


RESEARCH

Open Access



On almost B -summable double sequence spaces

Orhan Tuğ* 

*Correspondence:
orhan.tug@ishik.edu.iq
Department of Mathematics
Education, Ishik University, 100
meter street, Erbil, Iraq

Abstract

The concept of a four-dimensional generalized difference matrix and its domain on some double sequence spaces was recently introduced and studied by Tuğ and Başar (AIP Conference Proceedings, vol. 1759, 2016) and Tuğ (J. Inequal. Appl. 2017(1):149, 2017). In this present paper, as a natural continuation of (J. Inequal. Appl. 2017(1):149, 2017), we introduce new almost null and almost convergent double sequence spaces $B(\mathcal{C}_f)$ and $B(\mathcal{C}_{f_0})$ as the four-dimensional generalized difference matrix $B(r, s, t, u)$ domain in the spaces \mathcal{C}_f and \mathcal{C}_{f_0} , respectively. Firstly, we prove that the spaces $B(\mathcal{C}_f)$ and $B(\mathcal{C}_{f_0})$ of double sequences are Banach spaces under some certain conditions. Then we give an inclusion relation of these new almost convergent double sequence spaces. Moreover, we identify the α -dual, β (bp)-dual and γ -dual of the space $B(\mathcal{C}_f)$. Finally, we characterize some new matrix classes ($B(\mathcal{M}_u) : \mathcal{C}_f$), ($\mathcal{M}_u : B(\mathcal{C}_f)$), and we complete this work with some significant results.

MSC: 46A45; 40C05

Keywords: four-dimensional matrices; $B(r, s, t, u)$ -summable; four-dimensional matrix domain; almost convergent double sequence space; α -, β (ϑ)- and γ -dual; four-dimensional matrix mapping

1 Preliminaries, background and notation

We denote the set of all complex valued double sequence by Ω , which is a vector space with coordinatewise addition and scalar multiplication. Any subspace of Ω is called a double sequence space. A double sequence $x = (x_{mn})$ of complex numbers is called bounded if $\|x\|_\infty = \sup_{m,n \in \mathbb{N}} |x_{mn}| < \infty$, where $\mathbb{N} = \{0, 1, 2, \dots\}$. The space of all bounded double sequences is denoted by \mathcal{M}_u , which is a Banach space with the norm $\|\cdot\|_\infty$. Consider the double sequence $x = (x_{mn}) \in \Omega$. If for every $\epsilon > 0$ there exist a natural number $n_0 = n_0(\epsilon)$ and $l \in \mathbb{C}$ such that $|x_{mn} - l| < \epsilon$ for all $m, n > n_0$, then the double sequence x is said to be convergent in Pringsheim's sense to the limit point l says that $p - \lim_{m,n \rightarrow \infty} x_{mn} = l$; where \mathbb{C} indicates the complex field. The space \mathcal{C}_p denotes the set of all convergent double sequences in Pringsheim's sense. Although every convergent single sequence is bounded, this is not hold for double sequences in general. That is, there are such double sequences which are convergent in Pringsheim's sense but not bounded. Actually, Boos [3, p. 16]

defined the sequence $x = (x_{mn})$ by

$$x_{mn} = \begin{cases} n & m = 0, n \in \mathbb{N}; \\ 0 & m \geq 1, n \in \mathbb{N}. \end{cases}$$

Then it is clearly seen that $p - \lim_{m,n \rightarrow \infty} x_{mn} = 0$ but $\|x\|_\infty = \sup_{m,n \in \mathbb{N}} |x_{mn}| = \infty$, so $x \in C_p - \mathcal{M}_u$. Now, we may denote the space of all both convergent in Pringsheim's sense and bounded double sequences by the set C_{bp} , i.e., $C_{bp} = C_p \cap \mathcal{M}_u$. Hardy [4] showed that a double sequence $x = (x_{mn})$ is said to converge regularly to l if $x \in C_p$ and the limits $x_m := \lim_n x_{mn}, (m \in \mathbb{N})$ and $x_n := \lim_m x_{mn}, (n \in \mathbb{N})$ exist, and the limits $\lim_m \lim_n x_{mn}$ and $\lim_n \lim_m x_{mn}$ exist and are equal to the p -limit of x . Moreover, by C_{bp0} and C_{r0} , we may denote the spaces of all null double sequences contained in the sequence spaces C_{bp} and C_r , respectively. Móricz [5] proved that the double sequence spaces C_{bp}, C_{bp0}, C_r and C_{r0} are Banach spaces with the norm $\|\cdot\|_\infty$. The space \mathcal{L}_q of all absolutely q -summable double sequences corresponding to the space ℓ_q of q -summable single sequences was defined by Başar and Sever [6], that is,

$$\mathcal{L}_q := \left\{ x = (x_{kl}) \in \Omega : \sum_{k,l} |x_{kl}|^q < \infty \right\} \quad (1 \leq q < \infty),$$

which is a Banach space with the norm $\|\cdot\|_q$. Then the space \mathcal{L}_u , which is a special case of the space \mathcal{L}_q with $q = 1$, was introduced by Zeltser [7].

Let λ be a double sequence space and converging with respect to some linear convergence rule is $\vartheta - \lim : \lambda \rightarrow \mathbb{C}$. Then the sum of a double series $\sum_{i,j} x_{ij}$ relating to this rule is defined by $\vartheta - \sum_{i,j} x_{ij} = \vartheta - \lim_{m,n \rightarrow \infty} \sum_{i,j=0}^{m,n} x_{ij}$. Throughout, the summation from 0 to ∞ without limits, that is, $\sum_{i,j} x_{ij}$ means that $\sum_{i,j=0}^\infty x_{ij}$.

Here and below, unless stated otherwise we consider that ϑ denotes any of the symbols p, bp or r .

The α -dual λ^α , the $\beta(\vartheta)$ -dual $\lambda^{\beta(\vartheta)}$ with respect to the ϑ -convergence and the γ -dual λ^γ of the double sequence space λ are, respectively, defined by

$$\begin{aligned} \lambda^\alpha &:= \left\{ a = (a_{kl}) \in \Omega : \sum_{k,l} |a_{kl}x_{kl}| < \infty \text{ for all } x = (x_{kl}) \in \lambda \right\}, \\ \lambda^{\beta(\vartheta)} &:= \left\{ a = (a_{kl}) \in \Omega : \vartheta - \sum_{k,l} a_{kl}x_{kl} \text{ exists for all } x = (x_{kl}) \in \lambda \right\}, \\ \lambda^\gamma &:= \left\{ a = (a_{kl}) \in \Omega : \sup_{m,n \in \mathbb{N}} \left| \sum_{k,l=0}^{m,n} a_{kl}x_{kl} \right| < \infty \text{ for all } x = (x_{kl}) \in \lambda \right\}. \end{aligned}$$

It is easy to see for any two spaces λ and μ of double sequences that $\mu^\alpha \subset \lambda^\alpha$ whenever $\lambda \subset \mu$ and $\lambda^\alpha \subset \lambda^\gamma$. Additionally, it is well known that the inclusion $\lambda^\alpha \subset \lambda^{\beta(\vartheta)}$ holds, while the inclusion $\lambda^{\beta(\vartheta)} \subset \lambda^\gamma$ does not hold, since the ϑ -convergence of the double sequence of partial sum of a double series does not guarantee its boundedness.

Here, we shall be concerned with a four-dimensional matrix transformation from any double sequence space λ to any double sequence space μ . Given any four-dimensional infinite matrix $A = (a_{mnkl})$, where $m, n, k, l \in \mathbb{N}$, any double sequence $x = (x_{kl})$, we write

$Ax = \{(Ax)_{mn}\}_{m,n \in \mathbb{N}}$ for the A -transform of x , exists for every sequence $x = (x_{kl}) \in \lambda$ and it is in μ ; here

$$(Ax)_{mn} = \vartheta - \sum_{k,l} a_{mnkl} x_{kl} \quad \text{for each } m, n \in \mathbb{N}. \tag{1.1}$$

The four-dimensional matrix domain has fundamental importance for this article. Therefore, this concept is presented in this paragraph. The ϑ -summability domain $\lambda_A^{(\vartheta)}$ of A in a space λ of double sequences is described as

$$\lambda_A^{(\vartheta)} = \left\{ x = (x_{kl}) \in \Omega : Ax = \left(\vartheta - \sum_{k,l} a_{mnkl} x_{kl} \right)_{m,n \in \mathbb{N}} \text{ exists and is in } \lambda \right\}.$$

