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Monotonicity and inequalities for the gamma function

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Abstract

In this paper, by using the monotonicity rule for the ratio of two Laplace transforms, we prove that the function

$$x \mapsto \frac{1}{24x(\ln\Gamma(x+1/2) - x\ln x + x - \ln\sqrt{2\pi}) + 1} - \frac{120}{7}x^2$$

is strictly increasing from $(0,\infty)$ onto $(1, 1860/343)$. This not only yields some known and new inequalities for the gamma function, but also gives some completely monotonic functions related to the gamma function.

MSC: Primary 33B15; 26A48; secondary 26D15; 26A51

Keywords: gamma function; Laplace transform; complete monotonicity; inequality

1 Introduction

Stirling's formula

$$n! \sim \sqrt{2\pi n}n^n e^{-n} \tag{1.1}$$

has important applications in statistical physics, probability theory and number theory. Due to its practical importance, it has attracted much interest of many mathematicians and have motivated a large number of research papers concerning various generalizations and improvements.

Burnside's formula [1]

$$n! \sim \sqrt{2\pi}\left(\frac{n+1/2}{e}\right)^{n+1/2} := b_n \tag{1.2}$$

slightly improves (1.1). Gosper [2] replaced $\sqrt{2\pi n}$ by $\sqrt{2\pi(n+1/6)}$ in (1.1) to get

$$n! \sim \sqrt{2\pi\left(n+\frac{1}{6}\right)}\left(\frac{n}{e}\right)^n := g_n, \tag{1.3}$$

which is better than (1.1) and (1.2). Batir [3] obtained an asymptotic formula similar to (1.3):

$$n! \sim \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-1/6}} := b'_n, \tag{1.4}$$

which is stronger than (1.1) and (1.2). A more accurate approximation for the factorial function

$$n! \sim \sqrt{2\pi}\left(\frac{n^2+n+1/6}{e^2}\right)^{n/2+1/4} := m_n \tag{1.5}$$

was presented in [4] by Mortici.

The gamma function $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$ for $x > 0$ is closely related to Stirling's formula since $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}$. This inspires some authors to also pay attention to finding various better approximations for the gamma function. Here we list some more accurate approximations:

(i) Ramanujan's [5, p. 339] approximation formula as $x \to \infty$

$$\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^x\left(8x^3+4x^2+x+\frac{1}{30}\right)^{1/6} := R(x); \tag{1.6}$$

(ii) Windschitl's (see [6, Eq. (42)]) approximation formula

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(x\sinh\frac{1}{x}\right)^{x/2} := W(x); \tag{1.7}$$

(iii) Smith's [6, Eq. (42)] approximation formula

$$\Gamma\left(x+\frac{1}{2}\right) \sim \sqrt{2\pi}\left(\frac{x}{e}\right)^x\left(2x\tanh\frac{1}{2x}\right)^{x/2} := S(x); \tag{1.8}$$

(iv) Nemes' formula ([7, Corollary 4.1]) states that

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(1+\frac{1}{12x^2-1/10}\right)^x := N(x); \tag{1.9}$$

(v) Chen's [8] presented a new approximation

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(1+\frac{1}{12x^3+24x/7-1/2}\right)^{x^2+53/210} = C(x). \tag{1.10}$$

Remark 1 Let $A(x)$ be an approximation for $\Gamma(x+1)$ as $x \to \infty$. If there is $m > 0$ such that

$$\lim_{x\to\infty}\frac{\ln\Gamma(x+1)-\ln A(x)}{x^{-m}} = c \neq 0, \pm\infty, \tag{1.11}$$

then we say that the rate of $A(x)$ converging to $\Gamma(x+1)$ is like x^{-m} as $x \to \infty$. Evidently, the larger m is, the higher the accuracy of $A(x)$ approximating for $\Gamma(x+1)$ is. Since $(x-$

$1)/\ln x \to 1$ as $x \to 1$, the limit relation can be equivalently written as

$$\lim_{x\to\infty} \frac{\Gamma(x+1)/A(x) - 1}{x^{-m}} = c \neq 0, \pm\infty,$$

or

$$\frac{\Gamma(x+1)}{A(x)} = 1 + O\left(x^{-m}\right) \quad \text{as } x \to \infty.$$

Remark 2 It is easy to check that as $n \to \infty$ or $x \to \infty$,

$$n! = \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-1/6}}\left(1 + O\left(n^{-2}\right)\right),$$
$$n! = \sqrt{2\pi}\left(\frac{n^2+n+1/6}{e^2}\right)^{n/2+1/4}\left(1 + O\left(n^{-3}\right)\right),$$
$$\Gamma\left(x+\frac{1}{2}\right) = \sqrt{2\pi}\left(\frac{x}{e}\right)^x\left(2x\tanh\frac{1}{2x}\right)^{x/2}\left(1 + O\left(x^{-5}\right)\right).$$

These together with those shown in [8, (3.5)-(3.10)] indicate that Chen's one $C(x)$ is the best among approximation formulas listed above.

More results involving the approximation formulas for the factorial or gamma function can be found in [9–28] and the references cited therein.

