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Approximation properties of λ-Bernstein operators

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Abstract

In this paper, we introduce a new type λ-Bernstein operators with parameter $\lambda \in [-1, 1]$, we investigate a Korovkin type approximation theorem, establish a local approximation theorem, give a convergence theorem for the Lipschitz continuous functions, we also obtain a Voronovskaja-type asymptotic formula. Finally, we give some graphs and numerical examples to show the convergence of $B_{n,\lambda}(f;x)$ to $f(x)$, and we see that in some cases the errors are smaller than $B_n(f)$ to f.

MSC: 41A10; 41A25; 41A36

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1 Introduction

In 1912, Bernstein [1] proposed the famous polynomials called nowadays Bernstein polynomials to prove the Weierstrass approximation theorem as follows:

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) b_{n,k}(x), \tag{1}$$

where $x \in [0,1]$, $n = 1, 2, \ldots$, and Bernstein basis functions $b_{n,k}(x)$ are defined as:

$$b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}. \tag{2}$$

Based on this, there are many papers about Bernstein type operators [2–9]. In 2010, Ye et al. [10] defined new Bézier bases with shape parameter λ by

$$\begin{cases} \tilde{b}_{n,0}(\lambda;x) = b_{n,0}(x) - \frac{\lambda}{n+1} b_{n+1,1}(x), \\ \tilde{b}_{n,i}(\lambda;x) = b_{n,i}(x) + \lambda\left(\frac{n-2i+1}{n^2-1} b_{n+1,i}(x) - \frac{n-2i-1}{n^2-1} b_{n+1,i+1}(x)\right) \quad (1 \le i \le n-1), \\ \tilde{b}_{n,n}(\lambda;x) = b_{n,n}(x) - \frac{\lambda}{n+1} b_{n+1,n}(x), \end{cases} \tag{3}$$

where $\lambda \in [-1, 1]$. When $\lambda = 0$, they reduce to (2). It must be pointed out that we have more modeling flexibility when adding the shape parameter λ.

In this paper, we introduce the new λ-Bernstein operators,

$$B_{n,\lambda}(f;x) = \sum_{k=0}^{n} \tilde{b}_{n,k}(\lambda;x) f\left(\frac{k}{n}\right), \tag{4}$$

where $\tilde{b}_{n,k}(\lambda;x)$ $(k = 0,1,\ldots,n)$ are defined in (3) and $\lambda \in [-1,1]$.

This paper is organized as follows: In the following section, we estimate the moments and central moments of these operators (4). In Sect. 3, we investigate a Korovkin approximation theorem, establish a local approximation theorem, give a convergence theorem for the Lipschitz continuous functions, and obtain a Voronovskaja-type asymptotic formula. In Sect. 4, we give some graphs and numerical examples to show the convergence of $B_{n,\lambda}(f;x)$ to $f(x)$ with different parameters.

2 Some preliminary results

Lemma 2.1 *For λ-Bernstein operators, we have the following equalities*:

$$B_{n,\lambda}(1;x) = 1; \tag{5}$$

$$B_{n,\lambda}(t;x) = x + \frac{1 - 2x + x^{n+1} - (1-x)^{n+1}}{n(n-1)}\lambda; \tag{6}$$

$$B_{n,\lambda}\left(t^2;x\right) = x^2 + \frac{x(1-x)}{n} + \lambda\left[\frac{2x - 4x^2 + 2x^{n+1}}{n(n-1)} + \frac{x^{n+1} + (1-x)^{n+1} - 1}{n^2(n-1)}\right]; \tag{7}$$

$$\begin{aligned} B_{n,\lambda}\left(t^3;x\right) = {} & x^3 + \frac{3x^2(1-x)}{n} + \frac{2x^3 - 3x^2 + x}{n^2} + \lambda\left[\frac{-6x^3 + 6x^{n+1}}{n^2} + \frac{3x^2 - 3x^{n+1}}{n(n-1)}\right. \\ & \left. + \frac{-9x^2 + 9x^{n+1}}{n^2(n-1)} + \frac{-4x + 4x^{n+1}}{n^3(n-1)} + \frac{(1 - x^{n+1} - (1-x)^{n+1})(n+3)}{n^3(n^2-1)}\right]; \end{aligned} \tag{8}$$

$$\begin{aligned} B_{n,\lambda}\left(t^4;x\right) = {} & x^4 + \frac{6(x^3 - x^4)}{n} + \frac{7x^2 - 18x^3 + 11x^4}{n^2} + \frac{x - 7x^2 + 12x^3 - 6x^4}{n^3} \\ & + \left[\frac{6x^2 - 2x^3 - 8x^4 + 4x^{n+1}}{n^2} + \frac{-x^2 - 32x^3 + 16x^4 + 17x^{n+1}}{n^3} + \frac{x - x^{n+1}}{n^4}\right. \\ & \left. + \frac{7x^2 - 7x^{n+1}}{n^2(n-1)} + \frac{x - 23x^2 + 22x^{n+1}}{n^3(n-1)} + \frac{(1-x)^{n+1} + x - 1}{n^4(n-1)}\right]\lambda. \end{aligned} \tag{9}$$

