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The necessary and sufficient conditions for the existence of a kind of Hilbert-type multiple integral inequality with the non-homogeneous kernel and its applications

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Abstract

For $x = (x_1, \ldots, x_n)$, $u(x) = (\sum_{i=1}^{n} a_i x_i^{\rho})^{1/\rho}$, $v(y) = (\sum_{i=1}^{n} b_i y_i^{\rho})^{1/\rho}$, by using the methods and techniques of real analysis, the sufficient and necessary conditions for the existence of the Hilbert-type multiple integral inequality with the kernel $K(u(x), v(y)) = G(u^{\lambda_1}(x)v^{\lambda_2}(y))$ and the best possible constant factor are discussed. Furthermore, its application in the operator theory is considered.

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1 Introduction

For $n \geq 1$, $R_+^n = \{x = (x_1, \ldots, x_n) : x_i > 0, i = 1, \ldots, n\}$, $a_i, b_i > 0$ $(i = 1, \ldots, n)$, $\omega(x) > 0$ $(x \in R_+^n)$, and $\rho > 0$, we set

$$u(x) = \left(\sum_{i=1}^{n} a_i x_i^{\rho}\right)^{\frac{1}{\rho}}, \qquad v(y) = \left(\sum_{i=1}^{n} b_i y_i^{\rho}\right)^{\frac{1}{\rho}},$$

$$L_{\omega}^{p}(R_+^n) := \left\{f(x) \geq 0 : \|f\|_{p,\omega} = \left(\int_{R_+^n} \omega(x) f^p(x)\, dx\right)^{1/p} < +\infty\right\}.$$

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $K(u, v) \geq 0$ $(u, v > 0)$, then the Hilbert-type multiple integral inequality is of the form

$$\int_{R_+^n} \int_{R_+^n} K\big(u(x), v(y)\big) f(x) g(y)\, dx\, dy \leq M \|f\|_{p,u^{\alpha}} \|g\|_{q,v^{\beta}}. \tag{1}$$

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Define a singular integral operator T:

$$T(f)(y) := \int_{R_+^n} K\bigl(u(x), v(y)\bigr) f(x)\, dx, \quad y \in R_+^n, \tag{2}$$

then (1) may be rewritten as follows:

$$\int_{R_+^n} T(f)(y) g(y)\, dy \le M \|f\|_{p,u^\alpha} \|g\|_{q,v^\beta}.$$

It is easy to prove that (1) is equivalent to the following inequality:

$$\bigl\| T(f) \bigr\|_{p,v^\gamma} \le M \|f\|_{p,u^\alpha} \|g\|_{q,v^\beta}, \tag{3}$$

where $\gamma = \beta(1-p)$. When the operator T satisfies (3), T is called bounded operator from $L^p_{u^\alpha}(R_+^n)$ to $L^p_{v^\gamma}(R_+^n)$.

At present, there are lots of research results on Hilbert-type single integral inequality (cf. [1–14]). But there are relatively few studies on Hilbert-type multiple integral inequality. In particular, there are fewer studies on the necessary and sufficient conditions for the existence of the multiple integral inequality.

In this article, by using the methods and techniques of real analysis, we give the sufficient and necessary conditions for the existence of the Hilbert-type multiple integral inequality with the non-homogeneous kernel

$$K\bigl(u(x), v(y)\bigr) = G\bigl(u^{\lambda_1}(x) v^{\lambda_2}(y)\bigr),$$

and calculate the best possible constant factor. Furthermore, its application in the operator theory is considered.

2 Some lemmas

Lemma 1 *Suppose that* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $n \ge 1$, $\rho > 0$, $\lambda_1 \lambda_2 > 0$, $a_i, b_i > 0$ $(i = 1, \ldots, n)$, $u(x) = (\sum_{i=1}^n a_i x_i^\rho)^{\frac{1}{\rho}}$, $v(y) = (\sum_{i=1}^n b_i y_i^\rho)^{\frac{1}{\rho}}$.

If $K(u(x), v(y)) = G(u^{\lambda_1}(x) v^{\lambda_2}(y))$ *is a non-negative measurable function, setting*

$$W_1 := \int_{R_+^n} \bigl(v(t)\bigr)^{-\frac{\beta+n}{q}} K\bigl(1, v(t)\bigr)\, dt,$$

$$W_2 := \int_{R_+^n} \bigl(u(t)\bigr)^{-\frac{\alpha+n}{p}} K\bigl(u(t), 1\bigr)\, dt,$$

we have the following:

$$\omega_1(x) := \int_{R_+^n} \bigl(v(y)\bigr)^{-\frac{\beta+n}{q}} K\bigl(u(x), v(y)\bigr)\, dy = \bigl(u(x)\bigr)^{\frac{\lambda_1}{\lambda_2}(\frac{\beta+n}{q} - n)} W_1,$$

$$\omega_2(x) := \int_{R_+^n} \bigl(u(x)\bigr)^{-\frac{\alpha+n}{p}} K\bigl(u(x), v(y)\bigr)\, dx = \bigl(v(y)\bigr)^{\frac{\lambda_2}{\lambda_1}(\frac{\alpha+n}{p} - n)} W_2.$$

Proof Since $v(ay) = av(y)$ $(a > 0)$, in view of $K(tu, v) = K(u, t^{\frac{\lambda_1}{\lambda_2}} v)$, setting $t = u^{\frac{\lambda_1}{\lambda_2}}(x)y$, we find $dy = u^{\frac{-n\lambda_1}{\lambda_2}}(x)\,dt$ and

$$\begin{aligned}\omega_1(x) &= \int_{R_+^n} \big(v(y)\big)^{-\frac{\beta+n}{q}} K\big(1, u^{\frac{\lambda_1}{\lambda_2}}(x)v(y)\big)\,dy \\ &= \int_{R_+^n} \big(u^{-\frac{\lambda_1}{\lambda_2}}(x)v(t)\big)^{-\frac{\beta+n}{q}} K\big(1, v(t)\big)u^{\frac{-n\lambda_1}{\lambda_2}}(x)\,dt \\ &= \big(u(x)\big)^{\frac{\lambda_1}{\lambda_2}(\frac{\beta+n}{q}-n)} W_1.\end{aligned}$$

In the same way, we have

$$\omega_2(x) = \big(v(y)\big)^{\frac{\lambda_2}{\lambda_1}(\frac{\alpha+n}{p}-n)} W_2.$$

