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Sharp constant of Hardy operators corresponding to general positive measures

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Abstract

We investigate a new kind of Hardy operator H_μ with respect to arbitrary positive measures μ and prove that H_μ is bounded on $L^p(d\mu)$ with an upper constant $p/(p-1)$. Moreover, we characterize a sufficient condition about the measure which makes $p/(p-1)$ to be the L^p-norm of H_μ.

MSC: 42B20; 42B35

Keywords: Hardy operator; Best constant; Sharp problem

1 Introduction

Let μ be a positive measure on $[0,\infty)$ and f be a nonnegative μ-measurable function. Define Hardy operator with respect to the measure μ by

$$H_\mu f(x) = \frac{1}{\mu([0,x])}\int_{[0,x]} f(t)\,d\mu(t), \tag{1}$$

if $0 < \mu([0,x]) < \infty$, and set $H_\mu f(x) = 0$, if $\mu([0,x]) = 0$ or ∞.

Observe that if μ is Lebesgue measure, then H_μ becomes the classical Hardy operator

$$Hf(x) = \frac{1}{x}\int_0^x f(t)\,dt, \tag{2}$$

and if $\mu = \sum_{k=1}^{\infty}\delta_k$, then H_μ becomes the discrete Hardy operator

$$\mathcal{H}f(k) = \frac{f(1)+\cdots+f(k)}{k}.$$

For $1 < p < \infty$, reference [1] showed that the two operators are bounded on L^p and l^p respectively. Moreover, for both, the best constants are $p/(p-1)$ and the maximizing functions do not exist. We refer the reader to [2–6] for the background material and further references.

Hardy operator has a close relationship with Hardy–Littlewood maximal operator. From the point of rearrangement, Hf is equivalent to Mf (see reference [7]). In reference [8], Grafakos considered the L^p-boundedness for the maximal functions associated with general measures. In this paper, we shall discuss the sharp problems about H_μ. We will show that the operator H_μ is bounded on $L^p(d\mu)$ with an upper bound no more than

$p/(p-1)$. Furthermore, we will characterize a sufficient condition about μ such that $\|H_\mu\|_{L^p\to L^p} = p/(p-1)$.

From the definition about H_μ, it is not necessary to consider the points x such that $\mu([0,x]) = 0$ or ∞. Therefore, we let

$$a = \inf\bigl\{x : \mu\bigl([0,x]\bigr) > 0\bigr\},$$

and

$$b = \begin{cases} \infty & \text{if } B = \emptyset, \\ \inf B & \text{if } B \neq \emptyset, \end{cases}$$

where B denotes the set $\{x : \mu([0,x]) = \infty \text{ or } \mu([x,\infty)) = 0\}$. Then we call that the measure μ is supported in the interval $[a,b]$.

For the case of weak type inequality, the best constant from $L^p(d\mu)$ to $L^{p,\infty}(d\mu)$ is always 1.

Theorem 1.1 *Let μ be a positive measure on $[0,\infty]$ and $1 \le p < \infty$. Then we have*

$$\|H_\mu\|_{L^p(d\mu)\to L^{p,\infty}(d\mu)} = 1.$$

Theorem 1.2 *Suppose that μ is supported in $[a,b]$ and $f \in L^p(d\mu)$ with $1 < p < \infty$. For $f \neq 0$, define*

$$\mathcal{R}_\mu(f) = \frac{\|H_\mu f\|_{L^p(d\mu)}}{\|f\|_{L^p(d\mu)}}.$$

Then the following statements hold:

(i) $\|H_\mu f\|_{L^p(d\mu)} \le \frac{p}{p-1}\|f\|_{L^p(d\mu)}$ *holds for arbitrary positive measure μ.*

(ii) *There exists no function f such that $\mathcal{R}_\mu(f) = \frac{p}{p-1}$ holds.*

Theorem 1.3 *If μ satisfies one of the following conditions*:

Condition 1. $\mu([a,b]) = \infty$ *and*

$$\lim_{x\to b}\frac{\mu([a,x])}{\mu([a,x))} = 1;$$

Condition 2. $\{a\}$ *is not an atom of μ, and*

$$\lim_{x\to a}\frac{\mu([a,x])}{\mu([a,x))} = 1,$$

then we have

$$\sup_{f\in L^p(d\mu), f\neq 0} \mathcal{R}_\mu(f) = \frac{p}{p-1}.$$

We remark that there indeed exist some measures so that

$$\sup_{f\in L^p(d\mu), f\neq 0} \mathcal{R}_\mu(f) < \frac{p}{p-1}. \tag{3}$$

For example, it is easy to know that the Dirac measure δ_0 satisfies inequality (3). In this paper, we will give some more complex counterexamples.

2 Preliminary and lemmas

In the study of sharp problems, the rearrangement of function is a very useful tool. Let

$$d_f(s) = \mu\bigl(\bigl\{|f| > s\bigr\}\bigr).$$

Then the rearrangement of f is defined by

$$f^*(t) = \inf\bigl\{s > 0 : d_f(s) \le t\bigr\}.$$

By the properties of the rearrangement, we can easily have

$$\|f\|_{L^p(d\mu)} = \bigl\|f^*\bigr\|_{L^p(dm)}.$$

We refer the reader to [9] for more properties of rearrangement. In reference [1], Hardy gave the following result.

Lemma 2.1 (G.H. Hardy and J.E. Littlewood) *Let (X, μ) be a measurable space. If $f, g \in \mathcal{M}(X, \mu)$, then*

$$\int_X |fg|\,d\mu \le \int_0^\infty f^*(t)g^*(t)\,dt$$

holds.

Moreover, the theory of rearrangement plays an important role in proving the existence of maximizing function. This is because of the following lemma introduced by Lieb [10].