Proof From (4), it is easy to prove $\sum_{k=0}^{n} \tilde{b}_{n,k}(\lambda;x) = 1$, then we can obtain (5). Next,

$$\begin{aligned} & B_{n,\lambda}(t;x) \\ & \quad = \sum_{k=0}^{n} \frac{k}{n} \tilde{b}_{n,k}(\lambda;x) \\ & \quad = \sum_{k=0}^{n-1} \frac{k}{n}\left[b_{n,k}(x) + \lambda\left(\frac{n-2k+1}{n^2-1} b_{n+1,k}(x) - \frac{n-2k-1}{n^2-1} b_{n+1,k+1}(x)\right)\right] \\ & \qquad + b_{n,n}(x) - \frac{\lambda}{n+1} b_{n+1,n}(x) \\ & \quad = \sum_{k=0}^{n} \frac{k}{n} b_{n,k}(x) + \lambda\left(\sum_{k=0}^{n} \frac{k}{n} \frac{n-2k+1}{n^2-1} b_{n+1,k}(x) - \sum_{k=1}^{n-1} \frac{k}{n} \frac{n-2k-1}{n^2-1} b_{n+1,k+1}(x)\right), \end{aligned}$$

as is well known, the Bernstein operators (1) preserve linear functions, that is to say, $B_n(at+b;x)=ax+b$. We denote the latter two parts in the bracket of the last formula by $\triangle_1(n;x)$ and $\triangle_2(n;x)$, then we have

$$B_{n,\lambda}(t;x)=x+\lambda\big(\triangle_1(n;x)+\triangle_2(n;x)\big). \tag{10}$$

Now, we will compute $\triangle_1(n;x)$ and $\triangle_2(n;x)$,

$$\begin{aligned}
\triangle_1(n;x)&=\sum_{k=0}^{n}\frac{k}{n}\frac{n-2k+1}{n^2-1}b_{n+1,k}(x)\\
&=\frac{1}{n-1}\sum_{k=0}^{n}\frac{k}{n}b_{n+1,k}(x)-\frac{2}{n^2-1}\sum_{k=0}^{n}\frac{k^2}{n}b_{n+1,k}(x)\\
&=\frac{(n+1)x}{n(n-1)}\sum_{k=0}^{n-1}b_{n,k}(x)-\frac{2x^2}{n-1}\sum_{k=0}^{n-2}b_{n-1,k}(x)-\frac{2x}{n(n-1)}\sum_{k=0}^{n-1}b_{n,k}(x)\\
&=\frac{(n+1)x}{n(n-1)}\big(1-x^n\big)-\frac{2x^2}{n-1}\big(1-x^{n-1}\big)-\frac{2x}{n(n-1)}\big(1-x^n\big)\\
&=\frac{x}{n}-\frac{2x^2}{n-1}+\frac{x^{n+1}}{n}+\frac{2x^{n+1}}{n(n-1)},
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
\triangle_2(n;x)&=-\sum_{k=1}^{n-1}\frac{k}{n}\frac{n-2k-1}{n^2-1}b_{n+1,k+1}(x)\\
&=-\frac{x}{n}\sum_{k=1}^{n-1}b_{n,k}(x)+\frac{1}{n(n+1)}\sum_{k=1}^{n-1}b_{n+1,k+1}(x)+\frac{2x^2}{n-1}\sum_{k=0}^{n-2}b_{n-1,k}(x)\\
&\quad-\frac{2x}{n(n-1)}\sum_{k=1}^{n-1}b_{n,k}(x)+\frac{2}{n(n^2-1)}\sum_{k=1}^{n-1}b_{n+1,k+1}(x)\\
&=-\frac{x[1-(1-x)^n-x^n]}{n}+\frac{[1-(1-x)^{n+1}-(n+1)x(1-x)^n-x^{n+1}]}{n(n+1)}\\
&\quad+\frac{2x^2(1-x^{n-1})}{n-1}-\frac{2x[1-(1-x)^n-x^n]}{n(n-1)}\\
&\quad+\frac{2[1-(1-x)^{n+1}-(n+1)x(1-x)^n-x^{n+1}]}{n(n^2-1)}\\
&=\frac{2x^2-x-x^{n+1}}{n-1}+\frac{1-(1-x)^{n+1}-x}{n(n-1)}.
\end{aligned} \tag{12}$$