The lemma is proved. □

Lemma 2 (cf. [15]) *If $p_i > 0$, $a_i > 0$, $\alpha_i > 0$ $(i = 1, \ldots, n)$ and $\psi(t)$ is a measurable function, then we have the following*:

$$\begin{aligned}&\int \cdots \int_{\{x_i>0; \sum_{i=1}^n (\frac{x_i}{a_i})^{\alpha_i} \le 1\}} \psi\left(\sum_{i=1}^n \left(\frac{x_i}{a_i}\right)^{\alpha_i}\right) x_1^{p_1-1} \cdots x_n^{p_n-1}\,dx_1 \cdots dx_n \\ &\quad = \frac{a_1^{p_1} \cdots a_n^{p_n} \Gamma(\frac{p_1}{\alpha_1}) \cdots \Gamma(\frac{p_n}{\alpha_n})}{\alpha_1 \cdots \alpha_n \Gamma(\sum_{i=1}^n \frac{p_i}{\alpha_i})} \int_0^1 \psi(t) t^{\sum_{i=1}^n \frac{p_i}{\alpha_i} - 1}\,dt,\end{aligned}$$

where $\Gamma(t)$ is the gamma function. In particular, for $\alpha_i = \rho$, $p_i = 1$, $b_i = \frac{1}{a_i^\rho}$ $(i = 1, \ldots, n)$, we have

$$\begin{aligned}&\int \cdots \int_{\{x_i>0; \sum_{i=1}^n b_i x_i^\rho \le 1\}} \psi\left(\sum_{i=1}^n b_i x_i^\rho\right) dx_1 \cdots dx_n \\ &\quad = \frac{\prod_{i=1}^n b_i^{-\frac{1}{\rho}} \Gamma^n(\frac{1}{\rho})}{\rho^n \Gamma(\frac{n}{\rho})} \int_0^1 \psi(t) t^{\frac{n}{\rho}-1}\,dt.\end{aligned}$$

3 Main results

We set

$$\begin{aligned}\Omega(a < b) &= \big\{x = (x_1, \ldots, x_n); a < u(x) < b\big\}, \\ \Omega'(a < b) &= \big\{x = (x_1, \ldots, x_n); a < v(y) < b\big\}.\end{aligned}$$

Theorem 1 *Suppose that $n \ge 1$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\rho > 0$, $\alpha, \beta \in R$, $\lambda_1\lambda_2 > 0$, $a_i > 0$, $b_i > 0$ $(i = 1, \ldots, n)$, $u(x) = (\sum_{i=1}^\infty a_i x_i^\rho)^{1/\rho}$, $v(y) = (\sum_{i=1}^\infty b_i y_i^\rho)^{1/\rho}$, $K(u(x), v(y)) = G(u^{\lambda_1}(x)v^{\lambda_2}(y))$ is a non-negative measurable function,*

$$\begin{aligned}0 < W_1 &= \int_{R_+^n} \big(v(t)\big)^{-\frac{\beta+n}{q}} K\big(1, v(t)\big)\,dt < \infty, \\ 0 < W_2 &= \int_{R_+^n} \big(u(t)\big)^{-\frac{\alpha+n}{p}} K\big(u(t), 1\big)\,dt < \infty,\end{aligned}$$

and for $a = 0$, $b = 1$ (*or* $a = 1$, $b = +\infty$),

$$\int_{\Omega(a<b)} \bigl(v(t)\bigr)^{-\frac{\beta+n}{q}} K\bigl(1, v(t)\bigr)\,dt > 0,$$
$$\int_{\Omega'(a<b)} \bigl(u(t)\bigr)^{-\frac{\alpha+n}{p}} K\bigl(u(t), 1\bigr)\,dt > 0,$$

then we have the following: *There is a constant* M *such that*, *for* $f(x) \in L^p_{u^\alpha(x)}(R^n_+)$ *and* $g(y) \in L^p_{v^\gamma(y)}(R^n_+)$, *the following inequality*

$$\int_{R^n_+}\int_{R^n_+} K\bigl(u(x), v(y)\bigr) f(x) g(y)\,dx\,dy \le M\|f\|_{p,u^\rho}\|g\|_{q,v^\rho} \tag{4}$$

holds true if and only if the equality $\frac{n\lambda_1+\alpha\lambda_2}{p} = \frac{n\lambda_2+\beta\lambda_1}{q}$ *is valid.*

Proof We assume that (4) is valid and set $c = \frac{n\lambda_2+\beta\lambda_1}{q} - \frac{n\lambda_1+\alpha\lambda_2}{p}$.

(i) For $\lambda_1, \lambda_2 > 0$, if $c > 0$, putting $\varepsilon > 0$ small enough and

$$f(x) = \begin{cases} (u(x))^{(-\alpha-n-\lambda_1\varepsilon)/p}, & u(x) > 1, \\ 0, & 0 < u(x) \le 1, \end{cases}$$

$$g(y) = \begin{cases} (v(y))^{(-\beta-n+\lambda_2\varepsilon)/q}, & 0 < v(y) < 1, \\ 0, & v(y) \ge 1, \end{cases}$$

by Lemma 2, we have

$$\begin{aligned}
&\|f\|_{p,u^\rho}\|g\|_{q,v^\rho} \\
&\quad = \left(\int_{\Omega(1<+\infty)} \bigl(u(x)\bigr)^{-n-\lambda_1\varepsilon}\,dx\right)^{1/p}\left(\int_{\Omega'(0<1)} \bigl(v(y)\bigr)^{-n+\lambda_2\varepsilon}\,dy\right)^{1/q} \\
&\quad = \left(\frac{\Gamma^n(\frac{1}{\rho})}{a_1^{1/\rho}\cdots a_n^{1/\rho}\rho^{n-1}\Gamma(\frac{n}{\rho})}\frac{1}{\lambda_1\varepsilon}\right)^{1/p} \\
&\quad\quad \times \left(\frac{\Gamma^n(\frac{1}{\rho})}{b_1^{1/\rho}\cdots b_n^{1/\rho}\rho^{n-1}\rho^{n-1}\Gamma(\frac{n}{\rho})}\frac{1}{\lambda_2\varepsilon}\right)^{1/q} \\
&\quad = \frac{\Gamma^n(\frac{1}{\rho})}{\lambda_1^{1/p}\lambda_2^{1/q}\rho^{n-1}\Gamma(\frac{n}{\rho})\varepsilon}\left(\prod_{i=1}^n a_i^{-1/\rho}\right)^{1/p}\left(\prod_{i=1}^n b_i^{-1/\rho}\right)^{1/q},
\end{aligned} \tag{5}$$