The notation (1.1) says that A maps the space λ into the space μ if $\lambda \subset \mu_A^{(\vartheta)}$ and we denote the set of all four-dimensional matrices, transforming the space λ into the space μ , by $(\lambda : \mu)$. Thus, $A = (a_{mnkl}) \in (\lambda : \mu)$ if and only if the double series on the right side of (1.1) converges in the sense of ϑ for each $m, n \in \mathbb{N}$, i.e., $A_{mn} \in \lambda^{\beta(\vartheta)}$ for all $m, n \in \mathbb{N}$ and we have $Ax \in \mu$ for all $x \in \lambda$; where $A_{mn} = (a_{mnkl})_{k,l \in \mathbb{N}}$ for all $m, n \in \mathbb{N}$. Moreover, the following definitions are significant in order to classify the four-dimensional matrices. A four-dimensional matrix A is called \mathcal{C}_ϑ -conservative if $\mathcal{C}_\vartheta \subset (\mathcal{C}_\vartheta)_A$, and it is called \mathcal{C}_ϑ -regular if it is \mathcal{C}_ϑ -conservative and

$$\vartheta - \lim Ax = \vartheta - \lim_{m,n \rightarrow \infty} (Ax)_{mn} = \vartheta - \lim_{m,n \rightarrow \infty} x_{mn}, \quad \text{where } x = (x_{mn}) \in \mathcal{C}_\vartheta.$$

By using the notations of Zelster [8] we may define the double sequences $e^{kl} = (e_{mn}^{kl}), e^1, e_k$ and e by

$$e_{mn}^{kl} = \begin{cases} 1 & (k, l) = (m, n); \\ 0 & \text{otherwise,} \end{cases}$$

$$e^1 = \sum_k e^{kl}, \quad e_k = \sum_l e^{kl} \quad \text{and} \quad e = \sum_{kl} e^{kl}$$

for all $k, l, m, n \in \mathbb{N}$ and we may write the set Φ by $\Phi = \text{span}\{e^{kl} : k, l \in \mathbb{N}\}$.

In order to establish a new sequence space, special triangular matrices were previously used. These new spaces derived by the domain of matrices are expansions or contractions of the original space, in general. Adams [9] called the four-dimensional infinite matrix $A = (a_{mnkl})$ a triangular matrix if $a_{mnkl} = 0$ for $k > m$ or $l > n$ or both. We also see by [9] that an infinite matrix $A = (a_{mnkl})$ is said to be a triangular if $a_{mmmm} \neq 0$ for all $m, n \in \mathbb{N}$. Moreover, Cooke [10] showed that every triangular matrix has a unique inverse which is also a triangular matrix.

The four-dimensional generalized difference matrix $B(r, s, t, u) = \{b_{mnkl}(r, s, t, u)\}$ and matrix domain of it on some double sequence spaces was recently defined and studied

by Tuğ and Başar [1], and Tuğ [2]. The matrix $B(r, s, t, u) = \{b_{mnkl}(r, s, t, u)\}$ was defined by

$$b_{mnkl}(r, s, t, u) := \begin{cases} su & (k, l) = (m - 1, n - 1), \\ st & (k, l) = (m - 1, n), \\ ru & (k, l) = (m, n - 1), \\ rt & (k, l) = (m, n) \\ 0 & \text{otherwise,} \end{cases}$$

for $r, s, t, u \in \mathbb{R} \setminus \{0\}$ and for all $m, n, k, l \in \mathbb{N}$. Therefore, the $B(r, s, t, u)$ -transform of a double sequence $x = (x_{mn})$ was defined by

$$\begin{aligned} y_{mn} &:= \{B(r, s, t, u)x\}_{mn} = \sum_{k,l} b_{mnkl}(r, s, t, u)x_{kl} \\ &= sux_{m-1,n-1} + stx_{m-1,n} + rux_{m,n-1} + rtx_{mn} \end{aligned} \tag{1.2}$$

for all $m, n \in \mathbb{N}$. Moreover, the matrix $B^{-1}(r, s, t, u) = F(r, s, t, u) = \{f_{mnkl}(r, s, t, u)\}$, which is the inverse of the matrix $B(r, s, t, u)$, was calculated to be

$$f_{mnkl}(r, s, t, u) := \begin{cases} \frac{(-s/r)^{m-k}(-u/t)^{n-l}}{rt} & 0 \leq k \leq m, 0 \leq l \leq n, \\ 0 & \text{otherwise,} \end{cases}$$

for all $m, n, k, l \in \mathbb{N}$. Furthermore, Tuğ and Başar [1] obtained the relation between $x = (x_{mn})$ and $y = (y_{mn})$:

$$x_{mn} = \frac{1}{rt} \sum_{k,l=0}^{m,n} \left(\frac{-s}{r}\right)^{m-k} \left(\frac{-u}{t}\right)^{n-l} y_{kl} \quad \text{for all } m, n \in \mathbb{N}. \tag{1.3}$$

In this paper, as natural continuation of [2] and [11], we introduce new almost null and almost convergent double sequence spaces $B(C_f)$ and $B(C_{f_0})$ as the domain of four-dimensional generalized difference matrix $B(r, s, t, u)$ in the spaces C_f and C_{f_0} , respectively. Throughout the paper, we suppose that the terms of the double sequence $x = (x_{mn})$ and $y = (y_{mn})$ are connected with equation (1.3) and the four-dimensional generalized difference matrix $B(r, s, t, u) = (b_{mnkl}(r, s, t, u))$ will be presented with $B = (b_{mnkl})$.

2 The sequence space C_f of almost convergent double sequences

Lorentz [12] introduced the concept of almost convergence for a single sequence and Móricz and Rhoades [13] extended and studied this concept for a double sequence. A double sequence $x = (x_{kl})$ of complex numbers is said to be almost convergent to a generalized limit L if

$$p - \lim_{q,q' \rightarrow \infty} \sup_{m,n > 0} \left| \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} x_{kl} - L \right| = 0.$$

In this case, L is called the f_2 -limit of the double sequence x . Throughout the paper, \mathcal{C}_f denotes the space of all almost convergent double sequences, *i.e.*,

$$\mathcal{C}_f := \left\{ x = (x_{kl}) \in \Omega : \exists L \in \mathbb{C} \ni \right. \\ \left. p - \lim_{q, q' \rightarrow \infty} \sup_{m, n > 0} \left| \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} x_{kl} - L \right| = 0, \text{ uniformly in } m, n \right\}.$$

It is well known that a convergent double sequence need not be almost convergent. But it is well known that every bounded convergent double sequence is also almost convergent and every almost convergent double sequence is bounded. That is, the inclusion $\mathcal{C}_{bp} \subset \mathcal{C}_f \subset \mathcal{M}_u$ holds, and each inclusion is proper. A double sequence $x = (x_{kl})$ is called almost Cauchy, which was introduced by Čunjaló [14], if for every $\epsilon > 0$ there exists a positive integer K such that

$$\left| \frac{1}{(q_1+1)(q'_1+1)} \sum_{k=m_1}^{m_1+q_1} \sum_{l=n_1}^{n_1+q'_1} x_{kl} - \frac{1}{(q_2+1)(q'_2+1)} \sum_{k=m_2}^{m_2+q_2} \sum_{l=n_2}^{n_2+q'_2} x_{kl} \right| < \epsilon$$

for all $q_1, q'_1, q_2, q'_2 > K$ and $(m_1, n_1), (m_2, n_2) \in \mathbb{N} \times \mathbb{N}$. Mursaleen and Mohiuddine [15] proved that every double sequence is almost convergent if and only if it is almost Cauchy.

Móricz and Rhoades [13] considered that four-dimensional matrices transforming every almost convergent double sequence into a bp -convergent double sequence with the same limit. Almost conservative and almost regular matrices for single sequences were characterized by King [16] and almost \mathcal{C}_ϑ -conservative and almost \mathcal{C}_ϑ -regular four-dimensional matrices for double sequences were defined and characterized by Zeltser *et al.* [17]. Mursaleen [18] introduced the almost strongly regularity for double sequences. A four-dimensional matrix $A = (a_{mnkl})$ is called almost strongly regular if it transforms every almost convergent double sequence into an almost convergent double sequence with the same limit.

Definition 2.1 ([17]) A four-dimensional matrix $A = (a_{mnkl})$ is said to be almost \mathcal{C}_ϑ -conservative matrix if it transforms every ϑ -convergent double sequence $x = (x_{kl})$ into an almost convergent double sequence space, that is, $A = (a_{mnkl}) \in (\mathcal{C}_\vartheta : \mathcal{C}_f)$.

Definition 2.2 ([17]) A four-dimensional matrix $A = (a_{mnkl})$ is said to be almost \mathcal{C}_ϑ -regular if it is \mathcal{C}_ϑ -conservative and $f_2 - \lim Ax = \vartheta - \lim x$ for all $x \in \mathcal{C}_\vartheta$.

3 Spaces of almost B-summable double sequences

In this present section, we define new almost convergent double sequence spaces $B(\mathcal{C}_f)$ and $B(\mathcal{C}_{f_0})$ derived by the domain of four-dimensional matrix B in the spaces of all almost convergent and almost null double sequences \mathcal{C}_f and \mathcal{C}_{f_0} , respectively. Then we show that $B(\mathcal{C}_f)$ and $B(\mathcal{C}_{f_0})$ are Banach spaces with the norm $\|x\|_{B(\mathcal{C}_f)}$, and we prove an inclusion relation.