It is worth mentioning that Yang and Chu [9] proposed a new approach to construct asymptotic formulas by bivariate means. As applications, they offered in [9, Propositions 4 and 5] two asymptotic formulas: as $x \to \infty$,

$$\Gamma(x+1) \sim \sqrt{2\pi}\left(\frac{x+1/2}{e}\right)^{x+1/2}\exp\left(-\frac{1}{24}\frac{x+1/2}{x^2+x+37/120}\right) := Y_1(x),$$
$$\Gamma(x+1) \sim \sqrt{2\pi}\left(\frac{x+1/2}{e}\right)^{x+1/2}\exp\left(-\frac{1517}{44{,}640}\frac{1}{x+1/2} - \frac{343}{44{,}640}\frac{x+1/2}{x^2+x+111/196}\right)$$
$$:= Y_2(x),$$

which satisfy

$$\Gamma(x+1) = Y_1(x)\left(1 + O\left(x^{-5}\right)\right) \quad \text{and} \quad \Gamma(x+1) = Y_2(x)\left(1 + O\left(x^{-7}\right)\right),$$

and proved that the functions (replace x by $x - 1/2$)

$$f_4\left(x-\frac{1}{2}\right) = \ln\Gamma\left(x+\frac{1}{2}\right) - \frac{1}{2}\ln 2\pi - x\ln x + x + \frac{1}{24}\frac{x}{x^2+7/120},$$
$$f_5\left(x-\frac{1}{2}\right) = \ln\Gamma\left(x+\frac{1}{2}\right) - \frac{1}{2}\ln 2\pi - x\ln x + \frac{1}{1440}\frac{5880x^2+1517}{x(98x^2+31)}$$

are increasingly concave and decreasingly convex on $(0,\infty)$, respectively. Clearly, both $Y_1(x)$ and $Y_2(x)$ are accurate and simpler approximation formulas for the gamma function.

According to these inequalities given in [9, Corollary 7], it is natural to ask: What are the best α and β such that the double inequality

$$\exp\left[-\frac{1}{24x}\frac{120x^2+7(\alpha-1)}{120x^2+7\alpha}\right] < \frac{\Gamma(x+1/2)}{\sqrt{2\pi}(x/e)^x} < \exp\left[-\frac{1}{24x}\frac{120x^2+7(\beta-1)}{120x^2+7\beta}\right] \tag{1.12}$$

holds for all $x > 0$? This problem is equivalent to determining the monotonicity of the function

$$f(x) = \frac{1}{24x(\ln\Gamma(x+1/2) - x\ln x + x - \ln\sqrt{2\pi}) + 1} - \frac{120}{7}x^2 \tag{1.13}$$

on $(0, \infty)$.

The aim of this paper is to answer this problem. Our main result is the following theorem.

Theorem 1 *The function f defined by* (1.13) *is strictly increasing from* $(0, \infty)$ *onto* $(1, 1860/343)$.

As a consequence of the above theorem, the following corollary is immediate.

Corollary 1 *For $x > x_0 \geq 0$, the double inequality* (1.12) *holds if and only if $\alpha \geq f(\infty) =$ $1860/343$ and $1 \leq \beta \leq f(x_0)$. In particular, we have*

$$\exp\left[-\frac{1}{1440}\frac{5880x^2+1517}{x(98x^2+31)}\right] < \frac{\Gamma(x+1/2)}{\sqrt{2\pi}(x/e)^x} < \exp\left(-\frac{5x}{120x^2+7}\right)$$

holds for $x > 0$.

Replacing x by $n + 1/2$, then putting $x_0 = 1$ in Corollary 1, and noting that

$$\beta_1 := f\left(\frac{3}{2}\right) = \frac{1}{36\ln 2 - 54\ln 3 - 18\ln\pi + 55} - \frac{270}{7} \approx 4.7243, \tag{1.14}$$

we deduce the following statement.

Corollary 2 *The double inequality*

$$\begin{aligned}
&\exp\left[-\frac{1}{24(n+1/2)}\frac{120(n+1/2)^2+7(\alpha-1)}{120(n+1/2)^2+7\alpha}\right] \\
&\quad < \frac{n!}{\sqrt{2\pi}((n+1/2)/e)^{n+1/2}} \\
&\quad < \exp\left[-\frac{1}{24(n+1/2)}\frac{120(n+1/2)^2+7(\beta_1-1)}{120(n+1/2)^2+7\beta_1}\right]
\end{aligned}$$

holds with the best constants $\beta_1 \approx 4.7243$ given by (1.14) *and $\alpha = 1860/343 \approx 5.4227$.*

2 Tools

To prove our main result, we need some lemmas as tools. The first lemma is the convolution formula of the Laplace transform.

Lemma 1 ([29]) *Let $f_i(t)$ for $i = 1,2$ be piecewise continuous in arbitrary finite intervals included on $(0,\infty)$. If there exist some constants $M_i > 0$ and $c_i \geq 0$ such that $|f_i(t)| \leq M_i e^{c_i t}$ for $i = 1,2$, then*

$$\int_0^\infty f_1(u)e^{-su}\,du \int_0^\infty f_2(v)e^{-sv}\,dv = \int_0^\infty \left(\int_0^t f_1(u)f_2(t-u)\,du\right)e^{-st}\,dt. \tag{2.1}$$

The second one is a special monotonicity rule for the ratio of two power series, which first appeared in [30, Lemma 6.4] and was proved in [31], also see [32].

