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A more accurate Mulholland-type inequality in the whole plane

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Abstract

By introducing independent parameters, applying the weight coefficients, and Hermite-Hadamard's inequality, we give a more accurate Mulholland-type inequality in the whole plane with a best possible constant factor. Furthermore, the equivalent forms, the reverses, a few particular cases, and the operator expressions are considered.

MSC: 26D15; 47A07

Keywords: Mulholland's inequality; parameter; weight coefficient; equivalent form; operator expression

1 Introduction

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_m, b_n \geq 0$, $0 < \sum_{m=1}^{\infty} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then we have the following Hardy-Hilbert's inequality (*cf.* [1]):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}, \tag{1}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. In 1934, Hardy proved the following more accurate inequality of (1) with the same best possible constant factor (*cf.* Theorem 343 of [2]):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n-1} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}. \tag{2}$$

We still have the following Mulholland's inequality with the same best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (*cf.* Theorem 343 of [2], replacing $\frac{a_m}{m}$, $\frac{b_n}{n}$ by a_m, b_n):

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{\ln mn} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=2}^{\infty} \frac{a_m^p}{m}\right)^{\frac{1}{p}}\left(\sum_{n=2}^{\infty} \frac{b_n^q}{n}\right)^{\frac{1}{q}}. \tag{3}$$

Inequalities (1)-(3) are important in analysis and its applications (*cf.* [2, 3]). In 2007, Yang [4] first gave a Hilbert-type integral inequality in the whole plane. Many extensions of this type inequalities and (1)-(3) were provided in [5–20].

In 2016, Yang and Chen [21] gave a more accurate Hardy-Hilbert's inequality in the whole plane:

$$\sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty}\frac{a_m b_n}{(|m-\xi|+|n-\eta|)^{\lambda}} < 2B(\lambda_1,\lambda_2)\left[\sum_{|m|=1}^{\infty}|m-\xi|^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{|n|=1}^{\infty}|n-\eta|^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}, \tag{4}$$

where the constant factor $2B(\lambda_1,\lambda_2)$ ($0<\lambda_1,\lambda_2\le 1$, $\lambda_1+\lambda_2=\lambda$, $\xi,\eta\in[0,\frac{1}{2}]$) is the best possible.

In this paper, by introducing independent parameters and applying the weight coefficients and Hermite-Hadamard's inequality, we give a new more accurate extension of (3) in the whole plane with a best possible constant factor similar to (4). Furthermore, we consider the equivalent forms, the reverses, a few particular cases, and the operator expressions.

2 Some lemmas

In the following, we agree that $p\neq 0,1$, $\frac{1}{p}+\frac{1}{q}=1$, $\lambda_1,\lambda_2>0$, $\lambda_1+\lambda_2=\lambda$,

$$\alpha,\beta\in\left[\arccos\sqrt{\frac{1}{3}},\pi-\arccos\sqrt{\frac{1}{3}}\right]\quad\bigl(\subseteq(0,\pi)\bigr),$$

$\xi,\eta\in(-\frac{3}{2},\frac{3}{2})$, satisfying

$$\frac{1}{1-\cos\gamma}-\frac{3}{2}\le\xi,\eta\le\frac{-1}{1+\cos\gamma}+\frac{3}{2}, \tag{5}$$

and

$$h_{\gamma}(\lambda_1):=2B(\lambda_1,\lambda_2)\csc^2\gamma\quad(\gamma=\alpha,\beta). \tag{6}$$

Note 1 For $\alpha,\beta=\frac{\pi}{2}$, we find $\xi,\eta\in[-\frac{1}{2},\frac{1}{2}]$. If $\alpha,\beta\in[\arccos\sqrt{\frac{1}{3}},\pi-\arccos\sqrt{\frac{1}{3}}]$, then $\xi,\eta=0$ satisfy (5).

For $|x|,|y|\ge\frac{3}{2}$, we set $A_{\xi,\alpha}(x):=|x-\xi|+(x-\xi)\cos\alpha$,

$$B_{\eta,\beta}(y):=|y-\eta|+(y-\eta)\cos\beta,$$

and

$$k(x,y):=\frac{1}{(\ln A_{\xi,\alpha}(x)+\ln B_{\eta,\beta}(y))^{\lambda}}=\frac{1}{\ln^{\lambda}A_{\xi,\alpha}(x)B_{\eta,\beta}(y)}. \tag{7}$$