Now we may define the spaces $B(C_f)$ and $B(C_{f_0})$ by

$$B(C_f) := \left\{ x = (x_{kl}) \in \Omega : \exists L \in \mathbb{C} \ni \right. \\ \left. p - \lim_{q,q' \rightarrow \infty} \sup_{m,n > 0} \left| \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} (Bx)_{kl} - L \right| = 0, \text{ uniformly in } m, n \right\},$$

$$B(C_{f_0}) := \left\{ x = (x_{kl}) \in \Omega : \right. \\ \left. p - \lim_{q,q' \rightarrow \infty} \sup_{m,n > 0} \left| \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} (Bx)_{kl} \right| = 0, \text{ uniformly in } m, n \right\}.$$

Theorem 3.1 *The sequence spaces $B(C_f)$ and $B(C_{f_0})$ are Banach spaces and linearly norm isomorphic to the spaces C_f and C_{f_0} , respectively, with the norm defined by*

$$\|x\|_{B(C_f)} = \sup_{q,q',m,n \in \mathbb{N}} \left| \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} (Bx)_{kl} \right|. \tag{3.1}$$

Proof Since in other cases it can be similarly proved, we prove the theorem only for the space $B(C_f)$. Let us consider a Cauchy sequence $x^{(j)} = \{x_{kl}^{(j)}\}_{k,l \in \mathbb{N}} \in B(C_f)$. Then, for a given $\epsilon > 0$, there exists a positive integer $M(\epsilon) \in \mathbb{N}$ such that

$$\|x^{(j)} - x^{(i)}\|_{B(C_f)} = \sup_{q,q',m,n \in \mathbb{N}} \left| \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} [(Bx^{(j)})_{kl} - (Bx^{(i)})_{kl}] \right| < \epsilon \tag{3.2}$$

for all $i, j > M(\epsilon)$. Then we can read from equation (3.2) that $\{(Bx^{(j)})_{kl}\}_{j \in \mathbb{N}}$ is Cauchy in C_f for each $k, l \in \mathbb{N}$. Since C_f is complete with the norm $\|x\|_{C_f}$ (see [19]), it is convergent. Then we may say that there exists a double sequence $x = (x_{kl}) \in C_f$ such that

$$\frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} (Bx^{(j)})_{kl} \rightarrow \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} (Bx)_{kl}$$

as $j \rightarrow \infty$. Now, by taking the limit as $i \rightarrow \infty$ on the equality (3.2), for every $\epsilon > 0$ we have for all $k, l \in \mathbb{N}$

$$\left| \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} (Bx^{(j)})_{kl} - \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} (Bx)_{kl} \right| < \epsilon.$$

Moreover, since $\{(Bx^{(j)})_{kl}\} \in C_f$ and every almost convergent double sequence is bounded, there exists a positive real number K such that

$$\sup_{m,n \in \mathbb{N}} \left| \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} (Bx^{(j)})_{kl} \right| \leq K.$$

Therefore, we are enabled to write the following inequality:

$$\begin{aligned} \left| \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} (Bx)_{kl} \right| &\leq \left| \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} (Bx^{(j)})_{kl} \right. \\ &\quad \left. - \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} (Bx)_{kl} \right| \\ &\quad + \left| \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} (Bx^{(j)})_{kl} \right| \\ &< \epsilon + K. \end{aligned}$$

Now we can say by taking the supremum over $m, n \in \mathbb{N}$ and the p -limit as $q, q' \rightarrow \infty$ from the inequality acquired above that

$$\left(\sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} (Bx)_{kl} / [(q+1)(q'+1)] \right) \in C_f,$$

that is, $x \in B(C_f)$. We see from this approach that the space $B(C_f)$ is a Banach space with the norm $\| \cdot \|_{B(C_f)}$ defined by (3.1).

Now, we should show here that $B(C_f) \cong C_f$. To show this, we should prove the existence of a linear bijection between the spaces $B(C_f)$ and C_f . Let us consider the transformation T from $B(C_f)$ to C_f by $x \mapsto Tx = y = Bx$, with the notation of (1.2). The linearity and injectivity of T is clear. Let us take any $y = (y_{kl}) \in C_f$ and consider the sequence $x = (x_{kl})$ with respect to the sequence y by equation (1.3) for all $k, l \in \mathbb{N}$. Then we have the following equality:

$$\begin{aligned} (Bx)_{kl} &= sux_{k-1,l-1} + stx_{k-1,l} + rux_{k,l-1} + rtx_{kl} \\ &= su \sum_{i,j=0}^{k-1,l-1} \left(\frac{-s}{r}\right)^{k-i-1} \left(\frac{-u}{t}\right)^{l-j-1} \frac{y_{ij}}{rt} \\ &\quad + st \sum_{i,j=0}^{k-1,l} \left(\frac{-s}{r}\right)^{k-i-1} \left(\frac{-u}{t}\right)^{l-j} \frac{y_{ij}}{rt} \\ &\quad + ru \sum_{i,j=0}^{k,l-1} \left(\frac{-s}{r}\right)^{k-i} \left(\frac{-u}{t}\right)^{l-j-1} \frac{y_{ij}}{rt} \\ &\quad + rt \sum_{i,j=0}^{k,l} \left(\frac{-s}{r}\right)^{k-i} \left(\frac{-u}{t}\right)^{l-j} \frac{y_{ij}}{rt} \\ &= y_{kl} \end{aligned}$$

for all $k, l \in \mathbb{N}$. Thus, we arrive at the consequence that

$$p - \lim_{q,q' \rightarrow \infty} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} (Bx)_{kl} / [(q+1)(q'+1)] = p - \lim_{q,q' \rightarrow \infty} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} y_{kl} / [(q+1)(q'+1)].$$

This shows that $x = (x_{kl}) \in B(C_f)$. Then we may say that T is surjective. Moreover, one can obtain the following equality:

$$\begin{aligned} \|x\|_{B(C_f)} &= \sup_{q,q',m,n \in \mathbb{N}} \left| \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} (Bx)_{kl} \right| \\ &= \sup_{q,q',m,n \in \mathbb{N}} \left| \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} y_{kl} \right| = \|y\|_{C_f} < \infty. \end{aligned}$$

That is, T is norm preserving. Hence, T is linear bijection and $B(C_f)$ and C_f are linearly norm isomorphic. This is what we proposed. □

Theorem 3.2 *Let $s = -r, t = -u$. The inclusions $C_f \subset B(C_f)$ and $C_{f_0} \subset B(C_{f_0})$ strictly hold.*

Proof Firstly, we should prove that the inclusions $C_f \subset B(C_f)$ and $C_{f_0} \subset B(C_{f_0})$ hold. Since $s = -r, t = -u$, the four-dimensional matrix $B = (b_{mkl})$ satisfy the conditions of Lemma 4.8. Then we can say that, for all $x \in C_f$ (or C_{f_0}), $Bx \in C_f$ (or C_{f_0}) whenever $x \in B(C_f)$ (or $B(C_{f_0})$), which means that the inclusions $C_f \subset B(C_f)$ and $C_{f_0} \subset B(C_{f_0})$ hold.

In order to show that the inclusions are strict, we should show that the sets $B(C_f) \setminus C_f$ and $B(C_{f_0}) \setminus C_{f_0}$ are not empty, that is, there exists a double sequence $x = (x_{mn})$ which belongs to $B(C_f)$ but not in C_f . Let consider a double sequence $x = (x_{mn})$ by $x_{mn} = \frac{mn}{rt}$ for all $m, n \in \mathbb{N}$. Since it is not bounded, it is obvious that $x \notin C_f$. But if $s = -r, t = -u$, then we obtain the B -transform of x as

$$\begin{aligned} (Bx)_{mn} &= \{B(r, -r, t, -t)x\}_{mn} = rtx_{m-1,n-1} - rtx_{m-1,n} - rtx_{m,n-1} + rtx_{mn} \\ &= rt \frac{(m-1)(n-1)}{rt} - rt \frac{(m-1)n}{rt} \\ &\quad - rt \frac{m(n-1)}{rt} + rt \frac{mn}{rt} \\ &= 1. \end{aligned}$$

Therefore, we have the following equality with the above result:

$$\frac{1}{(q+1)(q'+1)} \left| \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} (Bx)_{kl} \right| = 1.$$

After taking the supremum over $m, n \in \mathbb{N}$ in the above equality and applying the p -limit as $q, q' \rightarrow \infty$ we see that $Bx \in C_f$. It can easily be shown that the sequence $x_{mn} = \frac{mn}{rt}$ for all $m, n \in \mathbb{N}$ is in $B(C_{f_0}) \setminus C_{f_0}$ by the same reasoning as above. So we omit the details. □

4 The α -, $\beta(\vartheta)$ - and γ -duals of the sequence space $B(C_f)$

In this section, firstly, we calculate the α -dual of the space $B(C_f)$. Then we give some known definitions and lemmas which will be used in the proof of $\beta(bp)$ -dual of the space $B(C_f)$ and in the fourth section of this paper. Moreover, we characterize a new four-dimensional matrix class $(C_f : \mathcal{M}_u)$ in order to calculate the γ -dual of the space $B(C_f)$.