Lemma 2 ([31, Corollary 2.3]) *Let $A(t) = \sum_{k=0}^\infty a_k t^k$ and $B(t) = \sum_{k=0}^\infty b_k t^k$ be two real power series converging on $\mathbb{R}$ with $b_k > 0$ for all k. If, for certain $m \in \mathbb{N}$, the sequence $\{a_k/b_k\}$ is increasing (decreasing) for $0 \leq k \leq m$ and decreasing (increasing) for $k \geq m$, then there is a unique $t_0 \in (0,\infty)$ such that the function A/B is increasing (decreasing) on $(0,t_0)$ and decreasing (increasing) on (t_0,∞).*

The third lemma is called L'Hospital piecewise monotonicity rule [33].

Lemma 3 ([33, Theorem 8]) *Let $-\infty \leq a < b \leq \infty$. Suppose that (i) f and g are differentiable functions on (a,b); (ii) $g' \neq 0$ on (a,b); (iii) $f(a^+) = g(a^+) = 0$; (iv) there is $c \in (a,b)$ such that f'/g' is increasing (decreasing) on (a,c) and decreasing (increasing) on (c,b). Then*

- (i) *when $\operatorname{sgn} g' \operatorname{sgn} H_{f,g}(b^-) \geq (\leq) 0$, f/g is increasing (decreasing) on (a,b), where $H_{f,g} = (f'/g')g - f$;*
- (ii) *when $\operatorname{sgn} g' \operatorname{sgn} H_{f,g}(b^-) < (>) 0$, there is a unique number $x_a \in (a,b)$ such that f/g is increasing (decreasing) on (a,x_a) and decreasing (increasing) on (x_a,b).*

The last one gives a monotonicity rule for the ratio of two Laplace transforms, which is crucial to proving our main result (see [34, Remark 3]).

Lemma 4 *Let the functions A, B be defined on $(0,\infty)$ such that their Laplace transforms exist with $B(t) \neq 0$ for all $t > 0$. Then the function*

$$x \mapsto U(x) = \frac{\int_0^\infty A(t)e^{-xt}\,dt}{\int_0^\infty B(t)e^{-xt}\,dt}$$

is decreasing (increasing) on $(0,\infty)$ if A/B is increasing (decreasing) on $(0,\infty)$.

Proof Differentiation yields

$$\begin{aligned}
&\left(\int_0^\infty B(t)e^{-xt}\,dt\right)^2 U'(x)\\
&\quad= -\int_0^\infty A(t)e^{-xt}\,dt \int_0^\infty B(t)e^{-xt}\,dt + \int_0^\infty A(t)e^{-xt}\,dt \int_0^\infty tB(t)e^{-xt}\,dt\\
&\quad= \int_0^\infty \int_0^\infty t\left[\frac{A(s)}{B(s)} - \frac{A(t)}{B(t)}\right]B(t)B(s)e^{-xt-xs}\,ds\,dt := D.
\end{aligned}$$

Exchanging the integral variables s and t, we have

$$D = \int_0^\infty \int_0^\infty s\left[\frac{A(t)}{B(t)} - \frac{A(s)}{B(s)}\right]B(s)B(t)e^{-xs-xt}\,dt\,ds,$$

then adding gives

$$2D = -\int_0^\infty \int_0^\infty [t-s]\left[\frac{A(t)}{B(t)} - \frac{A(s)}{B(s)}\right]B(s)B(t)e^{-xt-xs}\,dt\,ds.$$

By the assumptions, the desired assertions follow. □

3 Proof of Theorem 1

Before proving Theorem 1, we also need several concrete lemmas.

Lemma 5 ([28, Lemma 4]) *Let g_0 be defined on $(0,\infty)$ by*

$$g_0(x) = \ln\Gamma\left(x+\frac{1}{2}\right) - x\ln x + x - \frac{1}{2}\ln(2\pi). \tag{3.1}$$

Then $g_0(x)$ has the following integral representation:

$$g_0(x) = -\int_0^\infty h(t)e^{-xt}\,dt, \tag{3.2}$$

where

$$h(t) = \frac{1}{t^2} - \frac{1}{2t\sinh(t/2)}. \tag{3.3}$$

Lemma 6 *Let $h(t)$ be defined on $(0,\infty)$ by* (3.3). *Then we have*

$$x\int_0^\infty h(t)e^{-xt}\,dt = \frac{1}{24} + \int_0^\infty h'(t)e^{-xt}\,dt, \tag{3.4}$$

$$x\int_0^\infty h'(t)e^{-xt}\,dt = \int_0^\infty h''(t)e^{-xt}\,dt, \tag{3.5}$$

$$x\int_0^\infty h''(t)e^{-xt}\,dt = -\frac{7}{2880} + \int_0^\infty h'''(t)e^{-xt}\,dt. \tag{3.6}$$

Proof Integration by parts yields

$$x\int_0^\infty h(t)e^{-xt}\,dt = -\int_0^\infty h(t)\,de^{-xt} = -\left[h(t)e^{-xt}\right]_{t=0}^{t=\infty} + \int_0^\infty h'(t)e^{-xt}\,dt,$$

which, by a simple computation,

$$\lim_{t\to 0} h(t)e^{-xt} = \lim_{t\to 0}\left(\frac{1}{t^2} - \frac{1}{2t\sinh(t/2)}\right)e^{-xt} = \frac{1}{24},$$

$$\lim_{t\to\infty} h(t)e^{-xt} = \lim_{t\to\infty}\left(\frac{1}{t^2} - \frac{1}{2t\sinh(t/2)}\right)e^{-xt} = 0,$$

gives (3.4).