Combining (10), (11) and (12), we have

$$B_{n,\lambda}(t;x)=x+\frac{1-2x+x^{n+1}-(1-x)^{n+1}}{n(n-1)}\lambda.$$

Hence, (6) is proved. Finally, by (4), we have

$$\begin{aligned}
&B_{n,\lambda}\big(t^2;x\big) \\
&\quad= \sum_{k=0}^{n} \frac{k^2}{n^2}\tilde{b}_{n,k}(\lambda;x) \\
&\quad= \sum_{k=0}^{n-1} \frac{k^2}{n^2}\left[b_{n,k}(x) + \lambda\left(\frac{n-2k+1}{n^2-1}b_{n+1,k}(x) - \frac{n-2k-1}{n^2-1}b_{n+1,k+1}(x)\right)\right] \\
&\qquad + b_{n,n}(x) - \frac{\lambda}{n+1}b_{n+1,n}(x) \\
&\quad= \sum_{k=0}^{n} \frac{k^2}{n^2}b_{n,k}(x) + \lambda\left(\sum_{k=0}^{n} \frac{k^2}{n^2}\frac{n-2k+1}{n^2-1}b_{n+1,k}(x) - \sum_{k=1}^{n-1} \frac{k^2}{n^2}\frac{n-2k-1}{n^2-1}b_{n+1,k+1}(x)\right),
\end{aligned}$$

since $B_n(t^2;x) = \sum_{k=0}^{n} \frac{k^2}{n^2}b_{n,k}(x) = x^2 + \frac{x(1-x)}{n}$, and we denote the latter two parts in the bracket of last formula by $\triangle_3(n;x)$ and $\triangle_4(n;x)$, then we have

$$B_{n,\lambda}\big(t^2;x\big) = x^2 + \frac{x(1-x)}{n} + \lambda\big(\triangle_3(n;x) + \triangle_4(n;x)\big). \tag{13}$$

On the one hand,

$$\begin{aligned}
\triangle_3(n;x) &= \sum_{k=0}^{n} \frac{k^2}{n^2}\frac{n-2k+1}{n^2-1}b_{n+1,k}(x) \\
&= \frac{1}{n-1}\sum_{k=0}^{n} \frac{k^2}{n^2}b_{n+1,k}(x) - \frac{2}{n^2-1}\sum_{k=0}^{n} \frac{k^3}{n^2}b_{n+1,k}(x) \\
&= \frac{(n+1)x^2}{n(n-1)}\sum_{k=0}^{n-2} b_{n-1,k}(x) + \frac{(n+1)x}{n^2(n-1)}\sum_{k=0}^{n-1} b_{n,k}(x) - \frac{2x^3}{n}\sum_{k=0}^{n-3} b_{n-2,k}(x) \\
&\quad - \frac{6x^2}{n(n-1)}\sum_{k=0}^{n-2} b_{n-1,k}(x) - \frac{2x}{n^2(n-1)}\sum_{k=0}^{n-1} b_{n,k}(x) \\
&= \frac{(n+1)x^2(1-x^{n-1})}{n(n-1)} + \frac{(n+1)x(1-x^n)}{n^2(n-1)} - \frac{2x^3(1-x^{n-2})}{n} \\
&\quad - \frac{6x^2(1-x^{n-1})}{n(n-1)} - \frac{2x(1-x^n)}{n^2(n-1)} \\
&= \frac{2x^{n+1}-2x^3}{n} + \frac{x^2-x^{n+1}}{n-1} + \frac{x-5x^2+4x^{n+1}}{n(n-1)} + \frac{x^{n+1}-x}{n^2(n-1)}.
\end{aligned} \tag{14}$$