$$\begin{aligned}
&\int_{R^n_+}\int_{R^n_+} K\bigl(u(x), v(y)\bigr) f(x) g(y)\,dx\,dy \\
&\quad = \int_{\Omega(1<+\infty)} \bigl(u(x)\bigr)^{(-\alpha-n-\lambda_1\varepsilon)/p}\left(\int_{\Omega'(0<1)} K\bigl(u(x), v(y)\bigr)\bigl(v(y)\bigr)^{(-\beta-n+\lambda_2\varepsilon)/q}\,dy\right)dx \\
&\quad = \int_{\Omega(1<+\infty)} \bigl(u(x)\bigr)^{(-\alpha-n-\lambda_1\varepsilon)/p}\left(\int_{\Omega'(0<1)} K\bigl(1, v(u^{\frac{\lambda_1}{\lambda_2}}(x)y)\bigr)\bigl(v(y)\bigr)^{(-\beta-n+\lambda_2\varepsilon)/q}\,dy\right)dx \\
&\quad = \int_{\Omega(1<+\infty)} \bigl(u(x)\bigr)^{(-\alpha-n-\lambda_1\varepsilon)/p}\left(\int_{\Omega'(0<u^{\frac{\lambda_1}{\lambda_2}}(x))} K\bigl(1, v(t)\bigr)\right.
\end{aligned}$$

$$\times \left(u^{-\frac{\lambda_1}{\lambda_2}}(x)v(t)\right)^{(-\beta-n+\lambda_2\varepsilon)/q} u^{-\frac{n\lambda_1}{\lambda_2}}(x)\,dt\Bigg)\,dx$$
$$= \int_{\Omega(1<+\infty)} \left(u(x)\right)^{-n+\frac{c}{\lambda_2}-\lambda_1\varepsilon}\left(\int_{\Omega'(0<u^{\frac{\lambda_1}{\lambda_2}}(x))} K\left(1,v(t)\right)\left(v(t)\right)^{(-\beta-n+\lambda_2\varepsilon)/q}\,dt\right)dx$$
$$\geq \int_{\Omega(1<+\infty)} \left(u(x)\right)^{-n+\frac{c}{\lambda_2}-\lambda_1\varepsilon}\,dx \int_{\Omega'(0<1)} K\left(1,v(t)\right)\left(v(t)\right)^{(-\beta-n+\lambda_2\varepsilon)/q}\,dt. \tag{6}$$

Hence, by (4), (5) and (6), we have the following:

$$\int_{\Omega(1<+\infty)} \left(u(x)\right)^{-n+\frac{c}{\lambda_2}-\lambda_1\varepsilon}\,dx \int_{\Omega'(0<1)} K\left(1,v(t)\right)\left(v(t)\right)^{(-\beta-n+\lambda_2\varepsilon)/q}\,dt$$
$$\leq M\frac{\Gamma^n(\frac{1}{\rho})}{\lambda_1^{1/p}\lambda_2^{1/q}\rho^{n-1}\Gamma(\frac{n}{\rho})\varepsilon}\left(\prod_{i=1}^n a_i^{-1/\rho}\right)^{1/p}\left(\prod_{i=1}^n b_i^{-1/\rho}\right)^{1/q}. \tag{7}$$

For $\lambda_2 > 0$, $c > 0$, $\varepsilon > 0$ small enough, $-n + \frac{c}{\lambda_2} - \lambda_1\varepsilon > -n$, it follows that $\int_{\Omega(1<+\infty)}(u(x))^{-n+\frac{c}{\lambda_2}-\lambda_1\varepsilon}\,dx = +\infty$, which contradicts inequality (7) in view of $\int_{\Omega'(0<1)} K(1,v(t))(v(t))^{(-\beta-n+\lambda_2\varepsilon)/q}\,dt > 0$. Hence it is not valid for $c > 0$.

If $c < 0$, putting $\varepsilon > 0$ small enough and

$$f(x) = \begin{cases} (u(x))^{(-\alpha-n+\lambda_1\varepsilon)/p}, & 0 < u(x) < 1, \\ 0, & u(x) \geq 1, \end{cases}$$
$$g(y) = \begin{cases} (v(y))^{(-\beta-n+\lambda_2\varepsilon)/q}, & v(y) > 1, \\ 0, & 0 < v(y) \leq 1, \end{cases}$$

in the same way, we have the following:

$$\int_{\Omega'(1<+\infty)} \left(v(y)\right)^{-n-\frac{c}{\lambda_1}-\lambda_2\varepsilon}\,dy \int_{\Omega(0<1)} K\left(u(t),1\right)\left(u(t)\right)^{(-\alpha-n+\lambda_1\varepsilon)/p}\,dt$$
$$\leq M\frac{\Gamma^n(\frac{1}{\rho})}{\lambda_1^{\frac{1}{p}}\lambda_2^{\frac{1}{q}}\rho^{n-1}\Gamma(\frac{n}{\rho})\varepsilon}\left(\prod_{i=1}^n a_i^{-\frac{1}{\rho}}\right)^{\frac{1}{p}}\left(\prod_{i=1}^n b_i^{-\frac{1}{\rho}}\right)^{\frac{1}{q}}. \tag{8}$$

For $\lambda_2 > 0$, $c < 0$, $\varepsilon > 0$ small enough, hence $-n - \frac{c}{\lambda_1} - \lambda_2\varepsilon > -n$, it follows that $\int_{\Omega'(1<+\infty)}(v(y))^{-n-\frac{c}{\lambda_1}-\lambda_2\varepsilon}\,dy = +\infty$, which contradicts inequality (8) in view of $\int_{\Omega(0<1)} K(u(t),1)(u(t))^{(-\alpha-n+\lambda_1\varepsilon)/p}\,dt > 0$. Hence, it is not valid for $c < 0$.

Therefore, we prove that $c = 0$, namely $\frac{n\lambda_1+\alpha\lambda_2}{p} = \frac{n\lambda_2+\beta\lambda_1}{q}$ is valid.