Similarly, integration by parts and limit relations $\lim_{t\to 0} h'(t)e^{-xt} = 0$ and $\lim_{t\to\infty} h'(t)e^{-xt} = 0$ yield (3.5). Integration by parts in combination with $\lim_{t\to 0} h''(t)e^{-xt} = -7/2880$ and $\lim_{t\to\infty} h''(t)e^{-xt} = 0$ gives (3.6). □

Lemma 7 *Let $h(t)$ be defined by* (3.3). *Then* (i) $h'(t) < 0$ *for* $t > 0$; (ii) *there is* $t_0 > 0$ *such that the function* h'''/h' *is increasing on* $(0, t_0)$ *and decreasing on* (t_0, ∞). *Therefore, we have*

$$-\frac{31}{98} < \frac{h'''(t)}{h'(t)} < \lambda_0 \approx 0.051704,$$

where $\lambda_0 = h'''(t_0)/h'(t_0)$, *here* t_0 *is the unique solution of the equation* $[h'''(t)/h'(t)]' = 0$ *on* $(0, \infty)$.

Proof Differentiation yields

$$h'(t) = \frac{1}{4}\frac{2\sinh(t/2) + t\cosh(t/2)}{t^2\sinh^2(t/2)} - \frac{2}{t^3},$$

$$h''(t) = \frac{6}{t^4} - \frac{1}{16}\frac{t^2\cosh t + 8\cosh t + 4t\sinh t + 3t^2 - 8}{t^3\sinh^3(t/2)},$$

$$\begin{aligned} h'''(t) = {}& -\frac{24}{t^5} + \frac{1}{64t^4\sinh^4(t/2)}\biggl(6t^2\sinh\frac{3t}{2} + 48\sinh\frac{3t}{2} + t^3\cosh\frac{3t}{2} + 24t\cosh\frac{3t}{2} \\ & + 23t^3\cosh\frac{t}{2} - 24t\cosh\frac{t}{2} + 30t^2\sinh\frac{t}{2} - 144\sinh\frac{t}{2}\biggr). \end{aligned}$$

Simplifying and expanding in power series yield

$$\begin{aligned} -\biggl(4t^3\sinh^2\frac{t}{2}\biggr)h'(t) &= 4\cosh t - t^2\cosh\frac{t}{2} - 2t\sinh\frac{t}{2} - 4 \\ &= \sum_{n=3}^{\infty}\frac{2^{2n-2} - n^2}{2^{2n-4}(2n)!}t^{2n} > 0, \end{aligned}$$

which proves $h'(t) < 0$ for $t > 0$.

Then $h'''(t)/h'(t)$ can be expressed as

$$\begin{aligned} \frac{h'''(t)}{h'(t)} = {}& \frac{96\sinh^4 s - 3s^3\sinh 3s - 6s\sinh 3s - s^4\cosh 3s - 6s^2\cosh 3s}{16s^2(2\sinh^2 s - s\sinh s - s^2\cosh s)\sinh^2 s} \\ & + \frac{-15s^3\sinh s + 18s\sinh s - 23s^4\cosh s + 6s^2\cosh s}{16s^2(2\sinh^2 s - s\sinh s - s^2\cosh s)\sinh^2 s} := \frac{h_1(s)}{h_2(s)}, \end{aligned}$$

where $s = 2t$. Using 'product into sum' formula for hyperbolic functions and expanding in power series give

$$\begin{aligned} h_1(s) := {}& 12\cosh 4s - 48\cosh 2s - 3s^3\sinh 3s - 6s\sinh 3s - s^4\cosh 3s - 6s^2\cosh 3s \\ & - 15s^3\sinh s + 18s\sinh s - 23s^4\cosh s + 6s^2\cosh s + 36 \\ = {}& 12\sum_{n=0}^{\infty}\frac{4^{2n}}{(2n)!}s^{2n} - 48\sum_{n=0}^{\infty}\frac{2^{2n}}{(2n)!}s^{2n} - 3\sum_{n=2}^{\infty}\frac{3^{2n-3}}{(2n-3)!}s^{2n} \end{aligned}$$

$$\begin{aligned}
&-6\sum_{n=1}^{\infty}\frac{3^{2n-1}}{(2n-1)!}s^{2n}-\sum_{n=2}^{\infty}\frac{3^{2n-4}}{(2n-4)!}s^{2n}-6\sum_{n=1}^{\infty}\frac{3^{2n-2}}{(2n-2)!}s^{2n}-15\sum_{n=2}^{\infty}\frac{1}{(2n-3)!}s^{2n}\\
&+18\sum_{n=1}^{\infty}\frac{1}{(2n-1)!}s^{2n}-23\sum_{n=2}^{\infty}\frac{1}{(2n-4)!}s^{2n}+6\sum_{n=1}^{\infty}\frac{1}{(2n-2)!}s^{2n}\\
:=\ &4\sum_{n=2}^{\infty}\frac{a_n}{(2n)!}s^{2n},
\end{aligned}$$