On the other hand,

$$\begin{aligned}
\triangle_4(n;x) &= -\sum_{k=1}^{n-1} \frac{k^2}{n^2}\frac{n-2k-1}{n^2-1}b_{n+1,k+1}(x) \\
&= -\frac{1}{n+1}\sum_{k=1}^{n-1} \frac{k^2}{n^2}b_{n+1,k+1}(x) + \frac{2}{n^2-1}\sum_{k=1}^{n-1} \frac{k^3}{n^2}b_{n+1,k+1}(x)
\end{aligned}$$

$$\begin{aligned}
&= -\frac{x^2}{n}\sum_{k=0}^{n-2} b_{n-1,k}(x) + \frac{x}{n^2}\sum_{k=1}^{n-1} b_{n,k}(x) - \frac{1}{n^2(n+1)}\sum_{k=1}^{n-1} b_{n+1,k+1}(x) \\
&\quad + \frac{2x^3}{n}\sum_{k=0}^{n-3} b_{n-2,k}(x) + \frac{2x}{n^2(n-1)}\sum_{k=1}^{n-1} b_{n,k}(x) - \frac{2}{n^2(n^2-1)}\sum_{k=1}^{n-1} b_{n+1,k+1}(x) \\
&= -\frac{x^2(1-x^{n-1})}{n} + \frac{x[1-(1-x)^n - x^n]}{n^2} \\
&\quad - \frac{1-(1-x)^{n+1}-(n+1)x(1-x)^n - x^{n+1}}{n^2(n+1)} \\
&\quad + \frac{2x^3(1-x^{n-2})}{n} + \frac{2x[1-(1-x)^n - x^n]}{n^2(n-1)} \\
&\quad - \frac{2[1-(1-x)^{n+1}-(n+1)x(1-x)^n - x^{n+1}]}{n^2(n^2-1)} \\
&= \frac{(2x-1)x^2}{n} + \frac{x}{n(n-1)} + \frac{x-1+(1-x)^{n+1}}{n^2(n-1)} - \frac{x^{n+1}}{n-1}. \qquad (15)
\end{aligned}$$

Combining (13), (14) and (15), we obtain

$$B_{n,\lambda}\left(t^2;x\right) = x^2 + \frac{x(1-x)}{n} + \lambda\left[\frac{2x-4x^2+2x^{n+1}}{n(n-1)} + \frac{x^{n+1}+(1-x)^{n+1}-1}{n^2(n-1)}\right],$$

therefore, we get (7). Thus, Lemma 2.1 is proved.

Similarly, we can obtain (8) and (9) by some computations, here we omit these. □

Corollary 2.2 *For fixed $x \in [0,1]$ and $\lambda \in [-1,1]$, using Lemma* 2.1 *and by some easy computations, we have*

$$\begin{aligned}
&B_{n,\lambda}(t-x;x) \\
&\quad = \frac{1-2x+x^{n+1}-(1-x)^{n+1}}{n(n-1)}\lambda \le \frac{1+2x+x^{n+1}+(1-x)^{n+1}}{n(n-1)} := \phi_n(x); \qquad (16) \\
&B_{n,\lambda}\left((t-x)^2;x\right) \\
&\quad = \frac{x(1-x)}{n} + \left[\frac{2x(1-x)^{n+1}+2x^{n+1}-2x^{n+2}}{n(n-1)} + \frac{x^{n+1}+(1-x)^{n+1}-1}{n^2(n-1)}\right]\lambda \\
&\quad \le \frac{x(1-x)}{n} + \frac{2x(1-x)^{n+1}+2x^{n+1}+2x^{n+2}}{n(n-1)} + \frac{x^{n+1}+(1-x)^{n+1}+1}{n^2(n-1)} := \psi_n(x); \qquad (17)
\end{aligned}$$

$$\lim_{n\to\infty} nB_{n,\lambda}(t-x;x) = 0; \qquad (18)$$

$$\lim_{n\to\infty} nB_{n,\lambda}\left((t-x)^2;x\right) = x(1-x), \quad x \in (0,1); \qquad (19)$$

$$\lim_{n\to\infty} n^2B_{n,\lambda}\left((t-x)^4;x\right) = 3x^2-6x^3+3x^4+6\left(x^2-x^3\right)\lambda, \quad x \in (0,1). \qquad (20)$$