Lemma 2.2 *Suppose that (M, Σ, μ) and (M', Σ', μ') are two measure spaces. Let X and Y be $L^p(M, \Sigma, \mu)$ and $L^q(M', \Sigma', \mu')$ with $1 \le p \le q < \infty$. Let A be a bounded linear operator from X to Y. For $f \in X$ with $f \ne 0$, set*

$$\mathcal{R}(f) = \frac{\|Af\|_Y}{\|f\|_X}$$

and

$$N = \sup\bigl\{\mathcal{R}(f) : f \ne 0\bigr\}.$$

Let $\{f_j\}$ be a uniform norm-bounded maximizing sequence for N, and assume that $f_j \to f \ne 0$ and that $A(f_j) \to A(f)$ pointwise almost everywhere. Then f maximizes, i.e., $\mathcal{R}(f) = N$.

3 The boundedness of weak-L^p

In this section, we first prove Theorem 1.1. For the sake of clarity, we define a function as

$$F_\mu(x) := \mu\bigl([0, x]\bigr).$$

Obviously F_μ increases as $x \to \infty$. It follows from Lemma 2.1 and the definition of H_μ that

$$\begin{aligned} H_\mu f(x) &= \frac{1}{\mu([0,x])} \int_{[0,x]} f(t)\,d\mu(t) \\ &\le \frac{1}{F_\mu(x)} \int_{[0,F_\mu(x)]} f^*(t)\,dt \\ &= Hf^*\bigl(F_\mu(x)\bigr). \end{aligned} \tag{4}$$

Let

$$E_\mu^{f^*}(\lambda) := \bigl\{x : Hf^*\bigl(F_\mu(x)\bigr) > \lambda\bigr\}.$$

Note that f^* decreases, so we easily have that Hf^* decreases as well. If we take

$$x_0 = \sup\bigl\{x : Hf^*\bigl(F_\mu(x)\bigr) > \lambda\bigr\},$$

then it implies that

$$E_\mu^{f^*}(\lambda) = [0, x_0).$$

Thus, we can obtain that

$$\bigl\{x : Hf^*(x) > \lambda\bigr\} \supset \bigl[0, F_\mu(x_0)\bigr).$$

We conclude that

$$\mu\bigl(\bigl\{x : Hf^*\bigl(F_\mu(x)\bigr) > \lambda\bigr\}\bigr) \le F_\mu(x_0) \le \bigl|\bigl\{x : Hf^*(x) > \lambda\bigr\}\bigr|, \tag{5}$$

where $|\cdot|$ denotes the Lebesgue measure. It follows from inequalities (4) and (5) that

$$\begin{aligned} \frac{\sup_{\lambda>0} \lambda \mu(\{x : H_\mu f(x) > \lambda\})^{\frac{1}{p}}}{\|f\|_{L^p(d\mu)}} &\le \frac{\sup_{\lambda>0} \lambda \mu(\{x : Hf^*(F_\mu(x)) > \lambda\})^{\frac{1}{p}}}{\|f^*\|_{L^p(dm)}} \\ &\le \frac{\sup_{\lambda>0} \lambda |\{x : Hf^*(x) > \lambda\}|^{\frac{1}{p}}}{\|f^*\|_{L^p(dm)}}. \end{aligned} \tag{6}$$

Since $f^* \in L^p(dm)$, by Hölder's inequality, we have that

$$Hf^*(x) = \frac{1}{x} \int_0^x f^*(t)\,dt \le \left(\frac{1}{x} \int_0^x \bigl|f^*(t)\bigr|^p\,dt\right)^{\frac{1}{p}} \le x^{-\frac{1}{p}} \bigl\|f^*\bigr\|_{L^p(dm)}. \tag{7}$$

Thus it is obvious to obtain that

$$\bigl|\bigl\{x : Hf^*(x) > \lambda\bigr\}\bigr| \le \bigl|\bigl\{x : x^{-\frac{1}{p}} \bigl\|f^*\bigr\|_{L^p(dm)} > \lambda\bigr\}\bigr| = \frac{\|f^*\|_{L^p(dm)}^p}{\lambda^p}. \tag{8}$$

From inequality (6) and inequality (8), we have

$$\frac{\sup_{\lambda>0} \lambda \mu(\{x : H_\mu f(x) > \lambda\})^{\frac{1}{p}}}{\|f\|_{L^p(d\mu)}} \le 1.$$

That is,

$$\frac{\|H_\mu f\|_{L^{p,\infty}(d\mu)}}{\|f\|_{L^p(d\mu)}} \le 1 \tag{9}$$

holds. This is equivalent to

$$\|H_\mu\|_{L^p(d\mu)\to L^{p,\infty}(d\mu)} \le 1. \tag{10}$$

Next it suffices to show that the constant 1 is sharp for inequality (10).

Take $0 \le x_1 < x_2 < \infty$ such that $0 < \mu([x_1, x_2]) < \infty$. Let $g = \chi_{[x_1,x_2]}$. It is easy to obtain

$$\|H_\mu g\|_{L^{p,\infty}(d\mu)} = \|g\|_{L^p(d\mu)}.$$

The proof is completed.

4 L^p-boundedness of the operator H_μ with upper bound $p/(p-1)$

Now we will show the results (i) and (ii) of Theorem 1.2.

Proof Following the proof of (5), we obtain

$$\int_{[0,\infty]} \bigl(f\bigl(\mu\bigl([0,x]\bigr)\bigr)\bigr)^p \, d\mu(x) \le \int_{[0,\infty]} f^p(x)\, dx. \tag{11}$$

By inequality (11), we conclude that

$$\begin{aligned}
\|H_\mu f\|_{L^p(\mathbb{R}_+,d\mu)} &= \left(\int_{\mathbb{R}_+} \left|\frac{1}{\mu([0,x])}\int_{[0,x]} f(t)\, d\mu(t)\right|^p d\mu(x)\right)^{\frac{1}{p}} \\
&\le \left(\int_{\mathbb{R}_+} \left|\frac{1}{F_\mu(x)}\int_{[0,F_\mu(x)]} f^*(t)\, dt\right|^p d\mu(x)\right)^{\frac{1}{p}} \\
&= \left(\int_{\mathbb{R}_+} \bigl|Hf^*\bigl(F_\mu(x)\bigr)\bigr|^p \, d\mu(x)\right)^{\frac{1}{p}} \\
&\le \left(\int_{\mathbb{R}_+} \bigl|Hf^*(x)\bigr|^p \, dx\right)^{\frac{1}{p}}.
\end{aligned} \tag{12}$$