(ii) For $\lambda_1, \lambda_2 < 0$, we prove that $\frac{n\lambda_1+\alpha\lambda_2}{p} = \frac{n\lambda_2+\beta\lambda_1}{q}$ is valid as follows.

If $c > 0$, putting $\varepsilon > 0$ small enough and

$$f(x) = \begin{cases} (u(x))^{(-\alpha-n-\lambda_1\varepsilon)/p}, & 0 < u(x) < 1, \\ 0, & u(x) \geq 1, \end{cases}$$
$$g(y) = \begin{cases} (v(y))^{(-\beta-n+\lambda_2\varepsilon)/q}, & v(y) > 1, \\ 0, & 0 < v(y) \leq 1, \end{cases}$$

we have

$$\begin{aligned}
&\|f\|_{p,u^\rho}\|g\|_{q,v^\rho} \\
&\quad = \left(\int_{\Omega(0<1)}\big(u(x)\big)^{-n-\lambda_1\varepsilon}\,dx\right)^{1/p}\left(\int_{\Omega'(1<+\infty)}\big(v(y)\big)^{-n+\lambda_2\varepsilon}\,dy\right)^{1/q} \\
&\quad = \frac{\Gamma^n(\frac{1}{\rho})}{(-\lambda_1)^{1/p}(-\lambda_2)^{1/q}\rho^{n-1}\Gamma(\frac{n}{\rho})\varepsilon}\left(\prod_{i=1}^n a_i^{-1/\rho}\right)^{1/p}\left(\prod_{i=1}^n b_i^{-1/\rho}\right)^{1/q}, \qquad (9)
\end{aligned}$$

$$\begin{aligned}
&\int_{R_+^n}\int_{R_+^n} K\big(u(x),v(y)\big)f(x)g(y)\,dx\,dy \\
&\quad = \int_{\Omega(0<1)}\big(u(x)\big)^{(-\alpha-n-\lambda_1\varepsilon)/p}\left(\int_{\Omega'(1<+\infty)}K\big(u(x),v(y)\big)\big(v(y)\big)^{(-\beta-n+\lambda_2\varepsilon)/q}\,dy\right)dx \\
&\quad = \int_{\Omega(0<1)}\big(u(x)\big)^{(-\alpha-n-\lambda_1\varepsilon)/p}\left(\int_{\Omega'(u^{\frac{\lambda_1}{\lambda_2}}(x)<+\infty)}K\big(1,v(t)\big)\right. \\
&\qquad \left.\times\big(u^{-\frac{\lambda_1}{\lambda_2}}(x)v(t)\big)^{(-\beta-n+\lambda_2\varepsilon)/q}u^{-\frac{n\lambda_1}{\lambda_2}}(x)\,dt\right)dx \\
&\quad = \int_{\Omega(0<1)}\big(u(x)\big)^{-n+\frac{c}{\lambda_2}-\lambda_1\varepsilon}\left(\int_{\Omega'(u^{\frac{\lambda_1}{\lambda_2}}(x)<+\infty)}K\big(1,v(t)\big)\big(v(t)\big)^{(-\beta-n+\lambda_2\varepsilon)/q}\,dt\right)dx \\
&\quad \ge \int_{\Omega(0<1)}\big(u(x)\big)^{-n+\frac{c}{\lambda_2}-\lambda_1\varepsilon}\,dx\int_{\Omega'(1<+\infty)}K\big(1,v(t)\big)\big(v(t)\big)^{(-\beta-n+\lambda_2\varepsilon)/q}\,dt. \qquad (10)
\end{aligned}$$

Hence, by (4), (9) and (10), we have the following:

$$\begin{aligned}
&\int_{\Omega(0<1)}\big(u(x)\big)^{-n+\frac{c}{\lambda_2}-\lambda_1\varepsilon}\,dx\int_{\Omega'(1<+\infty)}K\big(1,v(t)\big)\big(v(t)\big)^{(-\beta-n+\lambda_2\varepsilon)/q}\,dt \\
&\quad \le M\frac{\Gamma^n(\frac{1}{\rho})}{(-\lambda_1)^{1/p}(-\lambda_2)^{1/q}\rho^{n-1}\Gamma(\frac{n}{\rho})\varepsilon}\left(\prod_{i=1}^n a_i^{-1/\rho}\right)^{1/p}\left(\prod_{i=1}^n b_i^{-1/\rho}\right)^{1/q}. \qquad (11)
\end{aligned}$$

It is obvious that $\int_{\Omega(0<1)}(u(x))^{-n+\frac{c}{\lambda_2}-\lambda_1\varepsilon}\,dx = +\infty$, which contradicts inequality (11) in view of $\int_{\Omega'(1<+\infty)}K(1,v(t))(v(t))^{(-\beta-n+\lambda_2\varepsilon)/q}\,dt > 0$. Hence it is not valid for $c > 0$.

If $c < 0$, putting $\varepsilon > 0$ small enough and

$$f(x) = \begin{cases}(u(x))^{(-\alpha-n+\lambda_1\varepsilon)/p}, & u(x) > 1,\\ 0, & 0 < u(x) \le 1,\end{cases}$$

$$g(y) = \begin{cases}(v(y))^{(-\beta-n-\lambda_2\varepsilon)/q}, & 0 < v(y) < 1,\\ 0, & v(y) \ge 1,\end{cases}$$

in the same way, we have

$$\begin{aligned}
&\int_{\Omega'(0<1)}\big(v(y)\big)^{-n-\frac{c}{\lambda_1}-\lambda_2\varepsilon}\,dy\int_{\Omega(1<+\infty)}K\big(u(t),1\big)\big(u(t)\big)^{(-\alpha-n+\lambda_1\varepsilon)/p}\,dt \\
&\quad \le M\frac{\Gamma^n(\frac{1}{\rho})}{(-\lambda_1)^{1/p}(-\lambda_2)^{1/q}\rho^{n-1}\Gamma(\frac{n}{\rho})\varepsilon}\left(\prod_{i=1}^n a_i^{-1/\rho}\right)^{1/p}\left(\prod_{i=1}^n b_i^{-1/\rho}\right)^{1/q}. \qquad (12)
\end{aligned}$$

In virtue of $\int_{\Omega'(0<1)}(\nu(y))^{-n-\frac{c}{\lambda_1}-\lambda_2\varepsilon}\,dy = +\infty$, (12) is a contradiction in view of $\int_{\Omega(1<+\infty)} K(u(t),1)(u(t))^{(-\alpha-n+\lambda_1\varepsilon)/p}\,dt > 0$. Hence, $c < 0$ is not valid.