Remark 2.3 For $\lambda \in [-1,1]$, $x \in [0,1]$, λ-Bernstein operators possess the endpoint interpolation property, that is,

$$B_{n,\lambda}(f;0) = f(0), \qquad B_{n,\lambda}(f;1) = f(1). \qquad (21)$$

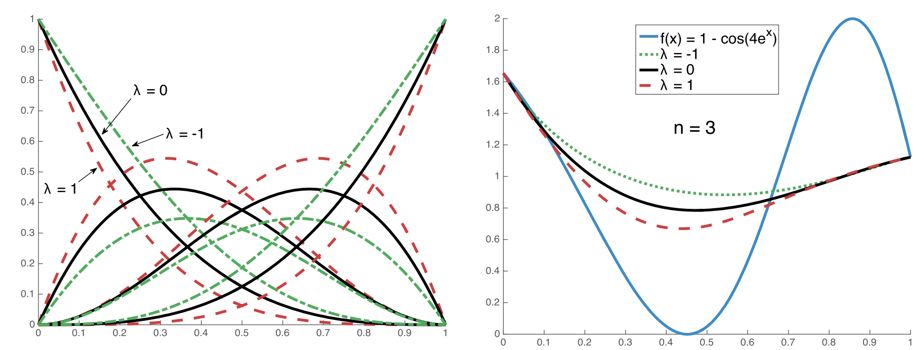

Figure 1 The graphs of $\tilde{b}_{3,k}(\lambda;x)$ with different values of λ (left) and their corresponding $B_{3,\lambda}(f;x)$ (right)

Proof We can obtain (21) easily by using the definition of λ-Bernstein operators (4) and

$$\tilde{b}_{n,k}(\lambda;0)=\begin{cases}0 & (k\neq 0),\\ 1 & (k=0),\end{cases}\qquad \tilde{b}_{n,k}(\lambda;1)=\begin{cases}0 & (k\neq n),\\ 1 & (k=n).\end{cases}$$

Remark 2.3 is proved. □

Example 2.4 The graphs of $\tilde{b}_{3,k}(\lambda;x)$ with $\lambda=-1,0,-1$ are shown in Fig. 1(left). The corresponding $B_{3,\lambda}(f;x)$ with $f(x)=1-\cos(4e^x)$ are shown in Fig. 1(right). The graphs show the λ-Bernstein operators' endpoint interpolation property, which is based on the interpolation property of $\tilde{b}_{n,k}(\lambda,x)$.

3 Convergence properties

As we know, the space $C[0,1]$ of all continuous functions on $[0,1]$ is a Banach space with sup-norm $\|f\|:=\sup_{x\in[0,1]}|f(x)|$. Now, we give a Korovkin type approximation theorem for $B_{n,\lambda}(f;x)$.

Theorem 3.1 *For* $f\in C[0,1]$, $\lambda\in[-1,1]$, λ-*Bernstein operators* $B_{n,\lambda}(f;x)$ *converge uniformly to* f *on* $[0,1]$.

Proof By the Korovkin theorem it suffices to show that

$$\lim_{n\to\infty}\left\|B_{n,\lambda}\left(t^i;x\right)-x^i\right\|=0,\quad i=0,1,2.$$

We can obtain these three conditions easily by (5), (6) and (7) of Lemma 2.1. Thus the proof is completed. □

The Peetre K-functional is defined by $K_2(f;\delta):=\inf_{g\in C^2[0,1]}\{\|f-g\|+\delta\|g''\|\}$, where $\delta>0$ and $C^2[0,1]:=\{g\in C[0,1]:g',g''\in C[0,1]\}$. By [11], there exists an absolute constant $C>0$ such that

$$K_2(f;\delta)\le C\omega_2(f;\sqrt{\delta}),\tag{22}$$

where $\omega_2(f;\delta) := \sup_{0<h\leq\delta}\sup_{x,x+h,x+2h\in[0,1]}|f(x+2h)-2f(x+h)+f(x)|$ is the second order modulus of smoothness of $f\in C[0,1]$. We also denote the usual modulus of continuity of $f\in C[0,1]$ by $\omega(f;\delta) := \sup_{0<h\leq\delta}\sup_{x,x+h\in[0,1]}|f(x+h)-f(x)|$.

Next, we give a direct local approximation theorem for the operators $B_{n,\lambda}(f;x)$.

Theorem 3.2 *For $f\in C[0,1]$, $\lambda\in[-1,1]$, we have*

$$\left|B_{n,\lambda}(f;x)-f(x)\right| \leq C\omega_2\bigl(f;\sqrt{\phi_n(x)+\psi_n(x)}/2\bigr)+\omega\bigl(f;\phi_n(x)\bigr), \tag{23}$$

where C is a positive constant, $\phi_n(x)$ and $\psi_n(x)$ are defined in (16) *and* (17).