It follows from the inequality of classical Hardy operator that

$$\left(\int_{\mathbb{R}_+} \bigl|Hf^*(x)\bigr|^p \, dx\right)^{\frac{1}{p}} \le \frac{p}{p-1}\bigl\|f^*\bigr\|_{L^p(dm)} = \frac{p}{p-1}\|f\|_{L^p(d\mu)}. \tag{13}$$

Combining inequality (12) with inequality (13), we have

$$\|H_\mu f\|_{L^p(\mathbb{R}_+,d\mu)} \le \frac{p}{p-1}\|f\|_{L^p(d\mu)}.$$

Since the sharp function for the classical Hardy operator does not exist, it is easy to know from inequality (12) that there exists no function f such that $\mathcal{R}_\mu(f) = \frac{p}{p-1}$. The proof of the result (ii) of Theorem 1.2 is completed. □

5 A characterization of the measure μ which ensures $\sup_{f\neq 0} \mathcal{R}_\mu(f) = p/(p-1)$

In this section, we try to characterize the measure μ which ensures $\sup_{f\neq 0} \mathcal{R}_\mu(f) = p/(p-1)$. We regard μ as a complete atom measure by giving an appropriate partition on $[0,\infty]$. We first present a partition on $[0,\infty]$ by the following two lemmas.

Lemma 5.1 *Let μ be a positive measure that is supported on $[0,\infty]$. If $\mu([0,\infty]) = \infty$ and*

$$\lim_{x\to\infty} \frac{\mu(\{x\})}{\mu([0,x])} = 0,$$

then there exists a partition on $[0,\infty]$ as

$$I_0 = [0,x_1], \qquad I_1 = (x_1,x_2], \ldots, \qquad I_k = (x_k,x_{k+1}], \ldots,$$

such that

$$\mu(I_{k+1}) \geq \mu(I_k),$$

and

$$\lim_{k\to\infty} \frac{\mu(I_k)}{\mu([0,x_{k+1}])} = 0.$$

Proof Let x_1 be any positive number. Denote $I_0 = [0,x_1]$. Since μ is supported on $[0,\infty]$, we have

$$\mu(I_0) > 0.$$

For $k = 2$, we let

$$x_2 = \inf\bigl\{x : \mu\bigl((x_1,x]\bigr) \geq \mu\bigl([0,x_1]\bigr)\bigr\}.$$

For $k > 2$, we let

$$x_k = \inf\bigl\{x : \mu\bigl((x_{k-1},x]\bigr) \geq \mu\bigl((x_{k-2},x_{k-1}]\bigr)\bigr\}.$$

Denote $I_k = [x_{k-1},x_k]$ with $k = 2,3,\ldots$. Since $\mu([0,\infty]) = \infty$, we easily have

$$\lim_{k\to\infty} x_k = \infty.$$

Thus, $\{I_k\}$ obviously constitutes a partition of $[0,\infty]$.

We first show that

$$\mu(I_k) \geq \mu(I_{k-1})$$

and

$$\mu\bigl((x_k,x_{k+1})\bigr) \leq \mu(I_{k-1}). \tag{14}$$

By our construction, for any $x > x_{k+1}$, it follows that

$$\mu\bigl((x_k, x]\bigr) \geq \mu\bigl((I_{k-1})\bigr).$$

Thus the property of measure implies that

$$\mu(I_k) = \lim_{\substack{x > x_{k+1} \\ x \to x_{k+1}}} \mu\bigl((x_k, x]\bigr) \geq \mu(I_{k-1}).$$

Moreover, if $x_k < x < x_{k+1}$, then $\mu([x_k, x]) < \mu(I_{k-1})$. Thus, it follows that

$$\mu\bigl((x_k, x_{k+1})\bigr) = \lim_{\substack{x < x_{k+1} \\ x \to x_{k+1}}} \mu\bigl((x_k, x]\bigr) \leq \mu(I_{k-1}).$$

To complete the proof, it remains to show that

$$\lim_{k\to\infty} \frac{\mu(I_{k-1})}{\mu([0, x_k])} = 0.$$

This is equivalent to prove that, for any $\epsilon > 0$, there is an integer $N > 0$ such that

$$\frac{\mu(I_{k-1})}{\mu([0, x_k])} \leq 2\epsilon$$

holds for $k \geq N$.

In order to prove this result, we divide the set $\mathbb{Z}^+ \setminus \{1\}$ into two parts:

$$F_\epsilon := \left\{k \in \mathbb{Z} : k \geq 2, \frac{\mu(\{x_k\})}{\mu((x_{k-1}, x_k))} < \epsilon\right\} \tag{15}$$

and

$$G_\epsilon := \left\{k \in \mathbb{Z} : k \geq 2, \frac{\mu(\{x_k\})}{\mu((x_{k-1}, x_k))} \geq \epsilon\right\}. \tag{16}$$

By definition (16), if $k \in G_\epsilon$, then we have

$$\mu(I_{k-1}) \leq \left(1 + \frac{1}{\epsilon}\right)\mu\bigl(\{x_k\}\bigr). \tag{17}$$

We discuss the problem in two cases:

Case I. G_ϵ is not a finite set.

Case II. G_ϵ is a finite set.