Therefore, we prove that $c = 0$ is valid.

On the other hand, we assume that $\frac{n\lambda_1+\alpha\lambda_2}{p} = \frac{n\lambda_2+\beta\lambda_1}{q}$ is valid.

Setting $a = \frac{\alpha}{pq} + \frac{n}{pq}$, $b = \frac{\beta}{pq} + \frac{n}{pq}$, by Holder's inequality with weight and Lemma 1, we find

$$\begin{aligned}
&\int_{R_+^n}\int_{R_+^n} K\big(u(x),\nu(y)\big)f(x)g(y)\,dx\,dy \\
&= \int_{R_+^n}\int_{R_+^n}\left(f(x)\frac{u^a(x)}{\nu^b(y)}\right)\left(g(y)\frac{\nu^b(y)}{u^a(x)}\right)K\big(u(x),\nu(y)\big)\,dx\,dy \\
&\le \left(\int_{R_+^n}\int_{R_+^n} f^p(x)\frac{u^{ap}(x)}{\nu^{bp}(y)}K\big(u(x),\nu(y)\big)\,dx\,dy\right)^{1/p} \\
&\quad\times \left(\int_{R_+^n}\int_{R_+^n} g^q(y)\frac{\nu^{bq}(y)}{u^{aq}(x)}K\big(u(x),\nu(y)\big)\,dx\,dy\right)^{1/q} \\
&= \left(\int_{R_+^n}\big(u(x)\big)^{\frac{\alpha+n}{q}}f^p(x)\omega_1(x)\,dx\right)^{1/p}\left(\int_{R_+^n}\big(\nu(y)\big)^{\frac{\beta+n}{p}}g^q(y)\omega_2(y)\,dy\right)^{1/q} \\
&= W_1^{1/p}W_2^{1/q}\left(\int_{R_+^n}\big(u(x)\big)^{\frac{\alpha+n}{q}+\frac{\lambda_1}{\lambda_2}(\frac{\beta+n}{q}-n)}f^p(x)\,dx\right)^{1/p} \\
&\quad\times \left(\int_{R_+^n}\big(\nu(y)\big)^{\frac{\beta+n}{p}+\frac{\lambda_2}{\lambda_1}(\frac{\alpha+n}{p}-n)}g^q(y)\,dy\right)^{1/q} \\
&= W_1^{1/p}W_2^{1/q}\left(\int_{R_+^n}\big(u(x)\big)^{\alpha}f^p(x)\,dx\right)^{1/p}\left(\int_{R_+^n}\big(\nu(y)\big)^{\beta}g^q(y)\,dy\right)^{1/q} \\
&= W_1^{1/p}W_2^{1/q}\|f\|_{p,u^\alpha}\|g\|_{q,\nu^\beta}.
\end{aligned}$$

Taking $M \ge W_1^{1/p}W_2^{1/q}$, we prove that (4) is valid. □

Theorem 2 *With regards to the assumption of Theorem* 1, *the best possible constant factor of* (4) *is* $\inf M = W_1^{1/p}W_2^{1/q}$ *when* (4) *holds true.*

Proof We assume that (4) is valid. If there exists a positive number $M_0 < W_1^{1/p}W_2^{1/q}$ such that (4) is still valid when replacing M by M_0, then, $\forall f(x) \in L^p_{u^\alpha(x)}(R_+^n)$ and $g(y) \in L^p_{\nu^\beta(y)}(R_+^n)$, we have

$$\int_{R_+^n}\int_{R_+^n} K\big(u(x),\nu(y)\big)f(x)g(y)\,dx\,dy \le M_0\|f\|_{p,u^\alpha}\|g\|_{q,\nu^\beta}. \tag{13}$$

Taking $\varepsilon > 0$ and $\delta > 0$ small enough and setting

$$f(x) = \begin{cases} (u(x))^{(-\alpha-n-|\lambda_1|\varepsilon)/p}, & u(x) > \delta, \\ 0, & 0 < u(x) \le \delta, \end{cases}$$

$$g(y) = \begin{cases} (\nu(y))^{(-\beta-n+|\lambda_2|\varepsilon)/q}, & 0 < \nu(y) < 1, \\ 0, & \nu(y) \ge 1, \end{cases}$$

we have

$$
\begin{aligned}
&\|f\|_{p,u^\alpha}\|g\|_{q,v^\beta}\\
&=\left(\int_{\Omega(\delta<+\infty)}\big(u(x)\big)^{-n-|\lambda_1|\varepsilon}\,dx\right)^{1/p}\left(\int_{\Omega'(0<1)}\big(v(y)\big)^{-n+|\lambda_2|\varepsilon}\,dy\right)^{1/q}\\
&=\frac{\Gamma^n(\frac{1}{\rho})(\frac{1}{\delta^{|\lambda_1|\varepsilon/\rho}})^{1/p}}{|\lambda_1|^{1/p}|\lambda_2|^{1/q}\rho^{n-1}\Gamma(\frac{n}{\rho})\varepsilon}\left(\prod_{i=1}^n a_i^{-1/\rho}\right)^{1/p}\left(\prod_{i=1}^n b_i^{-1/\rho}\right)^{1/q}.
\end{aligned}
\tag{14}
$$