where

$$\begin{aligned}
a_n &= 3\times 4^{2n}-12\times 2^{2n}-2n\bigl(2n^3+3n^2+19n+30\bigr)3^{2n-4}\\
&\quad -2n(2n-3)\bigl(23n^2-27n+10\bigr),\\
h_2(s) &:= 16s^2\bigl(2\sinh^2 s-s\sinh s-s^2\cosh s\bigr)\sinh^2 s\\
&= 4s^2\bigl(\cosh 4s-4\cosh 2s-s\sinh 3s-s^2\cosh 3s+s^2\cosh s+3s\sinh s+3\bigr)\\
&= 4s^2\left(\sum_{n=1}^{\infty}\frac{4^{2n}}{(2n)!}s^{2n}-4\sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}s^{2n}-\sum_{n=1}^{\infty}\frac{3^{2n-1}}{(2n-1)!}s^{2n}-\sum_{n=1}^{\infty}\frac{3^{2n-2}}{(2n-2)!}s^{2n}\right.\\
&\quad\left.+\sum_{n=1}^{\infty}\frac{1}{(2n-2)!}s^{2n}+3\sum_{n=1}^{\infty}\frac{1}{(2n-1)!}s^{2n}+3\right):=4\sum_{n=2}^{\infty}\frac{b_n}{(2n)!}s^{2n},
\end{aligned}$$

where

$$b_n = 2n(2n-1)\bigl(4^{2n-2}-2^{2n}-4n(n-1)3^{2n-4}+4n(n-1)\bigr).$$

Thus, if we prove the sequence $\{a_n/b_n\}_{n\geq 5}$ is increasing then decreasing, then by Lemma 2 we deduce that there is t_0 such that h''/h is increasing on $(0,t_0)$ and decreasing on (t_0,∞), and the proof is done. To this end, if $b_n>0$ for $n\geq 5$, then it suffices to show that there is $n_0>5$ such that $d_n=a_nb_{n+1}-b_na_{n+1}\leq 0$ for $5\leq n\leq n_0$ and $d_n\geq 0$ for $n\geq n_0$.

Now, it is easy to check that

$$\frac{b_{n+1}}{2(n+1)(2n+1)}-16\frac{b_n}{2n(2n-1)}=4n(7n-25)3^{2n-4}+12\times 2^{2n}-4n(15n-17)>0,$$

which together with $b_4=0$ yields $b_n>0$ for $n\geq 5$. On the other hand, by an elementary computation, we obtain

$$d_n=a_nb_{n+1}-b_na_{n+1}=\sum_{k=1,2,3,4,6,8,9,12,16}p_k(n)k^{2n},$$

where

$$\begin{aligned}
p_{16}(n) &= 6(4n+1),\\
p_{12}(n) &= -\frac{1}{324}n\bigl(28n^5+12n^4-1181n^3+9678n^2+3457n+1830\bigr),\\
p_9(n) &= \frac{64}{243}n^2(n+1)\bigl(n^3+8n^2+20n-2\bigr),
\end{aligned}$$

$$p_8(n) = 6\bigl(18n^2 - 41n - 8\bigr),$$
$$p_6(n) = -\frac{4}{81}n\bigl(20n^5 + 132n^4 + 185n^3 - 678n^2 - 997n - 822\bigr),$$
$$p_4(n) = -\frac{1}{4}\bigl(1380n^6 - 1804n^5 + 989n^4 - 3134n^3 + 1327n^2 - 2118n - 384\bigr),$$
$$p_3(n) = \frac{128}{27}n^2(n+1)\bigl(32n^5 - 32n^4 - 33n^3 + 48n^2 - 50n + 8\bigr),$$
$$p_2(n) = 4n\bigl(276n^5 - 508n^4 - 295n^3 + 106n^2 + 43n - 150\bigr),$$
$$p_1(n) = 192n^2(n+1)\bigl(9n^3 - 8n^2 + 2\bigr).$$

An easy verification yields

$$d_5 = -4{,}007{,}555{,}481{,}600,$$
$$d_6 = -3{,}910{,}448{,}396{,}574{,}720,$$
$$d_7 = -1{,}900{,}746{,}298{,}639{,}319{,}040,$$
$$d_8 = -630{,}125{,}315{,}460{,}849{,}991{,}680,$$
$$d_9 = -150{,}180{,}694{,}294{,}194{,}463{,}408{,}128,$$
$$d_{10} = -20{,}155{,}436{,}802{,}005{,}011{,}207{,}151{,}616,$$

and $d_{11} = 3{,}463{,}285{,}943{,}229{,}784{,}738{,}339{,}553{,}280 > 0$. It remains to show $d_n > 0$ for $n \geq 11$. To this end, we write d_n as

$$\begin{aligned} d_n &= \bigl[p_{16}(n) \times 16^{2n} + p_{12}(n) \times 12^{2n}\bigr] \\ &\quad + \bigl[p_9(n) \times 9^{2n} + p_6(n) \times 6^{2n}\bigr] + \bigl[p_8(n) \times 8^{2n} + p_4(n) \times 4^{2n}\bigr] \\ &\quad + \bigl[p_3(n) \times 3^{2n} + p_2(n) \times 2^{2n} + p_1(n)\bigr], \end{aligned}$$

and denote the expressions in the square brackets by d'_n, d''_n, d'''_n and d''''_n, respectively. We easily get the recurrence relation of d'_n

$$\begin{aligned} &\frac{p_{16}(n)d'_{n+1} - 16^2 p_{16}(n+1)d'_n}{12^{2n}} \\ &\quad = 144p_{16}(n)p_{12}(n+1) - 16^2 p_{16}(n+1)p_{12}(n) \\ &\quad = \frac{8}{27}\bigl(784n^7 - 3724n^6 - 51{,}008n^5 + 328{,}397n^4 + 10{,}762n^3 \\ &\qquad - 1{,}037{,}977n^2 - 650{,}802n - 124{,}416\bigr) \\ &\quad = \frac{8}{27}\bigl(784m^7 + 23{,}716m^6 + 248{,}872m^5 + 1{,}086{,}697m^4 + 1{,}666{,}702m^3 \\ &\qquad + 1{,}160{,}503m^2 + 10{,}500{,}078m + 20{,}928{,}024\bigr) > 0, \end{aligned}$$

where $m = n - 5 \geq 6$. This in combination with $p_{16}(n) > 0$ and $d'_{11} = 2^{45} \times 71{,}481{,}197{,}516{,}733 > 0$ leads us to $d'_n > 0$ for $n \geq 11$.