If G_ϵ is not a finite set, then by equality $\lim_{x\to\infty} \frac{\mu(\{x\})}{\mu([0,x])} = 0$, there exists an integer $N \in G_\epsilon$ such that, for any $k \geq N$,

$$\frac{\mu(\{x_k\})}{\mu([0, x_k])} < \frac{\epsilon^2}{1+\epsilon}. \tag{18}$$

Thus if $k > N$ and $k \in G_\epsilon$, then by inequalities (17) and (18), we have

$$\frac{\mu(I_{k-1})}{\mu([0,x_k])} \le \epsilon. \tag{19}$$

On the other hand, if $k > N$ and $k \in F_\epsilon$, since G_ϵ is not a finite integer and $N \in G_\epsilon$, we can find a series of integers $k_0, k_0+1, \dots, k$, such that $k_0 \in G_\epsilon$, and

$$k_0+1,\dots,k \in F_\epsilon.$$

By the definition of F_ϵ and inequality (14), we can conclude that if $i \in F_\epsilon$, then

$$\begin{aligned} \mu\bigl((x_{i-1},x_i]\bigr) &= \mu\bigl((x_{i-1},x_i)\bigr) + \mu\bigl(\{x_i\}\bigr) \\ &\le (1+\epsilon)\mu\bigl((x_{i-1},x_i)\bigr) \\ &\le (1+\epsilon)\mu\bigl((x_{i-2},x_{i-1}]\bigr). \end{aligned} \tag{20}$$

It immediately implies from inequality (20) that

$$\begin{aligned} \mu\bigl((x_{k_0},x_k]\bigr) &= \sum_{i=k_0+1}^{k} \mu\bigl((x_{i-1},x_i]\bigr) \\ &\ge \sum_{i=k_0+1}^{k} (1+\epsilon)^{i-k} \mu\bigl((x_{k-1},x_k]\bigr) \\ &= \mu\bigl((x_{k-1},x_k]\bigr)\frac{1-(\frac{1}{1+\epsilon})^{k-k_0}}{1-\frac{1}{1+\epsilon}}. \end{aligned} \tag{21}$$

Thus, by inequality (21), we have

$$\begin{aligned} \frac{\mu(I_{k-1})}{\mu([0,x_k])} &\le \frac{\mu((x_{k-1},x_k])}{\mu((x_{k_0-1},x_k])} \le \frac{1-\frac{1}{1+\epsilon}}{1-(\frac{1}{1+\epsilon})^{k-k_0}} \\ &\le \frac{\epsilon}{1-(\frac{1}{1+\epsilon})^{k-k_0}}. \end{aligned} \tag{22}$$

Since $k_0 \in G_\epsilon$, inequalities (14) and (20) imply

$$\frac{\mu(I_{k-1})}{\mu([0,x_k])} \le (1+\epsilon)^{k-k_0}\frac{\mu(I_{k_0-1})}{\mu([0,x_k])} \le (1+\epsilon)^{k-k_0}\epsilon. \tag{23}$$

If $(1+\epsilon)^{k-k_0} > 2$, by inequality (22), we have

$$\frac{\mu(I_{k-1})}{\mu([0,x_k])} \le 2\epsilon.$$

If $(1+\epsilon)^{k-k_0} \le 2$, by inequality (23), we have

$$\frac{\mu(I_{k-1})}{\mu([0,x_k])} \le 2\epsilon.$$

At last, we conclude that if $k > N$ and $k \in F_\epsilon$, then

$$\frac{\mu(I_{k-1})}{\mu([0,x_k])} \le 2\epsilon. \tag{24}$$

The proof of Case I is complete.

If G_ϵ is a finite set, then we can find an integer k_0 such that $k \in F_\epsilon$ for $k > k_0$. Then, by inequality (22), we can find a big enough integer N such that

$$\frac{\mu(I_{k-1})}{\mu([0,x_k])} \le 2\epsilon$$

if $k \ge N$. The proof is completed. □

Lemma 5.2 *Suppose that* μ *is supported in* $[0,\infty]$. *If* $\mu(\{0\}) = 0$ *and* $\lim_{x\to 0} \frac{\mu([0,x])}{\mu([0,x))} = 1$, *then there exists a partition on* $(0,1]$,

$$(x_1, 1], (x_2, x_1], \dots, (x_k, x_{k-1}], \dots,$$

such that $\lim_{k\to\infty} x_k = 0$ *and*

$$\lim_{k\to\infty} \frac{\mu((x_k, x_{k-1}])}{\mu([0,x_{k-1}])} = 0.$$

Proof Without loss of generality, suppose

$$\mu\bigl([0,1]\bigr) = \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

If $\mu(\{1\}) < 1$, then we set $k_0 = 0$. If $\mu(\{1\}) \ge 1$, then we set

$$k_0 = \max\left\{ m : \sum_{k=1}^{m} \frac{1}{k^2} \le \mu\bigl(\{1\}\bigr) \right\}.$$

It is easy to see that

$$\sum_{k=1}^{k_0+1} \frac{1}{k^2} > \mu\bigl(\{1\}\bigr).$$

Then we can find a positive real number $x_1 < 1$ such that

$$x_1 = \sup\left\{ x : \mu\bigl((x,1]\bigr) \ge \sum_{k=1}^{k_0+1} \frac{1}{k^2} \right\}.$$

Proceeding in this way, we set

$$k_i = \max\left\{ m : \sum_{k=1}^{m} \frac{1}{k^2} \le \mu\bigl([x_i,1]\bigr) \right\} \tag{25}$$

and

$$x_{i+1} = \sup\left\{x : \mu\big((x,1]\big) \geq \sum_{k=1}^{k_i+1} \frac{1}{k^2}\right\} \tag{26}$$

for $i \geq 1$. By (27), (25), and (26), we can conclude

$$\sum_{k=1}^{k_i} \frac{1}{k^2} \leq \mu\big([x_i,1]\big) \leq \mu\big((x_{i+1},1]\big) \leq \sum_{k=1}^{k_i+1} \frac{1}{k^2} \leq \mu\big([x_{i+1},1]\big). \tag{27}$$

It is easy to see that $x_i > x_{i+1}$ and

$$\lim_{i\to\infty} \mu\big([x_i,1]\big) \geq \lim_{i\to\infty} \sum_{k=1}^{k_i} \frac{1}{k^2} = \mu\big((0,1]\big).$$