Theorem 4.1 *Let $|s/r|, |u/t| < 1$. The α -dual of the space $B(C_f)$ is the space \mathcal{L}_u .*

Proof We shall prove that the equality $\{B(C_f)\}^\alpha = \mathcal{L}_u$ and that the inclusions $\mathcal{L}_u \subset \{B(C_f)\}^\alpha$ and $\{B(C_f)\}^\alpha \subset \mathcal{L}_u$ hold. For the first inclusion, let us take a sequence $a = (a_{kl}) \in \mathcal{L}_u$ and $x = (x_{kl}) \in B(C_f)$. Then there exists a double sequence $y = (y_{kl}) \in C_f$ with equation (1.2) such that

$$p - \lim_{q,q' \rightarrow \infty} \sup_{m,n > 0} \left| \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} y_{kl} \right| \text{ exists.}$$

Moreover, the inclusion $C_f \subset \mathcal{M}_u$ holds, and there exists a positive real number K such that $\sup_{k,l} |y_{kl}| \leq K$. Since $|s/r|, |u/t| < 1$, we have the following inequality:

$$\begin{aligned} \sum_{k,l} |a_{kl}x_{kl}| &= \sum_{k,l} |a_{kl}| \left| \sum_{i,j=0}^{k,l} \left(\frac{-s}{r}\right)^{k-i} \left(\frac{-u}{t}\right)^{l-j} \frac{y_{ij}}{rt} \right| \\ &\leq \frac{1}{|rt|} \sum_{k,l} |a_{kl}| \sum_{i,j=0}^{k,l} \left| \left(\frac{-s}{r}\right)^{k-i} \left(\frac{-u}{t}\right)^{l-j} \right| |y_{ij}| \\ &\leq \frac{K}{|rt|} \sum_{k,l} |a_{kl}| \sum_{i,j=0}^{k,l} \left| \frac{-s}{r} \right|^{k-i} \left| \frac{-u}{t} \right|^{l-j} \\ &= \frac{K}{|rt|} \sum_{k,l} |a_{kl}| \left(\frac{1 - \left|\frac{s}{r}\right|^{k+1}}{1 - \left|\frac{s}{r}\right|} \right) \left(\frac{1 - \left|\frac{u}{t}\right|^{l+1}}{1 - \left|\frac{u}{t}\right|} \right) \\ &= \frac{K}{|rt|} \left(\frac{1}{1 - \left|\frac{s}{r}\right|} \right) \left(\frac{1}{1 - \left|\frac{u}{t}\right|} \right) \sum_{k,l} |a_{kl}| \left(1 - \left| \frac{s}{r} \right|^{k+1} \right) \left(1 - \left| \frac{u}{t} \right|^{l+1} \right) \\ &\leq \frac{K}{|rt|} \left(\frac{1}{1 - \left|\frac{s}{r}\right|} \right) \left(\frac{1}{1 - \left|\frac{u}{t}\right|} \right) \sum_{k,l} |a_{kl}| \\ &< \infty, \end{aligned}$$

which says that $a = (a_{kl}) \in \{B(C_f)\}^\alpha$. Hence, the inclusion $\mathcal{L}_u \subset \{B(C_f)\}^\alpha$ holds.

Conversely, suppose that $(a_{kl}) \in \{B(C_f)\}^\alpha \setminus \mathcal{L}_u$. Then we have $\sum_{k,l} |a_{kl}x_{kl}| < \infty$ for all $x = (x_{kl}) \in B(C_f)$. When we define a double sequence $x = (x_{kl})$ in the special case of $x = (x_{kl}) = \{(-1)^{k+l}/(rt)\}$ for $r = \alpha s, t = \alpha u$ and $\alpha \in \mathbb{R} - [-1, 1]$, it is trivial to see that $x = (x_{kl}) = \{(-1)^{k+l}/(rt)\} \in B(C_f)$ but

$$\sum_{k,l} |a_{kl}x_{kl}| = \frac{1}{|rt|} \sum_{k,l} |a_{kl}| = \infty.$$

This means that $(a_{kl}) \notin \{B(C_f)\}^\alpha$, which is a contradiction. Therefore, (a_{kl}) must belong to the space \mathcal{L}_u . So, the inclusion $\{B(C_f)\}^\alpha \subset \mathcal{L}_u$ holds. This completes the proof. \square

Now we have the following significant lemmas, which will be used in this present section and the fifth section of this work.

Lemma 4.2 ([17]) *The following statements hold:*

(a) A four-dimensional matrix $A = (a_{mnkl})$ is almost C_{bp} -conservative, i.e., $A \in (C_{bp} : C_f)$, iff the following conditions hold:

$$\sup_{m,n \in \mathbb{N}} \sum_{k,l} |a_{mnkl}| < \infty, \tag{4.1}$$

$$\begin{aligned} \exists a_{ij} \in \mathbb{C} \ni bp - \lim_{q,q' \rightarrow \infty} a(i, j, q, q', m, n) = a_{ij}, \\ \text{uniformly in } m, n \in \mathbb{N} \text{ for each } i, j \in \mathbb{N}, \end{aligned} \tag{4.2}$$

$$\begin{aligned} \exists u \in \mathbb{C} \ni bp - \lim_{q,q' \rightarrow \infty} \sum_{ij} a(i, j, q, q', m, n) = u, \\ \text{uniformly in } m, n \in \mathbb{N}, \end{aligned} \tag{4.3}$$

$$\begin{aligned} \exists a_{ij} \in \mathbb{C} \ni bp - \lim_{q,q' \rightarrow \infty} \sum_i |a(i, j, q, q', m, n) - a_{ij}| = 0, \\ \text{uniformly in } m, n \in \mathbb{N} \text{ for each } j \in \mathbb{N}, \end{aligned} \tag{4.4}$$

$$\begin{aligned} \exists a_{ij} \in \mathbb{C} \ni bp - \lim_{q,q' \rightarrow \infty} \sum_j |a(i, j, q, q', m, n) - a_{ij}| = 0, \\ \text{uniformly in } m, n \in \mathbb{N} \text{ for each } i \in \mathbb{N}, \end{aligned} \tag{4.5}$$

where $a(i, j, q, q', m, n) = \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} a_{kl ij} / [(q+1)(q'+1)]$. In this case, $a = (a_{ij}) \in \mathcal{L}_u$ and

$$f_2 - \lim Ax = \sum_{ij} a_{ij} x_{ij} + \left(u - \sum_{ij} a_{ij} \right) bp - \lim_{ij \rightarrow \infty} x_{ij},$$

that is,

$$\begin{aligned} bp - \lim_{q,q' \rightarrow \infty} \sum_{ij} a(i, j, q, q', m, n) x_{ij} = \sum_{ij} a_{ij} x_{ij} + \left(u - \sum_{ij} a_{ij} \right) bp - \lim_{ij \rightarrow \infty} x_{ij}, \\ \text{uniformly in } m, n \in \mathbb{N}. \end{aligned}$$

(b) A four-dimensional matrix $A = (a_{mnkl})$ is almost C_{bp} -regular, i.e., $A \in (C_{bp} : C_f)_{reg}$, iff the conditions (4.1)-(4.5) hold with $a_{ij} = 0$ for all $i, j \in \mathbb{N}$ and $u = 1$

Lemma 4.3 ([17]) *The following statements hold:*

(a) A four-dimensional matrix $A = (a_{mnkl})$ is almost C_r -conservative, i.e., $A \in (C_r : C_f)$ iff (4.1)-(4.3) and the following conditions hold:

$$\begin{aligned} \exists j_0 \in \mathbb{N} \ni bp - \lim_{q,q' \rightarrow \infty} \sum_i a(i, j_0, q, q', m, n) = u_{j_0}, \\ \text{uniformly in } m, n \in \mathbb{N}, \end{aligned} \tag{4.6}$$

$$\begin{aligned} \exists i_0 \in \mathbb{N} \ni bp - \lim_{q,q' \rightarrow \infty} \sum_j a(i_0, j, q, q', m, n) = v_{i_0}, \\ \text{uniformly in } m, n \in \mathbb{N}, \end{aligned} \tag{4.7}$$

where $a(i, j, q, q', m, n)$ is defined as in Lemma 4.2. In this case, $a = (a_{ij}) \in \mathcal{L}_u$; $(u_j), (v_i) \in \ell_1$ and

$$f_2 - \lim Ax = \sum_{ij} a_{ij}x_{ij} + \sum_i \left(v_i - \sum_j a_{ij} \right) x_i + \sum_j \left(u_j - \sum_i a_{ij} \right) x_j + \left(u + \sum_{ij} a_{ij} - \sum_i v_i - \sum_j u_j \right) r - \lim_{ij \rightarrow \infty} x_{ij}.$$

(b) A four-dimensional matrix $A = (a_{mnkl})$ is almost C_r -regular, i.e., $A \in (C_r : C_f)_{\text{reg}}$, iff the conditions (4.1)-(4.7) hold with $a_{ij} = u_j = v_i = 0$ for all $i, j \in \mathbb{N}$ and $u = 1$.