Similarly, we have

$$\begin{aligned}
&\frac{p_9(n)d''_{n+1} - 81p_9(n+1)d''_n}{6^{2n}} \\
&\quad = 36p_9(n)p_6(n+1) - 81p_9(n+1)p_6(n) \\
&\quad = \frac{256}{2187}n(n+1)^2\bigl(100n^9 + 1960n^8 + 15{,}413n^7 + 55{,}819n^6 + 53{,}414n^5 \\
&\qquad - 273{,}428n^4 - 1{,}024{,}655n^3 - 1{,}559{,}511n^2 - 1{,}278{,}612n - 399{,}492\bigr) > 0
\end{aligned}$$

for $n \geq 3$. This together with $p_9(n) > 0$ and $d''_3 = 717{,}610{,}752 > 0$ yields $d''_n > 0$ for $n \geq 3$.

Also, we get

$$\begin{aligned}
&\frac{p_8(n)d'''_{n+1} - 64p_8(n+1)d'''_n}{4^{2n}} \\
&\quad = 16p_8(n)p_4(n+1) - 64p_8(n+1)p_4(n) \\
&\quad = 1{,}788{,}480n^8 - 5{,}219{,}424n^7 - 367{,}632n^6 + 8{,}703{,}096n^5 + 13{,}278{,}240n^4 \\
&\qquad + 9{,}974{,}760n^3 - 7{,}438{,}608n^2 + 1{,}718{,}592n + 423{,}936,
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
&1{,}788{,}480m^8 + 23{,}396{,}256m^7 + 126{,}870{,}192m^6 + 367{,}098{,}936m^5 + 619{,}910{,}160m^4 \\
&\quad + 687{,}582{,}120m^3 + 676{,}606{,}944m^2 + 635{,}328{,}864m + 311{,}091{,}840 > 0,
\end{aligned}$$

where $m = n - 2 \geq 9$. This in combination with $p_8(n) > 0$ for $n \geq 3$ and $d'''_7 = 2^{30} \times 6{,}089{,}535 > 0$ indicates that $d'''_n > 0$ for $n \geq 7$.

As far as $d''''_n > 0$ for $n \geq 11$, it is clear, since

$$\begin{aligned}
\frac{27}{128n^2(n+1)}p_3(n) &= \bigl(32n^5 - 32n^4 - 33n^3 + 48n^2 - 50n + 8\bigr) \\
&= 32m^5 + 288m^4 + 991m^3 + 1642m^2 + 1282m + 348 > 0,
\end{aligned}$$

where $m = n - 2 > 0$,

$$\begin{aligned}
\frac{p_2(n)}{4n} &= \bigl(276n^5 - 508n^4 - 295n^3 + 106n^2 + 43n - 150\bigr) \\
&= 276m^5 + 3632m^4 + 18{,}449m^3 + 44{,}539m^2 + 49{,}630m + 18{,}888 > 0
\end{aligned}$$

for $m = n-3 > 0$, $p_1(n) = 192n^2(n+1)(9n^3 - 8n^2 + 2) > 0$ for $n \geq 1$. This proves the piecewise monotonicity of h'''/h' on $(0,\infty)$.

It is easy to verify that

$$\lim_{t\to 0}\frac{h'''(t)}{h'(t)} = -\frac{31}{98} \quad \text{and} \quad \lim_{t\to\infty}\frac{h'''(t)}{h'(t)} = 0.$$

Solving the equation $[h'''(t)/h'(t)]' = 0$ yields $t = t_0 \approx 10.96011$, which gives $\lambda_0 = h'''(t_0)/h'(t_0) \approx 0.051704$.

By the piecewise monotonicity of h'''/h' on $(0,\infty)$, we conclude that

$$-\frac{7}{120} = \min\left(\lim_{t\to 0}\frac{h'''(t)}{h'(t)}, \lim_{t\to\infty}\frac{h'''(t)}{h'(t)}\right) < \frac{h'''(t)}{h'(t)} < \frac{h'''(t_0)}{h'(t_0)} = \lambda_0 \approx 0.051704,$$

which completes the proof. □

We now are in a position to prove Theorem 1.

Proof of Theorem 1 We first prove that

$$f(x) = -\frac{1}{168}\frac{\int_0^\infty (7+2880h''(t))e^{-xt}\,dt}{\int_0^\infty (\int_0^t h'(s)\,ds)e^{-xt}\,dt} := -\frac{1}{168}\frac{\int_0^\infty A(t)e^{-xt}\,dt}{\int_0^\infty B(t)e^{-xt}\,dt}, \tag{3.7}$$

where

$$A(t) = 7+2880h''(t) \quad \text{and} \quad B(t) = \int_0^t h'(s)\,ds.$$

In fact, by Lemma 5 and identities (3.4) and (3.5), $f(x)$ can be expressed as

$$\begin{aligned} f(x) &= -\frac{1}{24\int_0^\infty h'(t)e^{-xt}\,dt} - \frac{120}{7}x^2 \\ &= -\frac{7+2880x^2\int_0^\infty h'(t)e^{-xt}\,dt}{168\int_0^\infty h'(t)e^{-xt}\,dt} = -\frac{1}{168}\frac{7/x+2880\int_0^\infty h''(t)e^{-xt}\,dt}{(1/x)\int_0^\infty h'(t)e^{-xt}\,dt}. \end{aligned}$$

Application of the identity

$$\frac{1}{x^n} = \frac{1}{\Gamma(n)}\int_0^\infty t^{n-1}e^{-xt}\,dt \quad \text{for } n>0$$

and Lemma 1 give (3.7).