Thus we have $\lim_{i\to\infty} x_i = 0$. It is easy to see that

$$(x_1,1], (x_2,x_1], \ldots, (x_k,x_{k-1}], \ldots,$$

divide $(0,1]$. It can be implied from inequality (27) that

$$\mu\big([x_i,1]\big) + \frac{1}{(k_i+1)^2} \geq \sum_{k=1}^{k_i+1} \frac{1}{k^2} \geq \mu\big((x_{i+1},1]\big). \tag{28}$$

To prove this partition satisfying the requirement of the lemma, we define two integer sets:

$$F_\epsilon = \left\{k \geq 1 : \frac{\mu(\{x_k\})}{\mu((x_{k+1},x_k])} < \epsilon\right\}$$

and

$$G_\epsilon = \left\{k \geq 1 : \frac{\mu(\{x_k\})}{\mu((x_{k+1},x_k])} \geq \epsilon\right\},$$

where ϵ is an arbitrary positive real number. Since $\lim_{x\to 0} \frac{\mu([0,x])}{\mu([0,x))} = 1$, we have $\lim_{x\to 0} \frac{\mu(\{x\})}{\mu([0,x))} = 0$. It is easy to find an integer N such that

$$\frac{\mu(\{x_{i-1}\})}{\mu([0,x_{i-1}])} < 2\epsilon^2$$

for any integer $i > N$. Thus, by the construction of G_ϵ, if $i > N$ and $i \in G_\epsilon$, we have

$$\frac{\mu((x_i,x_{i-1}])}{\mu([0,x_{i-1}])} < 2\epsilon.$$

If $i \in F_\epsilon$, then we have

$$\mu\big((x_{i+1},x_i]\big) \leq \frac{1}{1-\epsilon}\mu\big((x_{i+1},x_i)\big). \tag{29}$$

By inequalities (28) and (29), we have

$$\begin{aligned}\frac{\mu((x_{i+1},x_i])}{\mu([0,x_i])} &\le \frac{1}{1-\epsilon}\frac{\mu((x_{i+1},x_i))}{\mu([0,x_i])}\\ &= \frac{1}{1-\epsilon}\frac{\mu((x_{i+1},1])-\mu([x_i,1])}{\mu((0,1])-\mu((x_i,1])}\\ &\le \frac{1}{1-\epsilon}\frac{1/(k_i+1)^2}{\sum_{k=k_i+2}^{\infty}1/k^2}.\end{aligned}$$

Thus we can find a sufficiently large integer which is still denoted by N such that, for any integer $i > N$ and $i \in F_\epsilon$, there is

$$\frac{\mu((x_i,x_{i-1}])}{\mu([0,x_{i-1}])} < 2\epsilon.$$

Since ϵ is an arbitrary real number, we have

$$\lim_{k\to\infty}\frac{\mu((x_k,x_{k-1}])}{\mu([0,x_{k-1}])} = 0.$$

The proof is completed. □

After finishing our preparations, we can give the proof of the result (iii) of the main theorem.

Proof Let

$$T_{a,b}(x) = \begin{cases}\tan(\frac{\pi}{2}(\frac{x-a}{b-a})), & 0 < b < \infty;\\ x-a, & b = \infty.\end{cases} \tag{30}$$

By equality (30), we can obtain a new measure denoted by μ_T which is supported in $[0,\infty]$ so that, for any open interval (x,y), we have

$$\mu_T\big((x,y)\big) = \mu\big(\big(T_{a,b}^{-1}(x),T_{a,b}^{-1}(y)\big)\big).$$

Then it is easy to get

$$\sup\{\mathcal{R}_\mu f \,|\, f\in L^p(d\mu)\} = \sup\{\mathcal{R}_{\mu_T} f \,|\, f\in L^p(d\mu_T)\}.$$

Thus it is enough to assume that the measure μ is supported in $[0,\infty]$.

We first consider Condition 1.

By Lemma 5.1, we can divide $\mathbb{R}^+$ into a series of intervals

$$[0,x_1],(x_1,x_2],\dots,(x_k,x_{k+1}],\dots,$$

such that

$$\lim_{k\to\infty}\frac{\mu((x_k,x_{k+1}])}{\mu([0,x_k])} = 0.$$

For any $\epsilon > 0$, if we can find a function f_ϵ such that $\mathcal{R}(f_\epsilon) \geq \frac{p}{p-1} - O(\epsilon)$, then the proof is completed.

By the property of the partition, there exists an integer N satisfying

$$\frac{\mu((x_k, x_{k+1}])}{\mu([0, x_k])} < \epsilon$$

for $k \geq N$. This inequality is equivalent to

$$\frac{\mu([0, x_{k+1}])}{\mu([0, x_k])} < 1 + \epsilon. \tag{31}$$

Let

$$f_\epsilon = \sum_{k=N}^{\infty} \mu\big([0, x_{k+1}]\big)^{-\frac{1}{p}-\epsilon} \chi_{(x_k, x_{k+1}]}.$$

First we estimate the norm of f_ϵ

$$\begin{aligned}
\|f_\epsilon\|_{L^p(d\mu)} &= \left(\sum_{k=N}^{\infty} \mu\big([0, x_{k+1}]\big)^{-1-p\epsilon} \mu\big((x_k, x_{k+1}]\big)\right)^{\frac{1}{p}} \\
&\geq \left(\frac{1}{1+\epsilon}\right)^{\frac{1}{p}+\epsilon} \left(\sum_{k=N}^{\infty} \mu\big([0, x_k]\big)^{-1-p\epsilon} \mu\big((x_k, x_{k+1}]\big)\right)^{\frac{1}{p}} \\
&= \left(\frac{1}{1+\epsilon}\right)^{\frac{1}{p}+\epsilon} \left(\sum_{k=N}^{\infty} \int_{\mu([0,x_k])}^{\mu([0,x_{k+1}])} \mu\big([0, x_k]\big)^{-1-p\epsilon}\, dt\right)^{\frac{1}{p}} \\
&\geq \left(\frac{1}{1+\epsilon}\right)^{\frac{1}{p}+\epsilon} \left(\int_{\mu([0,x_N])}^{\infty} t^{-1-p\epsilon}\, dt\right)^{\frac{1}{p}} \\
&\geq \left(\frac{1}{1+\epsilon}\right)^{\frac{1}{p}+\epsilon} \left(\frac{1}{p\epsilon}\right)^{\frac{1}{p}} \mu\big([0, x_N]\big)^{-\epsilon}.
\end{aligned} \tag{32}$$