Proof We define the auxiliary operators

$$\widetilde{B}_{n,\lambda}(f;x) = B_{n,\lambda}(f;x) - f\left(x+\frac{1-2x+x^{n+1}-(1-x)^{n+1}}{n(n-1)}\lambda\right)+f(x). \tag{24}$$

From (5) and (6), we know that the operators $\widetilde{B}_{n,\lambda}(f;x)$ are linear and preserve the linear functions:

$$\widetilde{B}_{n,\lambda}(t-x;x)=0. \tag{25}$$

Let $g\in C^2[0,1]$, by Taylor's expansion,

$$g(t)=g(x)+g'(x)(t-x)+\int_x^t (t-u)g''(u)\,du,$$

and (25), we get

$$\widetilde{B}_{n,\lambda}(g;x)=g(x)+\widetilde{B}_{n,\lambda}\left(\int_x^t (t-u)g''(u)\,du;x\right).$$

Hence, by (24) and (17), we have

$$\begin{aligned}
&\left|\widetilde{B}_{n,\lambda}(g;x)-g(x)\right| \\
&\quad\leq \left|\int_x^{x+\frac{1-2x+x^{n+1}-(1-x)^{n+1}}{n(n-1)}\lambda}\left(x+\frac{1-2x+x^{n+1}-(1-x)^{n+1}}{n(n-1)}\lambda-u\right)g''(u)\,du\right| \\
&\qquad+\left|B_{n,\lambda}\left(\int_x^t (t-u)g''(u)\,du;x\right)\right| \\
&\quad\leq \int_x^{x+\frac{1-2x+x^{n+1}-(1-x)^{n+1}}{n(n-1)}\lambda}\left|x+\frac{1-2x+x^{n+1}-(1-x)^{n+1}}{n(n-1)}\lambda-u\right|\left|g''(u)\right|du \\
&\qquad+B_{n,\lambda}\left(\left|\int_x^t (t-u)\left|g''(u)\right|du\right|;x\right) \\
&\quad\leq \left[B_{n,\lambda}\bigl((t-x)^2;x\bigr)+\frac{1+2x+x^{n+1}+(1-x)^{n+1}}{n(n-1)}\right]\left\|g''\right\| \\
&\quad\leq \bigl[\phi_n(x)+\psi_n(x)\bigr]\left\|g''\right\|.
\end{aligned}$$

On the other hand, by (24), (5) and (4), we have

$$\left|\widetilde{B}_{n,\lambda}(f;x)\right| \leq \left|B_{n,\lambda}(f;x)\right| + 2\|f\| \leq \|f\|B_{n,\lambda}(1;x) + 2\|f\| \leq 3\|f\|. \tag{26}$$

Now, (24) and (26) imply

$$\begin{aligned}\left|B_{n,\lambda}(f;x) - f(x)\right| &\leq \left|\widetilde{B}_{n,\lambda}(f-g;x) - (f-g)(x)\right| + \left|\widetilde{B}_{n,\lambda}(g;x) - g(x)\right| \\ &\quad + \left|f\left(x + \frac{1-2x+x^{n+1}-(1-x)^{n+1}}{n(n-1)}\lambda\right) - f(x)\right| \\ &\leq 4\|f-g\| + \left[\phi_n(x) + \psi_n(x)\right]\left\|g''\right\| + \omega\left(f;\phi_n(x)\right).\end{aligned}$$

Hence, taking infimum on the right hand side over all $g \in C^2[0,1]$, we get

$$\left|B_{n,\lambda}(f;x) - f(x)\right| \leq 4K_2\left(f;\frac{\phi_n(x)+\psi_n(x)}{4}\right) + \omega\left(f;\phi_n(x)\right).$$

By (22), we have

$$\left|B_{n,\lambda}(f;x) - f(x)\right| \leq C\omega_2\left(f;\sqrt{\phi_n(x)+\psi_n(x)}/2\right) + \omega\left(f;\phi_n(x)\right),$$

where $\phi_n(x)$ and $\psi_n(x)$ are defined in (16) and (17). This completes the proof of Theorem 3.2. □

Remark 3.3 For any $x \in [0,1]$, we have $\lim_{n\to\infty} \phi_n(x) = 0$ and $\lim_{n\to\infty} \psi_n(x) = 0$, these give us a rate of pointwise convergence of the operators $B_{n,\lambda}(f;x)$ to $f(x)$.

Now, we study the rate of convergence of the operators $B_{n,\lambda}(f;x)$ with the help of functions of Lipschitz class $\mathrm{Lip}_M(\alpha)$, where $M > 0$ and $0 < \alpha \leq 1$. A function f belongs to $\mathrm{Lip}_M(\alpha)$ if

$$\left|f(y) - f(x)\right| \leq M|y-x|^\alpha \quad (x,y \in \mathbb{R}). \tag{27}$$

We have the following theorem.