Lemma 4.4 ([17]) *The following statements hold:*

(a) A four-dimensional matrix $A = (a_{mnkl})$ is almost C_p -conservative, i.e., $A \in (C_p : C_f)$, iff (4.1)-(4.3) hold. In this case $a = (a_{ij}) \in \mathcal{L}_u$, $(a_{ij_0})_{i \in \mathbb{N}}, (a_{i_0j})_{j \in \mathbb{N}} \in \varphi$ where φ denotes the space of all finitely non-zero sequences and

$$f_2 - \lim Ax = \sum_{ij} a_{ij}x_{ij} + \left(u - \sum_{ij} a_{ij} \right) p - \lim_{ij \rightarrow \infty} x_{ij}.$$

(b) A four-dimensional matrix $A = (a_{mnkl})$ is almost C_p -regular, i.e., $A \in (C_r : C_f)_{\text{reg}}$, iff the conditions (4.1)-(4.3) hold with $a_{ij} = 0$ for all $i, j \in \mathbb{N}$ and $u = 1$.

Lemma 4.5 ([13]) *The following statements hold:*

(a) A four-dimensional matrix $A = (a_{mnkl}) \in (C_f : C_{bp})$ iff the condition (4.1) and the following conditions hold:

$$\exists a_{kl} \in \mathbb{C} \exists, bp - \lim_{m,n \rightarrow \infty} a_{mnkl} = a_{kl} \quad \text{for all } k, l \in \mathbb{N}, \tag{4.8}$$

$$\exists u \in \mathbb{C} \exists, bp - \lim_{m,n \rightarrow \infty} \sum_{k,l} a_{mnkl} = u, \tag{4.9}$$

$$\exists k_0 \in \mathbb{N} \exists, bp - \lim_{m,n \rightarrow \infty} \sum_l |a_{mn,k_0,l} - a_{k_0,l}| = 0 \quad \text{for all } l \in \mathbb{N}, \tag{4.10}$$

$$\exists l_0 \in \mathbb{N} \exists, bp - \lim_{m,n \rightarrow \infty} \sum_k |a_{mnk,l_0} - a_{k,l_0}| = 0 \quad \text{for all } k \in \mathbb{N}, \tag{4.11}$$

$$bp - \lim_{m,n \rightarrow \infty} \sum_k \sum_l |\Delta_{01} a_{mnkl}| = 0, \tag{4.12}$$

$$bp - \lim_{m,n \rightarrow \infty} \sum_k \sum_l |\Delta_{10} a_{mnkl}| = 0. \tag{4.13}$$

(b) A four-dimensional matrix $A = (a_{mnkl})$ is strongly regular, i.e., $A \in (C_f : C_{bp})_{\text{reg}}$, iff the conditions (4.1) and (4.8)-(4.13) hold with $a_{kl} = 0$ for all $k, l \in \mathbb{N}$ and $u = 1$, where $\Delta_{10} a_{mnkl} = a_{mnkl} - a_{m,n,k+1,l}$ and $\Delta_{01} a_{mnkl} = a_{mnkl} - a_{m,n,k,l+1}$, $(m, n, k, l = 0, 1, 2, \dots)$.

Lemma 4.6 ([20]) *The four-dimensional matrix $A = (a_{mnkl}) \in (\mathcal{M}_u : C_f)$ iff the condition (4.1) and the following conditions hold:*

$$\exists \beta_{kl} \in \mathbb{C} \exists f_2 - \lim_{m,n \rightarrow \infty} a_{mnkl} = \beta_{kl} \quad \text{for all } k, l \in \mathbb{N}, \tag{4.14}$$

$$\text{for every } m, n, j \in \mathbb{N}, \exists K \in \mathbb{N} \ni \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} a_{klj} = 0,$$

$$\text{for all } q, q', i > K, \tag{4.15}$$

$$\text{for every } m, n, i \in \mathbb{N}, \exists L \in \mathbb{N} \ni \frac{1}{(q+1)(q'+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+q'} a_{klij} = 0,$$

$$\text{for all } q, q', j > L. \tag{4.16}$$

Lemma 4.7 ([21]) *A four-dimensional matrix $A = (a_{mkl})$ is almost regular, i.e., $A \in (C_{bp} : C_f)_{\text{reg}}$, iff the condition (4.1) and the following conditions hold:*

$$\lim_{q, q' \rightarrow \infty} a(i, j, q, q', m, n) = 0,$$

$$\text{uniformly in } m, n \in \mathbb{N} \text{ for each } i, j \in \mathbb{N}, \tag{4.17}$$

$$\lim_{q, q' \rightarrow \infty} \sum_{ij} a(i, j, q, q', m, n) = 1,$$

$$\text{uniformly in } m, n \in \mathbb{N}, \tag{4.18}$$

$$\lim_{q, q' \rightarrow \infty} \sum_i |a(i, j, q, q', m, n)| = 0,$$

$$\text{uniformly in } m, n \in \mathbb{N} \text{ for each } j \in \mathbb{N}, \tag{4.19}$$

$$\lim_{q, q' \rightarrow \infty} \sum_j |a(i, j, q, q', m, n)| = 0,$$

$$\text{uniformly in } m, n \in \mathbb{N} \text{ for each } i \in \mathbb{N}, \tag{4.20}$$

where $a(i, j, q, q', m, n)$ is defined as in Lemma 4.2.

Lemma 4.8 ([18]) *A four-dimensional matrix $A = (a_{mkl})$ is almost strongly regular, i.e., $A \in (C_f : C_f)_{\text{reg}}$, iff A is almost regular and the following two conditions hold:*

$$\lim_{q, q' \rightarrow \infty} \sum_i \sum_j |\Delta_{10} a(i, j, q, q', m, n)| = 0 \quad \text{uniformly in } m, n \in \mathbb{N}, \tag{4.21}$$

$$\lim_{q, q' \rightarrow \infty} \sum_j \sum_i |\Delta_{01} a(i, j, q, q', m, n)| = 0 \quad \text{uniformly in } m, n \in \mathbb{N}, \tag{4.22}$$

where

$$\Delta_{10} a(i, j, q, q', m, n) = a(i, j, q, q', m, n) - a(i + 1, j, q, q', m, n),$$

$$\Delta_{01} a(i, j, q, q', m, n) = a(i, j, q, q', m, n) - a(i, j + 1, q, q', m, n).$$

Now let us define the sets d_k with $k \in \{1, 2, \dots, 7\}$, for the compressing of the following theorems and their proofs, as follows:

$$d_1 = \left\{ a = (a_{kl}) \in \Omega : \sup_{m, n \in \mathbb{N}} \sum_{k, l} \left| \sum_{j, i=k, l}^{m, n} \left(\frac{-s}{r} \right)^{j-k} \left(\frac{-u}{t} \right)^{i-l} \frac{a_{ji}}{rt} \right| < \infty \right\},$$

$$\begin{aligned}
 d_2 &= \left\{ a = (a_{kl}) \in \Omega : \right. \\
 &\quad \left. \exists \beta_{kl} \in \mathbb{C} \exists, \vartheta - \lim_{m,n \rightarrow \infty} \sum_{j,i=k,l}^{m,n} \left(\frac{-s}{r}\right)^{j-k} \left(\frac{-u}{t}\right)^{i-l} a_{ji} = \beta_{kl} \right\}, \\
 d_3 &= \left\{ a = (a_{kl}) \in \Omega : \right. \\
 &\quad \left. \exists u \in \mathbb{C} \exists, \vartheta - \lim_{m,n \rightarrow \infty} \sum_{k,l} \sum_{j,i=k,l}^{m,n} \left(\frac{-s}{r}\right)^{j-k} \left(\frac{-u}{t}\right)^{i-l} \frac{a_{ji}}{rt} = u \right\}, \\
 d_4 &= \left\{ a = (a_{kl}) \in \Omega : \exists l_0 \in \mathbb{N} \exists, \right. \\
 &\quad \left. \vartheta - \lim_{m,n \rightarrow \infty} \sum_k \left| \sum_{j,i=k,l_0}^{m,n} \left(\frac{-s}{r}\right)^{j-k} \left(\frac{-u}{t}\right)^{i-l_0} a_{ji} - \beta_{k,l_0} \right| = 0 \text{ for all } k \in \mathbb{N} \right\}, \\
 d_5 &= \left\{ a = (a_{kl}) \in \Omega : \exists k_0 \in \mathbb{N} \exists, \right. \\
 &\quad \left. \vartheta - \lim_{m,n \rightarrow \infty} \sum_l \left| \sum_{j,i=k_0,l}^{m,n} \left(\frac{-s}{r}\right)^{j-k_0} \left(\frac{-u}{t}\right)^{i-l} a_{ji} - \beta_{k_0,l} \right| = 0 \text{ for all } l \in \mathbb{N} \right\}, \\
 d_6 &= \left\{ a = (a_{kl}) \in \Omega : \right. \\
 &\quad \left. \vartheta - \lim_{m,n \rightarrow \infty} \sum_k \sum_l \left| \Delta_{01} \left\{ \sum_{j,i=k,l}^{m,n} \left(\frac{-s}{r}\right)^{j-k} \left(\frac{-u}{t}\right)^{i-l} \frac{a_{ji}}{rt} \right\} \right| = 0 \right\}, \\
 d_7 &= \left\{ a = (a_{kl}) \in \Omega : \right. \\
 &\quad \left. \vartheta - \lim_{m,n \rightarrow \infty} \sum_k \sum_l \left| \Delta_{10} \left\{ \sum_{j,i=k,l}^{m,n} \left(\frac{-s}{r}\right)^{j-k} \left(\frac{-u}{t}\right)^{i-l} \frac{a_{ji}}{rt} \right\} \right| = 0 \right\}.
 \end{aligned}$$