Next, we estimate the value of $H_\mu f_\epsilon(x)$. When $k \geq N$ and $x_k < x \leq x_{k+1}$, we have

$$\begin{aligned}
H_\mu f_\epsilon(x) &= \frac{1}{\mu([0, x])} \int_{[0,x]} f_\epsilon(t)\, d\mu(t) \\
&\geq \frac{1}{\mu([0, x_{k+1}])} \int_{[0,x_k]} f_\epsilon(t)\, d\mu(t) \\
&= \frac{1}{\mu([0, x_{k+1}])} \sum_{i=N}^{k-1} \mu\big([0, x_{i+1}]\big)^{-\frac{1}{p}-\epsilon} \mu\big((x_i, x_{i+1}]\big) \\
&\geq \left(\frac{1}{1+\epsilon}\right)^{\frac{1}{p}+\epsilon} \frac{1}{\mu([0, x_{k+1}])} \sum_{i=N}^{k-1} \mu\big([0, x_i]\big)^{-\frac{1}{p}-\epsilon} \mu\big((x_i, x_{i+1}]\big) \\
&= \left(\frac{1}{1+\epsilon}\right)^{\frac{1}{p}+\epsilon} \frac{1}{\mu([0, x_{k+1}])} \sum_{i=N}^{k-1} \int_{\mu([0,x_i])}^{\mu([0,x_{i+1}])} \mu\big([0, x_i]\big)^{-\frac{1}{p}-\epsilon}\, dt
\end{aligned}$$

$$\geq \left(\frac{1}{1+\epsilon}\right)^{1+\frac{1}{p}+\epsilon} \frac{1}{\mu([0,x_k])} \int_{\mu([0,x_N])}^{\mu([0,x_k])} t^{-\frac{1}{p}-\epsilon}\,dt$$
$$\geq \left(\frac{1}{1+\epsilon}\right)^{1+\frac{1}{p}+\epsilon} \frac{1}{1-\frac{1}{p}-\epsilon}\left(\mu\big([0,x_k]\big)^{-\frac{1}{p}-\epsilon} - \frac{\mu([0,x_N])^{1-\frac{1}{p}-\epsilon}}{\mu([0,x_k])}\right). \tag{33}$$

Set

$$f_\epsilon^{(1)} = \left(\frac{1}{1+\epsilon}\right)^{1+\frac{1}{p}+\epsilon} \frac{1}{1-\frac{1}{p}-\epsilon} \sum_{k=N}^{\infty} \mu\big([0,x_k]\big)^{-\frac{1}{p}-\epsilon} \chi_{(x_k,x_{k+1}]}$$
$$= \left(\frac{1}{1+\epsilon}\right)^{1+\frac{1}{p}+\epsilon} \frac{1}{1-\frac{1}{p}-\epsilon} f_\epsilon$$

and

$$f_\epsilon^{(2)} = \left(\frac{1}{1+\epsilon}\right)^{1+\frac{1}{p}+\epsilon} \frac{1}{1-\frac{1}{p}-\epsilon} \sum_{k=N}^{\infty} \frac{\mu([0,x_N])^{1-\frac{1}{p}-\epsilon}}{\mu([0,x_k])} \chi_{(x_k,x_{k+1}]}.$$

Then we have

$$\big\|f_\epsilon^{(2)}\big\|_{L^p(d\mu)} = \left(\frac{1}{1+\epsilon}\right)^{1+\frac{1}{p}+\epsilon} \frac{1}{1-\frac{1}{p}-\epsilon} \mu\big([0,x_N]\big)^{1-\frac{1}{p}-\epsilon} \left(\sum_{k=N}^{\infty} \mu\big([0,x_k]\big)^{-p} \mu\big((x_k,x_{k+1}]\big)\right)^{\frac{1}{p}}$$
$$\leq \left(\frac{1}{1+\epsilon}\right)^{\frac{1}{p}+\epsilon} \frac{1}{1-\frac{1}{p}-\epsilon} \mu\big([0,x_N]\big)^{1-\frac{1}{p}-\epsilon} \left(\sum_{k=N}^{\infty} \mu\big([0,x_{k+1}]\big)^{-p} \mu\big((x_k,x_{k+1}]\big)\right)^{\frac{1}{p}}$$
$$\leq \left(\frac{1}{1+\epsilon}\right)^{\frac{1}{p}+\epsilon} \frac{1}{1-\frac{1}{p}-\epsilon} \left(\frac{1}{p-1}\right)^{\frac{1}{p}} \mu\big([0,x_N]\big)^{-\epsilon}. \tag{34}$$

By inequality (33), we have

$$\|H_\mu f_\epsilon\|_{L^p(d\mu)} \geq \big\|f_\epsilon^{(1)}\big\|_{L^p(d\mu)} - \big\|f_\epsilon^{(2)}\big\|_{L^p(d\mu)}.$$

From this result and inequalities (32) and (34), we can get

$$\mathcal{R}(f_\epsilon) \geq \frac{\|f_\epsilon^{(1)}\|_{L^p(d\mu)} - \|f_\epsilon^{(2)}\|_{L^p(d\mu)}}{\|f_\epsilon\|_{L^p(d\mu)}}$$
$$= \left(\frac{1}{1+\epsilon}\right)^{1+\frac{1}{p}+\epsilon} \frac{1}{1-\frac{1}{p}-\epsilon} - \frac{\|f_\epsilon^{(2)}\|_{L^p(d\mu)}}{\|f_\epsilon\|_{L^p(d\mu)}}$$
$$\geq \left(\frac{1}{1+\epsilon}\right)^{1+\frac{1}{p}+\epsilon} \frac{1}{1-\frac{1}{p}-\epsilon} - \frac{(\frac{1}{1+\epsilon})^{\frac{1}{p}+\epsilon} \frac{1}{1-\frac{1}{p}-\epsilon} (\frac{1}{p-1})^{\frac{1}{p}}}{(\frac{1}{1+\epsilon})^{\frac{1}{p}+\epsilon} (\frac{1}{p\epsilon})^{\frac{1}{p}}}. \tag{35}$$

Since ϵ is arbitrary, it is easy to imply $\sup_{f\neq 0} \mathcal{R}(f) = \frac{p}{p-1}$.