And we have the following by using $\frac{n\lambda_1+\alpha\lambda_2}{p}=\frac{n\lambda_2+\beta\lambda_1}{q}$:

$$
\begin{aligned}
&\int_{R_+^n}\int_{R_+^n}K\big(u(x),v(y)\big)f(x)g(y)\,dx\,dy\\
&=\int_{\Omega'(0<1)}\big(v(y)\big)^{(-\beta-n+|\lambda_2|\varepsilon)/q}\left(\int_{\Omega(\delta<+\infty)}\big(u(x)\big)^{(-\alpha-n-|\lambda_1|\varepsilon)/p}K\big(u(x),v(y)\big)\,dx\right)dy\\
&=\int_{\Omega'(0<1)}\big(v(y)\big)^{(-\beta-n+|\lambda_2|\varepsilon)/q}\\
&\quad\times\left(\int_{\Omega(\delta<+\infty)}\big(u(x)\big)^{\frac{-\alpha-n-|\lambda_1|\varepsilon}{p}}K\big(u\big(v^{\frac{\lambda_2}{\lambda_1}}(y)x,1\big)\big)\,dx\right)dy\\
&=\int_{\Omega'(0<1)}\big(v(y)\big)^{(-\beta-n+|\lambda_2|\varepsilon)/q}\left(\int_{\Omega(\delta v^{\frac{\lambda_2}{\lambda_1}}(y)<+\infty)}\big(v^{-\frac{\lambda_2}{\lambda_1}}(y)u(t)\big)^{\frac{-\alpha-n-|\lambda_1|\varepsilon}{p}}\right.\\
&\quad\left.\times K\big(u(t),1\big)v^{-\frac{n\lambda_2}{\lambda_1}}(y)\,dt\right)dy\\
&=\int_{\Omega'(0<1)}\big(v(y)\big)^{-n+|\lambda_2|\varepsilon}\left(\int_{\Omega(\delta v^{\frac{\lambda_2}{\lambda_1}}(y)<+\infty)}\big(u(t)\big)^{\frac{-\alpha-n-|\lambda_1|\varepsilon}{p}}K\big(u(t),1\big)\,dt\right)dy\\
&\geq\int_{\Omega'(0<1)}\big(v(y)\big)^{-n+|\lambda_2|\varepsilon}\,dy\int_{\Omega(\delta<+\infty)}\big(u(t)\big)^{\frac{-\alpha-n-|\lambda_1|\varepsilon}{p}}K\big(u(t),1\big)\,dt\\
&=\frac{\Gamma^n(\frac{1}{\rho})\prod_{i=1}^n b_i^{-1/\rho}}{|\lambda_2|\rho^{n-1}\Gamma(\frac{n}{\rho})\varepsilon}\int_{\Omega(\delta<+\infty)}\big(u(t)\big)^{\frac{-\alpha-n-|\lambda_1|\varepsilon}{p}}K\big(u(t),1\big)\,dt.
\end{aligned}
\tag{15}
$$

Combining (13), (14) and (15), we have

$$
\begin{aligned}
&\int_{\Omega(\delta<+\infty)}K\big(u(t),1\big)\big(u(t)\big)^{\frac{-\alpha-n-|\lambda_1|\varepsilon}{p}}\,dt\\
&\leq M_0\left(\frac{1}{|\lambda_1|}\prod_{i=1}^n a_i^{-1/\rho}\right)^{1/p}\left(\frac{1}{|\lambda_2|}\prod_{i=1}^n b_i^{-1/\rho}\right)^{1/p}\left(\frac{1}{\delta^{|\lambda_1|\varepsilon/\rho}}\right)^{1/p}.
\end{aligned}
\tag{16}
$$

If we set

$$
f(x)=\begin{cases}(u(x))^{(-\alpha-n-|\lambda_1|\varepsilon)/p}, & 0<u(x)<1,\\ 0, & u(x)\geq 1,\end{cases}
$$

$$
g(y)=\begin{cases}(v(y))^{(-\beta-n+|\lambda_2|\varepsilon)/q}, & v(y)>\delta,\\ 0, & 0<v(y)\leq\delta,\end{cases}
$$

then, in the same way, we have

$$\begin{aligned}&\int_{\Omega'(\delta<+\infty)} K\big(1,v(t)\big)\big(v(y)\big)^{(-\beta-n+|\lambda_2|\varepsilon)/q}\,dt\\&\quad\le M_0\Bigg(\frac{1}{|\lambda_1|}\prod_{i=1}^{n}a_i^{-1/\rho}\Bigg)^{1/q}\Bigg(\frac{1}{|\lambda_2|}\prod_{i=1}^{n}b_i^{-1/\rho}\Bigg)^{1/q}\bigg(\frac{1}{\delta^{|\lambda_2|\varepsilon/\rho}}\bigg)^{1/q}.\end{aligned}\tag{17}$$

Hence, by (16) and (17), we have

$$\begin{aligned}&\bigg(\int_{\Omega'(\delta<+\infty)} K\big(1,v(t)\big)\big(v(y)\big)^{(-\beta-n+|\lambda_2|\varepsilon)/q}\,dt\bigg)^{1/p}\\&\quad\times\bigg(\int_{\Omega(\delta<+\infty)} K\big(u(t),1\big)\big(u(t)\big)^{\frac{-\alpha-n-|\lambda_1|\varepsilon}{p}}\,dt\bigg)^{1/q}\\&\quad\le M_0\bigg(\frac{1}{\delta^{|\lambda_2|\varepsilon/\rho}}\bigg)^{1/(pq)}\bigg(\frac{1}{\delta^{|\lambda_1|\varepsilon/\rho}}\bigg)^{1/(pq)}.\end{aligned}$$

For $\varepsilon\to0^+$, using Fatou's lemma, we obtain

$$\bigg(\int_{\Omega'(\delta<+\infty)} K\big(1,v(t)\big)\big(v(y)\big)^{-\frac{\beta+n}{q}}\,dt\bigg)^{\frac1p}\bigg(\int_{\Omega(\delta<+\infty)} K\big(u(t),1\big)\big(u(t)\big)^{-\frac{\alpha+n}{p}}\,dt\bigg)^{\frac1q}\le M_0,$$

and then it follows that, for $\delta\to0^+$,

$$W_1^{\frac1p}W_2^{\frac1q}=\bigg(\int_{R_+^n}\big(v(y)\big)^{-\frac{\beta+n}{q}}K\big(1,v(t)\big)\,dt\bigg)^{\frac1p}\bigg(\int_{R_+^n}\big(u(t)\big)^{-\frac{\alpha+n}{p}}K\big(u(t),1\big)\,dt\bigg)^{\frac1q}\le M_0.$$

This is a contradiction, which leads to the fact that $W_1^{1/p}W_2^{1/q}$ is the best possible constant factor of (4). □

4 Application in the operator theory

For $\gamma=\beta(1-p)$, there is $-\frac{\beta+n}{q}=\frac{\gamma+n}{p}-n$, and it follows that $\frac{n\lambda_1+\alpha\lambda_2}{p}=\frac{n\lambda_2+\beta\lambda_1}{q}$ is equivalent to $\lambda_1(n+\gamma)+\lambda_2(n+\alpha)=\lambda_2 np$. In view of the fact that (1) is equivalent to (3), by Theorems 1-2, we have the following.