Theorem 4.9 *The $\beta(bp)$ -dual of the space $B(C_f)$ is the set $\bigcap_{i=1}^7 d_i$*

Proof Suppose that $a = (a_{mn}) \in \Omega$ and $x = (x_{mn}) \in B(C_f)$. Then we have $y = Bx \in C_f$. Therefore, we have the equality (3.16), which is in [2, Theorem 3.11, p.14] with the m, n th partial sum of $\sum_{k,l} a_{kl}x_{kl}$ with $\sum_{k,l=0}^{m,n} a_{kl}x_{kl} = (Dy)_{mn}$. By taking the limit as $m, n \rightarrow \infty$ from this equality, we have the four-dimensional matrix $D = (d_{mnkl})$, which was also defined by Tuğ [2, p. 14] as

$$d_{mnkl} = \begin{cases} \sum_{j,i=k,l}^{m,n} \left(\frac{-s}{r}\right)^{j-k} \left(\frac{-u}{t}\right)^{i-l} \frac{a_{ji}}{rt} & 0 \leq k \leq m, 0 \leq l \leq n; \\ 0 & \text{otherwise,} \end{cases} \tag{4.23}$$

for all $k, l, m, n \in \mathbb{N}$. Then one can obtain from the above consequences $ax \in \mathcal{CS}_{bp}$ whenever $x = (x_{mn}) \in B(C_f)$ iff $Dy \in C_{bp}$ whenever $y = (y_{mn}) \in C_f$. This says that $a = (a_{mn}) \in$

$\{B(C_f)\}^{\beta(\vartheta)}$ iff $D \in (C_f : C_{bp})$. Thus, we can say that the conditions of Lemma 4.5(a) hold with d_{mnkl} instead of a_{mnkl} , i.e.,

$$\begin{aligned} & \sup_{m,n \in \mathbb{N}} \sum_{k,l} \left| \sum_{j,i=k,l}^{m,n} \left(\frac{-s}{r}\right)^{j-k} \left(\frac{-u}{t}\right)^{i-l} \frac{a_{ji}}{rt} \right| < \infty, \\ & \exists \beta_{kl} \in \mathbb{C} \exists, bp - \lim_{m,n \rightarrow \infty} \sum_{j,i=k,l}^{m,n} \left(\frac{-s}{r}\right)^{j-k} \left(\frac{-u}{t}\right)^{i-l} a_{ji} = \beta_{kl}, \\ & \exists u \in \mathbb{C} \exists, bp - \lim_{m,n \rightarrow \infty} \sum_{k,l} \sum_{j,i=k,l}^{m,n} \left(\frac{-s}{r}\right)^{j-k} \left(\frac{-u}{t}\right)^{i-l} \frac{a_{ji}}{rt} = u, \\ & \exists l_0 \in \mathbb{N} \exists, bp - \lim_{m,n \rightarrow \infty} \sum_k \left| \sum_{j,i=k,l_0}^{m,n} \left(\frac{-s}{r}\right)^{j-k} \left(\frac{-u}{t}\right)^{i-l_0} a_{ji} - \beta_{k,l_0} \right| = 0, \\ & \text{for all } k \in \mathbb{N}, \\ & \exists k_0 \in \mathbb{N} \exists, bp - \lim_{m,n \rightarrow \infty} \sum_l \left| \sum_{j,i=k_0,l}^{m,n} \left(\frac{-s}{r}\right)^{j-k_0} \left(\frac{-u}{t}\right)^{i-l} a_{ji} - \beta_{k_0,l} \right| = 0, \\ & \text{for all } l \in \mathbb{N}, \\ & bp - \lim_{m,n \rightarrow \infty} \sum_k \sum_l \left| \Delta_{01} \left\{ \sum_{j,i=k,l}^{m,n} \left(\frac{-s}{r}\right)^{j-k} \left(\frac{-u}{t}\right)^{i-l} \frac{a_{ji}}{rt} \right\} \right| = 0, \\ & bp - \lim_{m,n \rightarrow \infty} \sum_k \sum_l \left| \Delta_{10} \left\{ \sum_{j,i=k,l}^{m,n} \left(\frac{-s}{r}\right)^{j-k} \left(\frac{-u}{t}\right)^{i-l} \frac{a_{ji}}{rt} \right\} \right| = 0, \end{aligned}$$

which is the set $\bigcap_{i=1}^7 d_i$. This is the result we desired. □

Now we characterize a new four-dimensional matrix class $(C_f : \mathcal{M}_u)$, which will be used in the proof of γ -dual of C_f and in some corollaries of the fifth section of this work.

Theorem 4.10 *A four-dimensional matrix $A = (a_{mnkl}) \in (C_f : \mathcal{M}_u)$ if and only if $A_{mn} \in C_f^{\beta(\vartheta)}$ and condition (4.1) hold.*

Proof Suppose that $A = (a_{mnkl}) \in (C_f : \mathcal{M}_u)$. Then Ax exists and is in \mathcal{M}_u for all $x \in C_f$. Then $A_{mn} \in C_f^{\beta(\vartheta)}$ for all $m, n \in \mathbb{N}$. Moreover, it is well known from [15] that the inclusion $C_f \subset \mathcal{M}_u$ holds. So, we can say that the inclusion $(C_f : \mathcal{M}_u) \subset (\mathcal{M}_u : \mathcal{M}_u)$ holds and it gives us the result that the condition (4.1) is necessary.

Conversely, suppose that the condition (4.1) holds and $A_{mn} \in C_f^{\beta(\vartheta)}$. Let us take any sequence $x = (x_{kl}) \in C_f \subset \mathcal{M}_u$, so there exists an $M \in \mathbb{R}^+$ such that $\sup_{k,l \in \mathbb{N}} |x_{kl}| < M$. Since $A_{mn} \in C_f^{\beta(\vartheta)}$ for each $m, n \in \mathbb{N}$, then Ax exists. Since the inequality

$$\left| \sum_{k,l} a_{mnkl} x_{kl} \right| \leq \sum_{k,l} |a_{mnkl} x_{kl}|$$

holds for each fixed $m, n \in \mathbb{N}$, one can obtain by taking the supremum over $m, n \in \mathbb{N}$

$$\begin{aligned} \sup_{m,n \in \mathbb{N}} \left| \sum_{k,l} a_{mnkl} x_{kl} \right| &\leq \sup_{m,n \in \mathbb{N}} \sum_{k,l} |a_{mnkl}| |x_{kl}| \\ &\leq M \sup_{m,n \in \mathbb{N}} \sum_{k,l} |a_{mnkl}| < \infty. \end{aligned}$$

This shows the fact that $Ax \in \mathcal{M}_u$, which completes the proof. □

Theorem 4.11 *The γ -dual of the space $B(C_f)$ is the set $d_1 \cap CS_\vartheta$.*

Proof Let us suppose that $a = (a_{mn}) \in \Omega$ and $x = (x_{mn}) \in B(C_f)$. Then we have $y = Bx \in C_f$. Therefore, by following a similar way to that used in the proof of Theorem 4.9, we may say that $ax \in \mathcal{BS}$ whenever $x = (x_{mn}) \in B(C_f)$ if and only if $Dy \in \mathcal{M}_u$ whenever $y = (y_{mn}) \in C_f$, where the matrix $D = (d_{mnkl})$ was defined by (4.23). This means that $a = (a_{mn}) \in \{B(C_f)\}^\gamma$ if and only if $D \in (C_f : \mathcal{M}_u)$. Thus, one can be seen that the conditions of Theorem 4.10 hold for the matrix $D = (d_{mnkl})$. That is, $D_{mn} \in C_f^{\beta(\vartheta)}$ for each fixed $m, n \in \mathbb{N}$ and

$$\sup_{m,n \in \mathbb{N}} \sum_{k,l} \left| \sum_{j,i=k,l}^{m,n} \left(\frac{-s}{r}\right)^{j-k} \left(\frac{-u}{t}\right)^{i-l} \frac{a_{ji}}{rt} \right| < \infty.$$

This means that the γ -dual of the space $B(C_f)$ is the set $d_1 \cup CS_\vartheta$ as mentioned. □

5 Matrix transformations related to the sequence space $B(C_f)$

In this section, we characterize some new four-dimensional matrix classes $(B(\mathcal{M}_u) : C_f)$, $(\mathcal{M}_u : B(C_f))$. Then we complete this section with some significant results of four-dimensional matrix mapping via the dual summability methods for double sequences which have been introduced and studied by Başar [22] and Yeşilkayağil and Başar [23], and which have recently been applied in [2].