Theorem 3.4 *Let* $f \in \mathrm{Lip}_M(\alpha)$, $x \in [0,1]$ *and* $\lambda \in [-1,1]$, *then we have*

$$\left|B_{n,\lambda}(f;x) - f(x)\right| \leq M\left[\psi_n(x)\right]^{\frac{\alpha}{2}},$$

where $\psi_n(x)$ *is defined in* (17).

Proof Since $B_{n,\lambda}(f;x)$ are linear positive operators and $f \in \mathrm{Lip}_M(\alpha)$, we have

$$\begin{aligned}\left|B_{n,\lambda}(f;x) - f(x)\right| &\leq B_{n,\lambda}\left(\left|f(t) - f(x)\right|;x\right) \\ &= \sum_{k=0}^{n} \tilde{b}_{n,k}(\lambda;x)\left|f\left(\frac{k}{n}\right) - f(x)\right|\end{aligned}$$

$$\leq M\sum_{k=0}^{n}\tilde{b}_{n,k}(\lambda;x)\left|\frac{k}{n}-x\right|^{\alpha}$$
$$\leq M\sum_{k=0}^{n}\left[\tilde{b}_{n,k}(\lambda;x)\left(\frac{k}{n}-x\right)^{2}\right]^{\frac{\alpha}{2}}\left[\tilde{b}_{n,k}(\lambda;x)\right]^{\frac{2-\alpha}{2}}.$$

Applying Hölder's inequality for sums, we obtain

$$\begin{aligned}\left|B_{n,\lambda}(f;x)-f(x)\right| &\leq M\left[\sum_{k=0}^{n}\tilde{b}_{n,k}(\lambda;x)\left(\frac{k}{n}-x\right)^{2}\right]^{\frac{\alpha}{2}}\left[\sum_{k=0}^{n}\tilde{b}_{n,k}(\lambda;x)\right]^{\frac{2-\alpha}{2}}\\ &= M\left[B_{n,\lambda}\left((t-x)^{2};x\right)\right]^{\frac{\alpha}{2}}.\end{aligned}$$

Thus, Theorem 3.4 is proved. □

Finally, we give a Voronovskaja asymptotic formula for $B_{n,\lambda}(f;x)$.

Theorem 3.5 *Let $f(x)$ be bounded on $[0,1]$. Then, for any $x\in(0,1)$ at which $f''(x)$ exists, $\lambda\in[-1,1]$, we have*

$$\lim_{n\to\infty} n\left[B_{n,\lambda}(f;x)-f(x)\right]=\frac{f''(x)}{2}\left[x(1-x)\right]. \tag{28}$$

Proof Let $x\in[0,1]$ be fixed. By the Taylor formula, we may write

$$f(t)=f(x)+f'(x)(t-x)+\frac{1}{2}f''(x)(t-x)^{2}+r(t;x)(t-x)^{2}, \tag{29}$$

where $r(t;x)$ is the Peano form of the remainder, $r(t;x)\in C[0,1]$, using L'Hopital's rule, we have

$$\begin{aligned}\lim_{t\to x} r(t;x) &= \lim_{t\to x}\frac{f(t)-f(x)-f'(x)(t-x)-\frac{1}{2}f''(x)(t-x)^{2}}{(t-x)^{2}}\\ &= \lim_{t\to x}\frac{f'(t)-f'(x)-f''(x)(t-x)}{2(t-x)}=\lim_{t\to x}\frac{f''(t)-f''(x)}{2}=0.\end{aligned}$$

Applying $B_{n,\lambda}(f;x)$ to (29), we obtain

$$\begin{aligned}\lim_{n\to\infty} n\left[B_{n,\lambda}(f;x)-f(x)\right] &= f'(x)\lim_{n\to\infty} nB_{n,\lambda}(t-x;x)+\frac{f''(x)}{2}\lim_{n\to\infty} nB_{n,\lambda}\left((t-x)^{2};x\right)\\ &\quad+\lim_{n\to\infty} nB_{n,\lambda}\left(r(t;x)(t-x)^{2};x\right).\end{aligned} \tag{30}$$