Theorem 3 *Suppose that* $n\ge1$, $p>1$, $\rho>0$, $\alpha,\gamma\in R$, $\lambda_1\lambda_2>0$, $a_i>0$, $b_i>0$, $u(x)=(\sum_{i=1}^{\infty}a_ix_i^{\rho})^{1/\rho}$, $v(y)=(\sum_{i=1}^{\infty}b_iy_i^{\rho})^{1/\rho}$, $K(u(x),v(y))=G(u^{\lambda_1}(x)v^{\lambda_2}(y))$ *is a non-negative measurable function, the operator* T *is defined by* (2),

$$\begin{aligned}&0<\tilde W_1=\int_{R_+^n}\big(v(t)\big)^{\frac{\gamma+n}{p}-n}K\big(1,v(t)\big)\,dt<\infty,\\&0<\tilde W_2=\int_{R_+^n}\big(u(t)\big)^{-\frac{\alpha+n}{p}}K\big(u(t),1\big)\,dt<\infty,\end{aligned}$$

and for $a=0$, $b=1$ (*or* $a=1$, $b=+\infty$),

$$\int_{\Omega'(a<b)}\big(v(t)\big)^{\frac{\gamma+n}{p}-n}K\big(1,v(t)\big)\,dt>0,\qquad\int_{\Omega'(a<b)}\big(u(t)\big)^{-\frac{\alpha+n}{p}}K\big(u(t),1\big)\,dt>0,$$

then we have the following:

(i) T *is a bounded operator from* $L^p_{u^\alpha}(R^n_+)$ *to* $L^p_{v^\gamma}(R^n_+)$ *if and only if the equality* $\lambda_1(n+\gamma)+\lambda_2(n+\alpha)=\lambda_2 np$ *is valid*.

(ii) *If the operator* T *is a bounded operator from* $L^p_{u^\alpha}(R^n_+)$ *to* $L^p_{v^\gamma}(R^n_+)$, *then we obtain the norm of the operator* T *as follows*:

$$\|T\| := \sup_{f\in L^p_{u^\alpha}(R^n_+)} \frac{\|T(f)\|_{p,v^\gamma}}{\|f\|_{p,u^\rho}} = \tilde{W}_1^{\frac{1}{p}}\tilde{W}_2^{\frac{1}{q}}.$$

Taking $\alpha=\gamma=0$ in Theorem 3, we have the result as follows.

Corollary 1 *Suppose that* $n\geq 1$, $p>1$, $\rho>0$, $\lambda_1\lambda_2>0$, $a_i>0$, $b_i>0$ $(i=1,\dots,n)$, $u(x)=(\sum_{i=1}^\infty a_i x_i^\rho)^{1/\rho}$, $v(y)=(\sum_{i=1}^\infty b_i y_i^\rho)^{1/\rho}$, $K(u(x),v(y))=G(u^{\lambda_1}(x)v^{\lambda_2}(y))$ *is a non-negative measurable function, the operator* T *is defined by* (2),

$$0<\tilde{W}_1=\int_{R^n_+}\bigl(v(t)\bigr)^{\frac{n}{p}-n}K\bigl(1,v(t)\bigr)\,dt<\infty,$$
$$0<\tilde{W}_2=\int_{R^n_+}\bigl(u(t)\bigr)^{-\frac{n}{p}}K\bigl(u(t),1\bigr)\,dt<\infty,$$

and for $a=0$, $b=1$ (*or* $a=1$, $b=+\infty$),

$$\int_{\Omega'(a<b)}\bigl(v(t)\bigr)^{\frac{n}{p}-n}K\bigl(1,v(t)\bigr)\,dt>0,\qquad \int_{\Omega'(a<b)}\bigl(u(t)\bigr)^{-\frac{n}{p}}K\bigl(u(t),1\bigr)\,dt>0,$$

then we have the following:

(i) T *is a bounded operator in* $L^p(R^n_+)$ *if and only if* $\lambda_1=(p-1)\lambda_2$.

(ii) *If the operator* T *is a bounded operator in* $L^p(R^n_+)$, *then the norm of the operator* T *is*

$$\|T\|=\tilde{W}_1^{\frac{1}{p}}\tilde{W}_2^{\frac{1}{q}}.$$

Theorem 4 *Suppose that* $n\geq 1$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\rho>0$, $\lambda_1,\lambda_2>0$, $a_i>0$, $b_i>0$ $(i=1,\dots,n)$, $b>\frac{n}{\lambda_2 p}$, $a>b-\frac{n}{\lambda_2 p}$, $u(x)=(\sum_{i=1}^\infty a_i x_i^\rho)^{\frac{1}{\rho}}$, $v(y)=(\sum_{i=1}^\infty b_i y_i^\rho)^{\frac{1}{\rho}}$, *the operator* T *is defined by*

$$T(f)(y)=\int_{R^n_+}\frac{(u^{\lambda_1}(x)v^{\lambda_2}(y))^b}{(1+u^{\lambda_1}(x)v^{\lambda_2}(y))^a}f(x)\,dx,\quad y\in R^n_+,$$

then we have the following:

(i) T *is a bounded operator in* $L^p(R^n_+)$ *if and only if* $\frac{\lambda_1}{p}=\frac{\lambda_2}{q}$.