To prove condition (ii), by Lemma 5.2, we can part the intervals $(0,1]$ to

$$(x_1,1], (x_2,x_1], \ldots, (x_{k+1},x_k], \ldots$$

such that

$$\lim_{k\to\infty}\frac{\mu((x_{k+1},x_k])}{\mu((0,x_k])}=0.$$

Then, for any $\epsilon>0$, there is a sufficiently large integer N such that

$$\frac{\mu((x_{k+1},x_k])}{\mu((0,x_k])}<\epsilon$$

for $k\geq N$.

Thus, we have

$$\frac{\mu((0,x_{k+1}])}{\mu((0,x_k])}\geq 1-\epsilon.$$

Let $f_\epsilon=\sum_{k=N}^{\infty}\mu((0,x_k])^{-\frac{1}{p}+\epsilon}\chi_{(x_{k+1},x_k]}$. Then, for $x_{k+1}<x\leq x_k$ and $k\geq N$, we have

$$\begin{aligned}
H_\mu f_\epsilon(x)&=\frac{1}{\mu((0,x])}\int_{(0,x]}f_\epsilon(t)\,d\mu(t)\\
&\geq\frac{1}{\mu((0,x_k])}\int_{(0,x_{k+1}]}f_\epsilon(t)\,d\mu(t)\\
&=\frac{1}{\mu((0,x_k])}\sum_{i=k+1}^{\infty}\mu\bigl((0,x_i]\bigr)^{-\frac{1}{p}+\epsilon}\mu\bigl((x_{i+1},x_i]\bigr)\\
&\geq\frac{1}{\mu((0,x_k])}\frac{\mu((0,x_{k+1}])^{1-\frac{1}{p}+\epsilon}}{1-\frac{1}{p}+\epsilon}\\
&\geq\frac{(1-\epsilon)^{1-\frac{1}{p}+\epsilon}}{1-\frac{1}{p}+\epsilon}\mu\bigl((0,x_k]\bigr)^{-\frac{1}{p}+\epsilon} &(36)\\
&=\frac{(1-\epsilon)^{1-\frac{1}{p}+\epsilon}}{1-\frac{1}{p}+\epsilon}f_\epsilon(x). &(37)
\end{aligned}$$

It follows from inequality (36) that

$$\mathcal{R}(f_\epsilon)\geq\frac{(1-\epsilon)^{1-\frac{1}{p}+\epsilon}}{1-\frac{1}{p}+\epsilon}.$$

Because ϵ is arbitrary, it is easy to know $\sup_f\mathcal{R}(f)\geq\frac{p}{p-1}$. The proof of the suffice part of Theorem 1.2 is then completed. □

6 Counterexample

In this section we give some counterexamples that make $\sup_{f\neq 0,f\in L^p}\mathcal{R}(f)<p/(p-1)$. The following two lemmas tell us that we can limit our discussion to a special function set.

Lemma 6.1 *Suppose μ is a positive measure on $\mathbb{R}_+$ and it has an atom x_0. If $\{f_n\}$, $n=1,2,\ldots$, is a series of functions satisfying $f_n(x_0)=1$ and*

$$\lim_{n\to\infty}\mathcal{R}_\mu(f_n)=\frac{p}{p-1},$$

then we have

$$\lim_{n\to\infty} \|f_n\|_{L^p(d\mu)} = \infty.$$

Proof Without loss of generality, we assume that $\mu(\{x_0\}) = 1$. If the assertion does not hold, then we can assume that there exists a constant C satisfying $\|f_n\|_{L^p(d\mu)} \le C$. Let f_n^* be the decreasing rearrangement of f_n, then it is easy to get $\|f_n^*\|_{L^p(dm)} \le C$ and $f_n^*(1) \ge 1$. Thus we have $f^*(x) \ge 1$ for $0 < x \le 1$. By Helly's theorem, we can assume $\lim_{n\to\infty} f_n^* = f^*$ almost everywhere. Since f_n^* is decreasing, we have

$$C^p \ge \|f_n^*\|_{L^p(dm)}^p \ge \int_{[0,R]} |f_n^*(t)|^p \, dt \ge R|f_n^*(R)|^p,$$

which is equivalent to $f_n^*(R) \le CR^{-\frac{1}{p}}$. Thus, by the control convergence theorem,

$$\lim_{n\to\infty} Hf_n^*(x) = \lim_{n\to\infty} \frac{1}{x} \int_{[0,x]} f_n^*(t) \, dt = Hf^*(x). \tag{38}$$

However, by inequality (12), we have

$$\mathcal{R}_m(f_n^*) \ge \mathcal{R}_\mu(f_n),$$

it obviously shows that $\{f_n^*\}$ is a maximizing sequence for H, i.e.,

$$\lim_{n\to\infty} \mathcal{R}(f_n^*) = \frac{p}{p-1}.$$

By $\lim_{n\to\infty} f_n^* = f^*$ and equality (38), using Lemma 2.2, we get $\mathcal{R}_m(f^*) = \frac{p}{p-1}$, which contradicts the result about Hardy operator we have known. The proof is completed. □