Now, to prove f is strictly increasing on $(0,\infty)$, it suffices to prove $t\mapsto A(t)/B(t)$ is increasing on $(0,\infty)$ by Lemma 4. Similar to the proof of Theorem 1, we easily see that

$$\begin{aligned} &\lim_{t\to 0}A(t) = \lim_{t\to 0}\bigl(7+2880h''(t)\bigr) = \lim_{t\to 0}\left[7+2880\left(\frac{1}{t^2}-\frac{1}{2t\sinh(t/2)}\right)''\right] = 0, \\ &\lim_{t\to 0}B(t) = \lim_{t\to 0}\left(\int_0^t h'(s)\,ds\right) = 0, \end{aligned}$$

and the function $A'/B' = 2880h'''/h'$ is increasing on $(0,t_0)$ and decreasing on (t_0,∞) by Lemma 7. Then by Lemma 3 it is enough to check that $\operatorname{sgn} B'(t)\operatorname{sgn} H_{A,B}(\infty) > 0$. In fact, $B'(t) = h'(t) < 0$ for $t>0$ in view of Lemma 7, and

$$\begin{aligned} &\lim_{t\to\infty}\frac{A'(t)}{B'(t)} = \lim_{t\to\infty}\frac{2800h'''(t)}{h'(t)} = 0, \\ &\lim_{t\to\infty}B(t) = \lim_{t\to\infty}\int_0^t h'(s)\,ds = h(\infty)-h\bigl(0^+\bigr) = -\frac{1}{24}, \\ &\lim_{t\to\infty}A(t) = \lim_{t\to\infty}\bigl(7+2880h''(t)\bigr) = 7, \end{aligned}$$

which imply that

$$H_{A,B}(t) = \frac{A'(t)}{B'(t)}B(t) - A(t) \to -7 \quad \text{as } t \to \infty.$$

This indicates $\operatorname{sgn} B'(t) \operatorname{sgn} H_{A,B}(\infty) > 0$.

Using the asymptotic formula [35, p. 32, (5)]

$$\ln\Gamma\left(x + \frac{1}{2}\right) = x\ln x - x + \frac{1}{2}\ln(2\pi) - \sum_{k=1}^{\infty} \frac{(1-2^{1-2k})B_{2k}}{2k(2k-1)} \frac{1}{x^{2k-1}} \tag{3.8}$$

as $x \to \infty$, we find that

$$\begin{aligned} f(x) &\sim \frac{1}{24x(-\frac{1}{24x} + \frac{7}{2880x^3} - \frac{31}{40{,}320x^5}) + 1} - \frac{120}{7}x^2 \\ &= \frac{3720}{7}\frac{x^2}{98x^2 - 31} \to \frac{1860}{343} \quad \text{as } x \to \infty. \end{aligned}$$

While $f(0^+) = 1$ is clear. This completes the proof. □

4 Concluding remarks

Remark 3 In this paper, we investigate the monotonicity of the function $f(x)$. In general, it is difficult to deal with such monotonicity since the gamma function Γ occurs in denominator. However, by the aid of Lemma 5, $f(x)$ is equivalently changed into the ratio of two Laplace transformations of $A(x)$ and $B(x)$. While Lemma 4 provides exactly an approach to confirm the monotonicity of such ratio. Undoubtedly, it is a novel idea.

Moreover, it is known that Laplace transformation is related to the completely monotonic function. A function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and satisfies

$$(-1)^n f^{(n)}(x) \geq 0 \quad \text{for all } x \in I \text{ and } n = 0, 1, 2, \ldots. \tag{4.1}$$

If inequality (4.1) is strict, then f is said to be strictly completely monotonic on I. The classical Bernstein's theorem [36, 37] states that a function f is completely monotonic on $(0, \infty)$ if and only if it is a Laplace transform of some nonnegative measure μ, that is,

$$f(x) = \int_0^\infty e^{-xt}\, d\mu(t),$$

where $\mu(t)$ is non-decreasing and the integral converges for $0 < x < \infty$.

Remark 4 Let $\alpha > \beta$. If $B(t) > 0$ for $t > 0$ and

$$\beta < \frac{\int_0^\infty A(t)e^{-xt}\, dt}{\int_0^\infty B(t)e^{-xt}\, dt} < \alpha,$$

then, by Bernstein's theorem, both the functions

$$x \mapsto \int_0^\infty \left[A(t) - \beta B(t)\right]e^{-xt}\, dt \quad \text{and} \quad x \mapsto \int_0^\infty \left[\alpha B(t) - A(t)\right]e^{-xt}\, dt$$

are completely monotonic on $(0,\infty)$. And then, by Theorem 1, we immediately get the following.