(ii) *If the operator* T *is a bounded operator in* $L^p(R^n_+)$, *then the norm of the operator* T *is as follows*:

$$\|T\|=\frac{\Gamma^n(\frac{1}{\rho})}{\lambda_1^{\frac{1}{q}}\lambda_2^{\frac{1}{p}}\rho^{n-1}\Gamma(\frac{n}{\rho})\Gamma(a)}\left(\prod_{i=1}^n a_i^{-\frac{1}{\rho}}\right)^{\frac{1}{q}}\left(\prod_{i=1}^n b_i^{-\frac{1}{\rho}}\right)^{\frac{1}{p}}\Gamma\left(b-\frac{n}{\lambda_2 p}\right)\Gamma\left(a-b+\frac{n}{\lambda_2 p}\right).$$

Proof (ii) In view of $\frac{\lambda_1}{p} = \frac{\lambda_2}{q}$, we have the following by using Lemma 2:

$$\begin{aligned}
\tilde{W}_1 &= \int_{R_+^n} \big(\nu(t)\big)^{-\frac{n}{q}} K\big(1, \nu(t)\big)\, dt \\
&= \int_{R_+^n} \left(\sum_{i=1}^n b_i t_i^\rho\right)^{\frac{\lambda_2 b}{\rho} - \frac{n}{q\rho}} \frac{1}{[1 + (\sum_{i=1}^n b_i t_i^\rho)^{\lambda_2/\rho}]^a}\, dt \\
&= \prod_{i=1}^n b_i^{-\frac{1}{\rho}} \int_{R_+^n} \left(\sum_{i=1}^n x_i^\rho\right)^{\frac{\lambda_2 b}{\rho} - \frac{n}{q\rho}} \frac{1}{[1 + (\sum_{i=1}^n x_i^\rho)^{\lambda_2/\rho}]^a}\, dt \\
&= \prod_{i=1}^n b_i^{-\frac{1}{\rho}} \lim_{r\to\infty} \int \cdots \int_{x_i>0; x_1^p + \cdots + x_n^p \le r^p} \frac{r^{\lambda_2 b - n/q}}{[1 + r^{\lambda_2} (\sum_{i=1}^n (\frac{x_i}{r})^\rho)^{\lambda_2/\rho}]^a} \\
&\quad \times \left(\sum_{i=1}^n \left(\frac{x_i}{r}\right)^\rho\right)^{\frac{\lambda_2 b}{\rho} - \frac{n}{q\rho}} x_1^{1-1} \cdots x_n^{1-1}\, dx_1 \cdots dx_n \\
&= \prod_{i=1}^n b_i^{-\frac{1}{\rho}} \lim_{r\to\infty} r^{\lambda_2 b - \frac{n}{q}} \frac{r^n \Gamma^n(\frac{1}{\rho})}{\rho^n \Gamma(\frac{n}{\rho})} \int_0^1 \frac{u^{\frac{\lambda_2 b}{\rho} - \frac{n}{q\rho} + \frac{n}{\rho} - 1}}{(1 + r^{\lambda_2} u^{\lambda_2/\rho})^a}\, du \\
&= \prod_{i=1}^n b_i^{-\frac{1}{\rho}} \lim_{r\to\infty} \frac{\Gamma^n(\frac{1}{\rho})}{\lambda_2 \rho^{n-1} \Gamma(\frac{n}{\rho})} \int_0^{r^{\lambda_2}} \frac{1}{(1+t)^a} t^{b - \frac{n}{\lambda_2 p} - 1}\, dt \\
&= \frac{\Gamma^n(\frac{1}{\rho})}{\lambda_2 \rho^{n-1} \Gamma(\frac{n}{\rho})} \prod_{i=1}^n b_i^{-\frac{1}{\rho}} \int_0^{+\infty} \frac{1}{(1+t)^a} t^{b - \frac{n}{\lambda_2 p} - 1}\, dt \\
&= \frac{\Gamma^n(\frac{1}{\rho})}{\lambda_2 \rho^{n-1} \Gamma(\frac{n}{\rho})} \prod_{i=1}^n b_i^{-\frac{1}{\rho}} B\left(b - \frac{n}{\lambda_2 p}, a - \left(b - \frac{n}{\lambda_2 p}\right)\right) \\
&= \frac{\Gamma^n(\frac{1}{\rho})}{\lambda_2 \rho^{n-1} \Gamma(\frac{n}{\rho}) \Gamma(a)} \prod_{i=1}^n b_i^{-\frac{1}{\rho}} \Gamma\left(b - \frac{n}{\lambda_2 p}\right) \Gamma\left(a - b + \frac{n}{\lambda_2 p}\right).
\end{aligned}$$

In the same way, we still have the following:

$$\begin{aligned}
\tilde{W}_2 &= \int_{R_+^n} \big[u(t)\big]^{-\frac{n}{p}} K\big(u(t), 1\big)\, dt \\
&= \frac{\Gamma^n(\frac{1}{\rho})}{\lambda_1 \rho^{n-1} \Gamma(\frac{n}{\rho}) \Gamma(a)} \prod_{i=1}^n a_i^{-\frac{1}{\rho}} \Gamma\left(b - \frac{n}{\lambda_1 q}\right) \Gamma\left(a - b + \frac{n}{\lambda_1 q}\right) \\
&= \frac{\Gamma^n(\frac{1}{\rho})}{\lambda_1 \rho^{n-1} \Gamma(\frac{n}{\rho}) \Gamma(a)} \prod_{i=1}^n a_i^{-\frac{1}{\rho}} \Gamma\left(b - \frac{n}{\lambda_2 p}\right) \Gamma\left(a - b + \frac{n}{\lambda_2 p}\right).
\end{aligned}$$

It follows that

$$\tilde{W}_1^{\frac{1}{p}} \tilde{W}_2^{\frac{1}{q}} = \frac{\Gamma^n(\frac{1}{\rho})}{\lambda_1^{\frac{1}{q}} \lambda_2^{\frac{1}{p}} \rho^{n-1} \Gamma(\frac{n}{\rho}) \Gamma(a)} \left(\prod_{i=1}^n a_i^{-\frac{1}{\rho}}\right)^{\frac{1}{q}} \left(\prod_{i=1}^n b_i^{-\frac{1}{\rho}}\right)^{\frac{1}{p}} \Gamma\left(b - \frac{n}{\lambda_2 p}\right) \Gamma\left(a - b + \frac{n}{\lambda_2 p}\right).$$

Hence, we prove that (ii) is valid by Corollary 1. □

5 Conclusions

In this paper, by using the methods and techniques of real analysis, the sufficient and necessary conditions for the existence of the Hilbert-type multiple integral inequality with the kernel $K(u(x), v(y)) = G(u^{\lambda_1}(x)v^{\lambda_2}(y))$ and the best possible constant factor are discussed in Theorems 1-2. Furthermore, its application in the operator theory is considered in Theorems 3-4. The method of real analysis is very important as itis the key to prove the equivalent inequalities with the best possible constant factor. The lemmas and theorems provide an extensive account of this type of inequalities.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
YH carried out the mathematical studies, participated in the sequence alignment and drafted the manuscript. QH, BY and JL participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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