Lemma 6.2 *Suppose μ is a positive measure on $\mathbb{R}_+$ and it has an atom x_0. If*

$$\sup\{\mathcal{R}_\mu f | f \in L^p(d\mu)\} = \frac{p}{p-1},$$

then there exists a series of functions $\{f_k\}$, $k = 1, 2, \ldots$, and $f_k(x_0) = 0$ such that

$$\lim_{k\to\infty} \mathcal{R}_\mu(f_k) = \frac{p}{p-1}.$$

Proof It is obvious that we can assume there exists a series of functions g_k, $g_k(x_0) = 1$, such that

$$\lim_{k\to\infty} \mathcal{R}_\mu(g_k) = \frac{p}{p-1}.$$

Let

$$f_k(x) = \begin{cases} g_k(x), & x \ne x_0, \\ 0, & x = x_0. \end{cases}$$

Then we have

$$H_\mu f_k(x) = \begin{cases} H_\mu g_k(x), & x < x_0; \\ H_\mu g_k(x) - \mu(\{x_0\})/\mu([0,x]), & x \geq x_0. \end{cases} \tag{39}$$

By equality (39), we can get

$$\|H_\mu(f_k)\|_{L^p} \geq \|H_\mu(g_k)\|_{L^p} - \left\| \frac{\mu(\{x_0\})}{\mu([0,\cdot])} \chi_{[x_0,\infty]} \right\|_{L^p}. \tag{40}$$

On the other hand, it is easy to obtain

$$\|f_k\|_{L^p} \leq \|g_k\| + \mu\big(\{x_0\}\big)^{\frac{1}{p}}. \tag{41}$$

By Lemma 6.1, we know that $\lim_{k\to\infty} \|g_k\|_{L^p(d\mu)} = \infty$ and $\lim_{k\to\infty} \mathcal{R}_\mu(g_k) = p/(p-1)$. By this result, together with inequalities (40) and (41), we can have

$$\lim_{k\to\infty} \mathcal{R}_\mu(f_k) = \frac{p}{p-1}. \qquad \square$$

Now we can give some counterexamples.

Example 6.3 Suppose that μ is supported in $[a,b)$, $\mu(\{a\}) > 0$, and $\mu(\mathbb{R}_+) < \infty$. Then $\sup_{f\neq 0} \mathcal{R}_\mu(f) < p/(p-1)$.

Proof Suppose that the result is not valid. By Lemma 6.2, we can find a series of functions $\{f_k\}, f_k(a) = 0$, such that

$$\lim_{k\to\infty} \mathcal{R}_\mu(f_k) = \frac{p}{p-1}.$$

Let $A = \mu(\{a\})$, $B = \mu(\mathbb{R}_+)$, and $\mu_1 = \mu - A\delta_a$. Then we have

$$H_\mu f_k(x) = \frac{\mu_1([0,x])}{\mu([0,x])} \frac{1}{\mu_1([0,x])} \int_{[0,x]} f_k \, d\mu_1 \leq \frac{B-A}{B} H_{\mu_1} f_k(x) \tag{42}$$

and

$$\|f_k\|_{L^p(d\mu)} = \|f_k\|_{L^p(d\mu_1)}. \tag{43}$$

By inequalities (42) and (43), we obtain

$$\mathcal{R}_\mu(f_k) \leq \frac{B-A}{B} \mathcal{R}_{\mu_1}(f_k) \leq \frac{B-A}{B} \frac{p}{p-1}.$$

It contradicts with $\lim_{k\to\infty} \mathcal{R}_\mu(f_k) = p/(p-1)$. Then the counterexample is valid. $\square$

Example 6.4 If $\mu = \sum_{k=-\infty}^{\infty} \lambda^k \delta_{\lambda^k}$ with $\lambda > 1$, then $\sup_{f\in L^p(d\mu)} \mathcal{R}f < \frac{p}{p-1}$.

Proof By the definition of μ, we have

$$\begin{aligned} H_\mu f(\lambda_k) &= \frac{1}{\mu([0,\lambda_k])}\int_{[0,\lambda_k]} f(t)\,d\mu(t) \\ &= \frac{(\lambda-1)\sum_{i=-\infty}^{k}\lambda^i f(\lambda^i)}{\lambda^{k+1}} \\ &= \frac{\lambda-1}{\lambda}\sum_{i=-\infty}^{0}\lambda^i f\bigl(\lambda^{i+k}\bigr) \end{aligned} \tag{44}$$

and

$$\begin{aligned} \bigl\|f\bigl(\lambda^i\cdot\bigr)\bigr\|_{L^p(d\mu)} &= \left(\sum_{k=-\infty}^{\infty}\bigl|f\bigl(\lambda^{i+k}\bigr)\bigr|^p\lambda^k\right)^{\frac{1}{p}} \\ &= \left(\sum_{k=-\infty}^{\infty}\bigl|f\bigl(\lambda^{k}\bigr)\bigr|^p\lambda^{k-i}\right)^{\frac{1}{p}} \\ &= \lambda^{-\frac{i}{p}}\|f\|_{L^p(d\mu)}. \end{aligned} \tag{45}$$

By inequalities (44), (45), and Minkowski's inequality, it follows

$$\begin{aligned} \|H_\mu f\|_{L^p(d\mu)} &= \left\|\frac{\lambda-1}{\lambda}\sum_{i=-\infty}^{0}\lambda^i f\bigl(\lambda^i\cdot\bigr)\right\|_{L^p(d\mu)} \\ &\le \frac{\lambda-1}{\lambda}\sum_{i=-\infty}^{0}\lambda^i\bigl\|f\bigl(\lambda^i\cdot\bigr)\bigr\|_{L^p(d\mu)} \\ &= \frac{\lambda-1}{\lambda}\sum_{i=-\infty}^{0}\lambda^{i-\frac{i}{p}}\|f\|_{L^p(d\mu)} \\ &= \frac{\lambda-1}{\lambda-\lambda^{\frac{1}{p}}}\|f\|_{L^p(d\mu)}. \end{aligned} \tag{46}$$

It is easy to get $\frac{\lambda-1}{\lambda-\lambda^{\frac{1}{p}}} < \frac{p}{p-1}$. The proof is completed. □

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