Proposition 1 *Both the functions*

$$g_1(x) = 2880\left(x^2 + \frac{31}{98}\right)\left[\ln\Gamma(x+1/2) - x\ln x + x - \frac{1}{2}\ln(2\pi)\right] + 120x + \frac{1517}{49x},$$
$$g_2(x) = -\left(x^2 + \frac{7}{120}\right)\left[\ln\Gamma(x+1/2) - x\ln x + x - \frac{1}{2}\ln(2\pi)\right] - \frac{1}{24}x$$

are completely monotonic on $(0,\infty)$.

Furthermore, by Bernstein's theorem and Lemma 7, Proposition 1 can be improved as follows.

Theorem 2 *The function*

$$g(x) = 24x\left(\frac{120}{7}x^2 + a\right)\left[\ln\Gamma(x+1/2) - x\ln x + x - \ln\sqrt{2\pi}\right] + \frac{120}{7}x^2 + a - 1$$

is completely monotonic on $(0,\infty)$ *if and only if* $a \geq 1860/343$, *and so is* $-g(x)$ *on* $(0,\infty)$ *if and only if* $a \leq -120\lambda_0/7 \approx -0.88635$, *where* λ_0 *is defined in Lemma* 7.

Proof By Lemma 5 and identities (3.4), (3.5) and (3.6), $g(x)$ can be written as

$$\begin{aligned}
g(x) &= -24\left(\frac{120}{7}x^2 + a\right)\left(\frac{1}{24} + \int_0^\infty h'(t)e^{-xt}\,dt\right) + \frac{120}{7}x^2 + a - 1 \\
&= -1 - \frac{2880}{7}x^2\int_0^\infty h'(t)e^{-xt}\,dt - 24a\int_0^\infty h'(t)e^{-xt}\,dt \\
&= -24\int_0^\infty \left[a - \left(-\frac{120}{7}\frac{h'''(t)}{h'(t)}\right)\right]h'(t)e^{-xt}\,dt.
\end{aligned}$$

Since $h'(t) < 0$ for $t > 0$, by Bernstein's theorem and Lemma 7, g is completely monotonic on $(0,\infty)$ if and only if

$$a \geq \frac{120}{7}\sup_{t\in(0,\infty)}\left(-\frac{h'''(t)}{h'(t)}\right) = -\frac{120}{7}\inf_{t\in(0,\infty)}\frac{h'''(t)}{h'(t)} = -\frac{120}{7}\left(-\frac{31}{98}\right) = \frac{1860}{343},$$

and so is $-g$ on $(0,\infty)$ if and only if

$$a \leq \frac{120}{7}\inf_{t\in(0,\infty)}\left(-\frac{h'''(t)}{h'(t)}\right) = -\frac{120}{7}\sup_{t\in(0,\infty)}\frac{h'''(t)}{h'(t)} = -\frac{120}{7}\lambda_0 \approx -0.88635.$$

This ends the proof. □

Remark 5 The expression of $f(x)$ reminds us to consider the asymptotic expansion of

$$\frac{1}{24x(\ln\Gamma(x+1/2) - x\ln x + x - \ln\sqrt{2\pi}) + 1} := x^2\sum_{n=0}^\infty \frac{c_n}{x^{2n}}.$$

Using asymptotic expansion (3.8), we have

$$-\left(24\sum_{n=2}^{\infty}\frac{(1-2^{1-2n})B_{2n}}{2n(2n-1)}\frac{1}{x^{2n-2}}\right)\left(x^2\sum_{n=0}^{\infty}c_nx^{-2n}\right)=1,$$

that is,

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(1-2^{-2k-3})B_{2k+4}}{2(k+2)(2k+3)}c_{n-k}\right)\frac{1}{x^{2n}}=-\frac{1}{24}.$$

Comparing coefficients gives

$$-\frac{7}{2880}c_0=-\frac{1}{24},$$
$$\sum_{k=0}^{n}\frac{(1-2^{-2k-3})B_{2k+4}}{2(k+2)(2k+3)}c_{n-k}=0\quad\text{for } n\geq 1,$$

which show that c_n has the recurrence formula

$$c_n=\frac{2880}{7}\sum_{k=1}^{n}\frac{(1-2^{-2k-3})B_{2k+4}}{2(k+2)(2k+3)}c_{n-k}\quad\text{and}\quad c_0=\frac{120}{7},$$

from which we obtain a new asymptotic expansion for the gamma function:

$$\frac{\Gamma(x+1/2)}{\sqrt{2\pi}(x/e)^x}\sim\exp\left(-\frac{1}{24x}+\frac{1}{24x^3}\frac{1}{\frac{120}{7}+\frac{1860}{343}x^{-2}+\cdots+c_nx^{-2n}+\cdots}\right)\quad\text{as } x\to\infty.$$

Moreover, it is easy to prove the inequalities

$$\begin{aligned}\exp\left(-\frac{1}{1440}\frac{5880x^2+1517}{x(98x^2+31)}\right)&=\exp\left(-\frac{1}{24x}+\frac{1}{24x^3}\frac{1}{\frac{120}{7}+\frac{1860}{343}x^{-2}}\right)\\&<\frac{\Gamma(x+1/2)}{\sqrt{2\pi}(x/e)^x}<\exp\left(-\frac{1}{24x}+\frac{1}{24x^3}\frac{1}{\frac{120}{7}}\right)\\&=\exp\left(-\frac{120x^2-7}{2880x^3}\right)\end{aligned}$$

hold for $x\geq 1/2$.

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The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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