By the Cauchy–Schwarz inequality, we have

$$B_{n,\lambda}\left(r(t;x)(t-x)^{2};x\right)\leq\sqrt{B_{n,\lambda}\left(r^{2}(t;x);x\right)}\sqrt{B_{n,\lambda}\left((t-x)^{4};x\right)}, \tag{31}$$

since $r^{2}(x;x)=0$, then we can obtain

$$\lim_{n\to\infty} nB_{n,\lambda}\left(r(t;x)(t-x)^{2};x\right)=0 \tag{32}$$

by (31) and (20). Finally, using (18), (19), (32) and (30), we get

$$\lim_{n\to\infty} n\big[B_{n,\lambda}(f;x)-f(x)\big]=\frac{f''(x)}{2}\big[x(1-x)\big].$$

Theorem 3.5 is proved. □

4 Graphical and numerical analysis

In this section, we give several graphs and numerical examples to show the convergence of $B_{n,\lambda}(f;x)$ to $f(x)$ with different values of λ and n.

Let $f(x)=1-\cos(4e^x)$, the graphs of $B_{n,-1}(f;x)$ and $B_{n,1}(f;x)$ with different values of n are shown in Figs. 2 and 3. In Table 1, we give the errors of the approximation of $B_{n,\lambda}(f;x)$ to $f(x)$. We can see from Table 1 that in some special cases (such as $n=10,20$ and $\lambda>0$), the errors of $\|f-B_{n,\lambda}(f)\|_\infty$ are smaller than $\|f-B_{n,0}(f)\|_\infty$ (where $B_{n,0}(f;x)$ are classical Bernstein operators). Figure 4 shows the graphs of $B_{n,\lambda}(f;x)$ with $n=10$ and different values of λ.

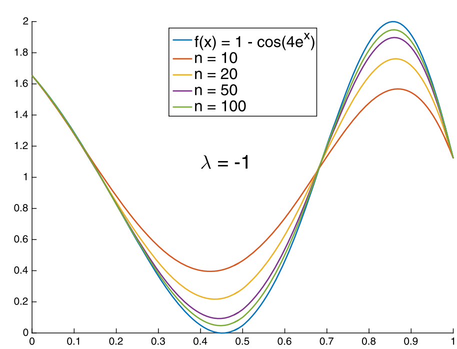

Figure 2 The graphs of $B_{n,-1}(f;x)$ with different values of n

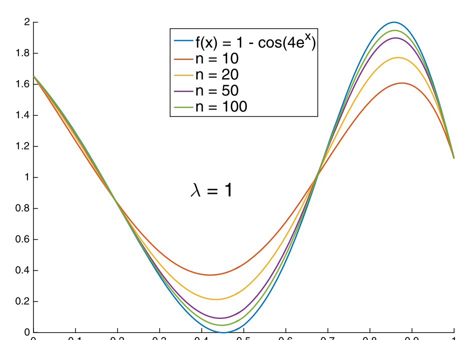

Figure 3 The graphs of $B_{n,1}(f;x)$ with different values of n

Table 1 The errors of the approximation of $B_{n,\lambda}(f;x)$ to $f(x)$ with different values of n and λ

λ	$\|f-B_{n,\lambda}(f)\|_\infty$				
	$n=10$	$n=20$	$n=50$	$n=100$	$n=150$
−1	0.437813	0.242921	0.104883	0.054106	0.036478
−0.5	0.430221	0.241337	0.104880	0.054136	0.036496
0	0.422857	0.239850	0.104884	0.054166	0.036513
0.5	0.415719	0.238458	0.104897	0.054196	0.036531
1	0.408808	0.237158	0.104918	0.054229	0.036550

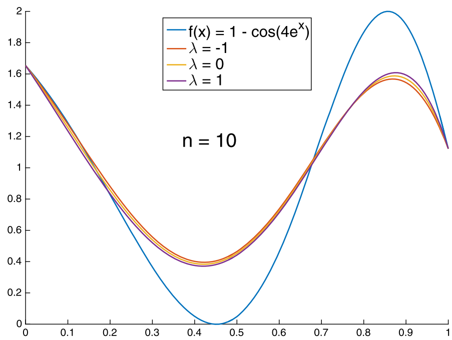

Figure 4 The graphs of $B_{10,\lambda}(f;x)$ with different values of λ

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
QBC, BYL and GZ carried out the molecular genetic studies, participated in the sequence alignment and drafted the manuscript. QBC, BYL and GZ carried out the immunoassays. QBC. BYL and GZ participated in the sequence alignment. QBC, BYL and GZ participated in the design of the study and performed the statistical analysis. QBC, BYL and GZ conceived of the study, and participated in its design and coordination and helped to draft the manuscript. All authors read and approved the final manuscript.

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