Theorem 5.1 *A four-dimensional matrix $A = (a_{mnkl}) \in (B(C_f) : \mathcal{M}_u)$ if and only if $A_{mn} \in \{B(C_f)\}^{\beta(\vartheta)}$ and the following condition holds:*

$$\sup_{m,n \in \mathbb{N}} \sum_{k,l} \left| \sum_{i,j=k,l}^{m,n} \left(\frac{-s}{r}\right)^{i-k} \left(\frac{-u}{t}\right)^{j-l} \frac{a_{mnij}}{rt} \right| < \infty. \tag{5.1}$$

Proof Suppose that $A = (a_{mnkl}) \in (B(C_f) : \mathcal{M}_u)$. Then Ax exists and is in \mathcal{M}_u for all $x = (x_{mn}) \in B(C_f)$, which implies that $A_{mn} \in \{B(C_f)\}^{\beta(\vartheta)}$ for all $m, n \in \mathbb{N}$. Thus, we may have the following equality derived from the partial sum of the series $\sum_{k,l} a_{mnkl} x_{kl}$:

$$\begin{aligned} \sum_{k,l=0}^{m,n} a_{mnkl} x_{kl} &= \sum_{k,l=0}^{m,n} a_{mnkl} \sum_{j,i=0}^{k,l} \left(\frac{-s}{r}\right)^{k-j} \left(\frac{-u}{t}\right)^{l-i} \frac{y_{ji}}{rt} \\ &= \sum_{k,l=0}^{m,n} \sum_{j,i=k,l}^{m,n} \left(\frac{-s}{r}\right)^{j-k} \left(\frac{-u}{t}\right)^{i-l} \frac{a_{mnji}}{rt} y_{kl} \\ &= (Ey)_{mn}, \end{aligned} \tag{5.2}$$

where the four-dimensional matrix $E = (e_{mnkl})$ is defined by

$$e_{mnkl} = \begin{cases} \sum_{j,i=k,l}^{m,n} \left(\frac{-s}{r}\right)^{j-k} \left(\frac{-u}{t}\right)^{i-l} \frac{a_{mnij}}{rt} & 0 \leq k \leq m, 0 \leq l \leq n; \\ 0 & \text{otherwise,} \end{cases}$$

for all $m, n \in \mathbb{N}$. Then, by taking the ϑ -limit on (5.2) as $m, n \rightarrow \infty$, we may say that $Ax = Ey$. Hence, $Ey \in \mathcal{M}_u$ whenever $y \in \mathcal{C}_f$, that is, $E \in (\mathcal{C}_f : \mathcal{M}_u)$. In this instance, the conditions of Theorem 4.10 hold with $E = (e_{mnkl})$ instead of $A = (a_{mnkl})$, i.e., $E_{mn} \in \{\mathcal{C}_f\}^{\beta(\vartheta)}$ and $\sup_{m,n \in \mathbb{N}} \sum_{k,l} |e_{mnkl}| < \infty$. This completes the proof. \square

Theorem 5.2 *A four-dimensional matrix $A = (a_{mnkl}) \in (\mathcal{C}_f : B(\mathcal{M}_u))$ if and only if $A_{mn} \in \{\mathcal{C}_f\}^{\beta(\vartheta)}$ and the following condition hold:*

$$\sup_{m,n \in \mathbb{N}} \sum_{k,l} \left| \sum_{i,j=0}^{m,n} b_{mnij}(r, s, t, u) a_{ijkl} \right| < \infty. \tag{5.3}$$

Proof The proof can be shown by the same method as is followed in Theorem 5.1 by using equation (4.6) [24, Theorem 4.7] between the elements of the four-dimensional matrices $A = (a_{mnkl})$ and $G = (g_{mnkl})$. So we omit the details. \square

Tuğ [2] has recently applied the dual summability methods for double sequences which has been introduced and studied by Başar [22], and Yeşilkayağil and Başar [23]. In this work, we use the relation between the four-dimensional matrices $E = (e_{mnkl})$, $e(m, n)$, $G = (g_{mnkl})$ and $H = (h_{mnkl})$ with $A = (a_{mnkl})$, which has been proved and studied in [2, Lemma 4.2, Theorem 4.5].

Now, we may give the relation between the four-dimensional matrices $E = (e_{mnkl})$, $e(m, n)$, $G = (g_{mnkl})$ and $H = (h_{mnkl})$ by

$$\begin{aligned} e_{mnkl} &= \sum_{i,j=k,l}^{m,n} \left(\frac{-s}{r}\right)^{i-k} \left(\frac{-u}{t}\right)^{j-l} \frac{a_{mnij}}{rt}, \\ e(m, n) &= \sum_{k,l=0}^{m,n} \sum_{i,j=k,l}^{\infty} \left(\frac{-s}{r}\right)^{i-k} \left(\frac{-u}{t}\right)^{j-l} \frac{a_{mnij}}{rt}, \\ g_{mnkl} &= \sum_{i,j=0}^{m,n} b_{mnij} a_{ijkl}, \quad \text{and} \\ h_{mnkl} &= \sum_{i,j=k,l}^{m,n} b_{mnij} e_{ijkl} \end{aligned}$$

for all $m, n, k, l \in \mathbb{N}$.

Now we may give the following new significant results for the four-dimensional infinite matrix $A = (a_{mnkl})$.

Corollary 5.3 *The following statements hold.*

- (i) $A \in (B(\mathcal{C}_{bp}) : \mathcal{C}_f)$ iff (4.1)-(4.5) hold with e_{mnkl} instead of a_{mnkl} .
- (ii) $A \in (B(\mathcal{C}_r) : \mathcal{C}_f)$ iff (4.1)-(4.3) and (4.6)-(4.7) hold with e_{mnkl} instead of a_{mnkl} .

- (iii) $A \in (B(C_p) : C_f)$ iff (4.1)-(4.3) hold with e_{mnkl} instead of a_{mnkl} .
- (iv) $A \in (B(\mathcal{M}_u) : C_f)$ iff (4.1) and (4.14)-(4.16) hold with e_{mnkl} instead of a_{mnkl} .
- (v) $A \in (B(C_f) : C_{bp})$ iff (4.1) and (4.8)-(4.13) hold with e_{mnkl} instead of a_{mnkl} .

Corollary 5.4 *The following statements hold.*

- (i) $A \in (C_p : B(C_f))$ iff (4.1)-(4.3) hold with g_{mnkl} instead of a_{mnkl} .
- (ii) $A \in (C_{bp} : B(C_f))$ iff (4.1)-(4.5) hold with g_{mnkl} instead of a_{mnkl} .
- (iii) $A \in (C_r : B(C_f))$ iff (4.1)-(4.3) and (4.6)-(4.7) hold with g_{mnkl} instead of a_{mnkl} .
- (iv) $A \in (\mathcal{M}_u : B(C_f))$ iff (4.1) and (4.14)-(4.16) hold with g_{mnkl} instead of a_{mnkl} .
- (v) $A \in (C_f : B(C_{bp}))$ iff (4.1) and (4.8)-(4.13) hold with g_{mnkl} instead of a_{mnkl} .

Corollary 5.5 *The following statements hold.*

- (i) $A \in (B(C_p) : B(C_f))$ iff (4.1)-(4.3) hold with h_{mnkl} instead of a_{mnkl} .
- (ii) $A \in (B(C_{bp}) : B(C_f))$ iff (4.1)-(4.5) hold with h_{mnkl} instead of a_{mnkl} .
- (iii) $A \in (B(C_r) : B(C_f))$ iff (4.1)-(4.3) and (4.6)-(4.7) hold with h_{mnkl} instead of a_{mnkl} .
- (iv) $A \in (B(\mathcal{M}_u) : B(C_f))$ iff (4.1) and (4.14)-(4.16) hold with h_{mnkl} instead of a_{mnkl} .
- (v) $A \in (B(C_f) : B(C_{bp}))$ iff (4.1) and (4.8)-(4.13) hold with h_{mnkl} instead of a_{mnkl} .
- (vi) $A \in (B(C_f) : B(\mathcal{M}_u))$ iff (4.1) hold with h_{mnkl} instead of a_{mnkl} .

Corollary 5.6 *The following statements hold.*

- (i) $A \in (B(C_f) : CS_{bp})$ iff (4.1) and (4.8)-(4.13) hold with $e(m, n)$ instead of a_{mnkl} .
- (ii) $A \in (B(C_f) : BS)$ iff (4.1) hold with $e(m, n)$ instead of a_{mnkl} .

Corollary 5.7 *The following statements hold.*

- (i) $A \in (B(C_f) : C_f; p)$ iff (4.1), (4.17)-(4.22) hold with e_{mnkl} instead of a_{mnkl} .
- (ii) $A \in (C_f : B(C_f); p)$ iff (4.1), (4.17)-(4.22) hold with g_{mnkl} instead of a_{mnkl} .
- (iii) $A \in (B(C_f) : B(C_f); p)$ iff (4.1), (4.17)-(4.22) hold with h_{mnkl} instead of a_{mnkl} .

6 Conclusion

The concept of almost convergence of single sequence was introduced by Lorentz [12]. In 2010, Mursaleen [25] investigated the certain properties of the space of almost convergent sequences denoted by f . Then many mathematicians have studied the matrix domain on almost null and almost convergent sequences spaces (see [26–29]).

