+1} dt \left| \psi''\left(\frac{\int_0^1 t^{\alpha+1} \{tb_1 + (1 - t)b_2\} dt}{\int_0^1 t^{\alpha+1} dt}\right) \right| \right] \\ & = \frac{(b_2 - b_1)^2}{\alpha + 1} \left[\frac{1}{2(\alpha + 1)} \left| \psi''\left(\frac{3\alpha b_1 + 2b_1 + b_2}{3(\alpha + 1)}\right) \right| + \frac{1}{\alpha + 2} \left| \psi''\left(\frac{\alpha b_1 + 2b_1 + b_2}{\alpha + 3}\right) \right| \right]. \end{aligned}$$

□

Remark 2.12 In Theorem 2.11, if we take $\alpha = 1$, then we obtain

$$\begin{aligned} & \left| \psi\left(\frac{b_1 + b_2}{2}\right) - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \psi(t) dt \right| \\ & \leq \frac{(b_2 - b_1)^2}{2} \left[\frac{1}{4} \left| \psi''\left(\frac{5b_1 + b_2}{6}\right) \right| + \frac{1}{3} \left| \psi''\left(\frac{3b_1 + b_2}{4}\right) \right| \right]. \end{aligned}$$

Theorem 2.13 *Assume that $\psi \in C^2([b_1, b_2])$ and $|\psi''|$ is a concave function. Then, for any $\alpha > 0$, we have the inequality*

$$\begin{aligned} & \left| \frac{\alpha \psi(b_1) + \psi(b_2)}{\alpha + 1} - \frac{\Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} J_{b_1^+}^\alpha \psi(b_2) \right| \\ & \leq \frac{\alpha(b_2 - b_1)^2}{2(\alpha + 1)(\alpha + 2)} \left| \psi''\left(\frac{2\alpha b_1 + \alpha b_2 + 4b_1 + 5b_2}{3(\alpha + 3)}\right) \right|. \end{aligned}$$

Proof From (2.21) one has

$$\begin{aligned} & \frac{\alpha\psi(b_1) + \psi(b_2)}{\alpha + 1} - \frac{\Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} J_{b_1^+}^\alpha \psi(b_2) \\ &= \frac{(b_2 - b_1)^2}{\alpha + 1} \int_0^1 (t - t^{\alpha+1}) \psi''(tb_1 + (1 - t)b_2) dt. \end{aligned}$$

Taking absolute on both sides and using Jensen’s integral inequality, we get

$$\begin{aligned} & \left| \frac{\alpha\psi(b_1) + \psi(b_2)}{\alpha + 1} - \frac{\Gamma(\alpha + 1)}{(b_2 - b_1)^\alpha} J_{b_1^+}^\alpha \psi(b_2) \right| \\ & \leq \frac{(b_2 - b_1)^2}{\alpha + 1} \int_0^1 (t - t^{\alpha+1}) dt \left| \psi'' \left(\frac{\int_0^1 (t - t^{\alpha+1}) \{tb_1 + (1 - t)b_2\} dt}{\int_0^1 (t - t^{\alpha+1}) dt} \right) \right| \\ & = \frac{\alpha(b_2 - b_1)^2}{2(\alpha + 1)(\alpha + 2)} \left| \psi'' \left(\frac{2\alpha b_1 + \alpha b_2 + 4b_1 + 5b_2}{3(\alpha + 3)} \right) \right|. \quad \square \end{aligned}$$

Remark 2.14 Let $\alpha = 1$. Then Theorem 2.13 leads to

$$\left| \frac{\psi(b_1) + \psi(b_2)}{2} - \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \psi(t) dt \right| \leq \frac{(b_2 - b_1)^2}{12} \left| \psi'' \left(\frac{b_1 + b_2}{2} \right) \right|.$$

Remark 2.15 We can also get the same results as in this article if we use the Green’s function G_2 given by (2.2) instead of the Green’s function G_1 given by (2.1). But we can only get previously known results if we use the idea of the article and the Green’s function G_3 given by (2.3) or G_4 given by (2.4). This is the reason why we only deal with the Green’s function G_1 in the article. As exercises, interested readers can use other three Green’s functions to give their corresponding results.

3 Results and discussion

In the article, we use Green’s function method to establish the left Riemann–Liouville fractional Hermite–Hadamard type inequalities. The given idea by using the Green’s function (2.1) or any other new Green’s function may be furthered to research the Hermite–Hadamard inequality for fractional integrals as presented in [38] and to research the Hermite–Hadamard inequality for pre-invx, s -convex, co-ordinate convex functions etc.

4 Conclusion

In the article, we establish the left Riemann–Liouville fractional Hermite–Hadamard type inequalities and the generalized Hermite–Hadamard type inequalities by using Green’s function and Jensen’s inequality. The given idea and results are novel and interesting, they may stimulate further research in the theory of fractional integrals and generalized convex functions.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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