The almost convergence for double sequence was introduced by Moricz and Rhoades [13] and studied by many researchers (see [15, 21, 30–38]). Yeşilkayagil and Başar [19] recently studied the topological properties of the spaces of almost null and almost convergent double sequences.

In this work, we studied the domain of the four-dimensional generalized difference matrix $B = (b_{mnkl})$ in the spaces of almost null and almost convergent double sequences and examined some topological properties. Moreover, we determined the α -, $\beta(bp)$ - and γ -duals of the space $B(C_f)$ and characterized some new classes of four-dimensional matrix mappings related with the sequence space $B(C_f)$. The characterization of the matrix classes $(C_f : C_p)$, $(C_f : C_r)$, $(B(C_f) : C_p)$ and $(B(C_f) : C_r)$, and $(\mathcal{L}_{s'} : C_f)$, $(B(\mathcal{L}_{s'}) : C_f)$, $(\mathcal{L}_{s'} : B(C_f))$ in the two cases $0 < s' < 1$ and $1 < s' < \infty$, and the $\beta(p)$ -dual and $\beta(r)$ -dual of the space $B(C_f)$ are still open problems.

Acknowledgements

We would like to thank Hüsametdin Çapan and Prof. Dr. Feyzi Başar for their careful reading and valuable suggestions on the revised version of this paper, which improved the presentation and readability. I also would like to thank the audience of ICAAM-2016 for their valuable suggestions to improve the quality of this study. This work was supported by the Research Center of Ishik University, Erbil-Iraq.

Competing interests

The author declares that there are no competing interests.

Authors' contributions

The author defined new almost convergent and almost null double sequence spaces which were derived as the domain of four-dimensional generalized difference matrix $B = (b_{mki})$ and proved some topological related theorems. Moreover, the author computed the α -, β (bp)- and γ -duals of this new almost convergent double sequence spaces and lastly, characterized some new matrix classes. In the last section, some studies were summarized and some open problems were given by the author. The author read and approved the final manuscript.

Article information

The extended abstract of this article has been published in the special issue of the collection of articles from Springer book series 'Proceedings in Mathematics and Statistics' with the materials of the conference 'Functional analysis in interdisciplinary applications' which is organized in the framework of the VI Congress of the Turkic World Mathematical Society (Astana, Kazakhstan, October 2-5, 2017).

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 18 October 2017 Accepted: 28 December 2017 Published online: 08 January 2018

References

1. Tuğ, O, Başar, F: Four-Dimensional Generalized Difference Matrix and Some Double Sequence Spaces. AIP Conference Proceedings, vol. 1759. AIP, New York (2016)
2. Tuğ, O: Four-dimensional generalized difference matrix and some double sequence spaces. *J. Inequal. Appl.* **2017**(1), 149 (2017)
3. Boos, J: Classical and Modern Methods in Summability. Oxford University Press, New York (2000)
4. Hardy, GH: On the convergence of certain multiple series. *Math. Proc. Camb. Philos. Soc.* **19**, 86-95 (1916-1919)
5. Móricz, F: Extensions of the spaces c and c_0 from single to double sequence. *Acta Math. Hung.* **57**, 129-136 (1991)
6. Başar, F, Sever, Y: The space \mathcal{L}_α of double sequences. *Math. J. Okayama Univ.* **51**, 149-157 (2009)
7. Zeltser, M: On conservative matrix methods for double sequence spaces. *Acta Math. Hung.* **95**(3), 225-242 (2002)
8. Zeltser, M: Investigation of Double Sequence Spaces by Soft and Hard Analytical Methods. Tartu Univ. Press, Tartu (2001)
9. Adams, CR: On non-factorable transformations of double sequences. *Proc. Natl. Acad. Sci. USA* **19**(5), 564-567 (1933)
10. Cooke, RC: Infinite Matrices and Sequence Spaces. Macmillan & Co., London (1950)
11. Tuğ, O: Four-dimensional generalized difference matrix and almost convergent double sequence spaces. In: Kalmenov, T, Nursultanov, E, Ruzhansky, M, Sadybekov, M (eds.) *Functional Analysis in Interdisciplinary Applications. FAIA 2017. Springer Proceedings in Mathematics & Statistics*, vol. 216. Springer, Cham (2017)
12. Lorentz, GG: A contribution to the theory of divergent sequences. *Acta Math.* **80**(1), 167-190 (1948)
13. Moricz, F, Rhoades, BE: Almost convergence of double sequences and strong regularity of summability matrices. *Math. Proc. Camb. Philos. Soc.* **104**, 283-294 (1988)
14. Čunjalo, F: Almost convergence of double sequences-some analogies between measure and category. *Math. Maced.* **5**, 21-24 (2007)
15. Mursaleen, M, Mohiuddine, SA: *Convergence Methods for Double Sequence and Applications*. Springer, New Delhi (2014)
16. King, JP: Almost summable sequences. *Proc. Am. Math. Soc.* **17**, 1219-1225 (1966)
17. Zeltser, M, Mursaleen, M, Mohiuddine, SA: On almost conservative matrix methods for double sequence spaces. *Publ. Math. (Debr.)* **75**, 378-399 (2009)
18. Mursaleen, M: Almost strongly regular matrices and a core theorem for double sequences. *J. Math. Anal. Appl.* **293**(2), 523-531 (2004)
19. Yeşilkayağil, M, Başar, F: Some topological properties of the spaces of almost null and almost convergent double sequences. *Turk. J. Math.* **40**(3), 624-630 (2016)
20. Yeşilkayağil, M, Başar, F: On the characterization of a class of four dimensional matrices and Steinhaus type theorems. *Kragujev. J. Math.* **40**(1), 35-45 (2016)
21. Mursaleen, M, Savaş, E: Almost regular matrices for double sequences. *Studia Sci. Math. Hung.* **40**, 205-212 (2003)
22. Başar, F: *Summability Theory and Its Applications*. Bentham Science Publishers, e-books, Monographs, Istanbul (2012)
23. Yeşilkayağil, M, Başar, F: Four dimensional dual and dual of some new sort summability methods. *Contemp. Anal. Appl. Math.* **3**(1), 13-29 (2015)
24. Yeşilkayağil, M, Başar, F: On the domain of Riesz mean in the space L_s . *Filomat* **31**(4), 925-940 (2017)
25. Mursaleen, M: Almost convergence and some related methods. In: *Modern Methods of Analysis and Its Applications*, pp. 1-10. Anamaya Publ., New Delhi (2010)
26. Başar, F, Kirişçi, M: Almost convergence and generalized difference matrix. *Comput. Math. Appl.* **61**(3), 602-611 (2011)
27. Tuğ, O, Başar, F: On the spaces of Nörlund almost null and Nörlund almost convergent sequences. *Filomat* **30**(3), 773-783 (2016)

28. Kayaduman, K, Şengönül, M: The spaces of Cesàro almost convergent sequences and core theorems. *Acta Math. Sci. Ser. B Engl. Ed.* **32**(6), 2265-2278 (2012)
29. Şengönül, M, Kayaduman, K: On the Riesz almost convergent sequences space. *Abstr. Appl. Anal.* **2012**, Article ID 691694 (2012)
30. Móricz, F, Rhoades, BE: Some characterizations of almost convergence for single and double sequences. *Proceedings of the Third Annual Meeting of the International Workshop in Analysis and its Applications. Publ. Inst. Math. (Belgr.)* **48**(62), 61-68 (1990)
31. Mohiuddine, SA: An application of almost convergence in approximation theorems. *Appl. Math. Lett.* **24**, 1856-1860 (2011)
32. Mursaleen, M, Mohiuddine, SA: Banach limit and some new spaces of double sequences. *Turk. J. Math.* **36**, 121-130 (2012)
33. Mohiuddine, SA, Alotaibi, A: Almost conservative four-dimensional matrices through de la Vallée-Poussin mean. *Abstr. Appl. Anal.* **2014**, Article ID 412974 (2014)
34. Mursaleen, M, Mohiuddine, SA: Regularly-conservative and-coercive four dimensional matrices. *Comput. Math. Appl.* **56**(6), 1580-1586 (2008)
35. Mohiuddine, SA, Alotaibi, A: Generalized almost convergence and core theorems of double sequences. *Abstr. Appl. Anal.* **2014**, Article ID 152910 (2014)
36. Mursaleen, M, Mohiuddine, SA: Almost bounded variation of double sequences and some four dimensional summability matrices. *Publ. Math. (Debr.)* **75**(3-4), 495-508 (2009)
37. Mursaleen, M, Mohiuddine, SA: Some new double sequence spaces of invariant means. *Glas. Mat.* **45**(1), 139-153 (2010)
38. Mursaleen, M, Mohiuddine, SA: Invariant mean and some core theorems for double sequences. *Taiwan. J. Math.* **14**(1), 21-33 (2010)

Submit your manuscript to a SpringerOpen[®] journal and benefit from:

- ▶ Convenient online submission
- ▶ Rigorous peer review
- ▶ Open access: articles freely available online
- ▶ High visibility within the field
- ▶ Retaining the copyright to your article

Submit your next manuscript at ▶ springeropen.com