Definition 1 Define two weight coefficients as follows:

$$\omega(\lambda_2,m):=\sum_{|n|=2}^{\infty}\frac{k(m,n)}{B_{\eta,\beta}(n)}\cdot\frac{\ln^{\lambda_1}A_{\xi,\alpha}(m)}{\ln^{1-\lambda_2}B_{\eta,\beta}(n)},\quad |m|\in\mathbb{N}\setminus\{1\}, \tag{8}$$

$$\varpi(\lambda_1,n) := \sum_{|m|=2}^{\infty} \frac{k(m,n)}{A_{\xi,\alpha}(m)} \cdot \frac{\ln^{\lambda_2} B_{\eta,\beta}(n)}{\ln^{1-\lambda_1} A_{\xi,\alpha}(m)}, \quad |n| \in \mathbb{N} \setminus \{1\}, \tag{9}$$

where $\sum_{|j|=2}^{\infty} \cdots = \sum_{j=-2}^{-\infty} \cdots \sum_{j=2}^{\infty} \cdots$ $(j = m, n)$.

Lemma 1 *For* $\lambda_2 \leq 1$, *we have the following inequalities*:

$$k_\beta(\lambda_1)\big(1 - \theta(\lambda_2, m)\big) < \omega(\lambda_2, m) < k_\beta(\lambda_1), \quad |m| \in \mathbb{N} \setminus \{1\}, \tag{10}$$

where

$$\theta(\lambda_2, m) := \frac{1}{B(\lambda_1,\lambda_2)} \int_0^{\frac{\ln[(2+\eta)(1+\cos\beta)]}{\ln A_{\xi,\alpha}(m)}} \frac{u^{\lambda_2-1}}{(1+u)^\lambda}\, du = O\biggl(\frac{1}{\ln^{\lambda_2} A_{\xi,\alpha}(m)}\biggr) \in (0,1). \tag{11}$$

Proof For $|m| \in \mathbb{N} \setminus \{1\}$, we set

$$k^{(1)}(m,y) := \frac{1}{\ln^\lambda[A_{\xi,\alpha}(m)(y-\eta)(\cos\beta - 1)]}, \quad y < -\frac{3}{2},$$
$$k^{(2)}(m,y) := \frac{1}{\ln^\lambda[A_{\xi,\alpha}(m)(y-\eta)(\cos\beta + 1)]}, \quad y > \frac{3}{2},$$

wherefrom

$$h^{(1)}(m,-y) = \frac{1}{\ln^\lambda[A_{\xi,\alpha}(m)(y+\eta)(1-\cos\beta)]}, \quad y > \frac{3}{2}.$$

We find

$$\begin{aligned}
\omega(\lambda_2, m) &= \sum_{n=-2}^{-\infty} \frac{k^{(1)}(m,n)\ln^{\lambda_1} A_{\xi,\alpha}(m)}{(n-\eta)(\cos\beta - 1)\ln^{1-\lambda_2}[(n-\eta)(\cos\beta-1)]} \\
&\quad + \sum_{n=2}^{\infty} \frac{k^{(2)}(m,n)\ln^{\lambda_1} A_{\xi,\alpha}(m)}{(n-\eta)(1+\cos\beta)\ln^{1-\lambda_2}[(n-\eta)(1+\cos\beta)]} \\
&= \frac{\ln^{\lambda_1} A_{\xi,\alpha}(m)}{1-\cos\beta} \sum_{n=2}^{\infty} \frac{k^{(1)}(m,-n)}{(n+\eta)\ln^{1-\lambda_2}[(n+\eta)(1-\cos\beta)]} \\
&\quad + \frac{\ln^{\lambda_1} A_{\xi,\alpha}(m)}{1+\cos\beta} \sum_{n=2}^{\infty} \frac{k^{(2)}(m,n)}{(n-\eta)\ln^{1-\lambda_2}[(n-\eta)(1+\cos\beta)]}.
\end{aligned} \tag{12}$$

For fixed $|m| \in \mathbb{N} \setminus \{1\}$, since $\lambda > 0$, $0 < \lambda_2 \leq 1$,we find that, for $y > \frac{3}{2}$,

$$\frac{d}{dy} \frac{k^{(i)}(m,(-1)^i y)}{(y-(-1)^i\eta)\ln^{1-\lambda_2}[(y-(-1)^i\eta)(1+(-1)^i\cos\beta)]} < 0,$$
$$\frac{d^2}{dy^2} \frac{k^{(i)}(m,(-1)^i y)}{(y-(-1)^i\eta)\ln^{1-\lambda_2}[(y-(-1)^i\eta)(1+(-1)^i\cos\beta)]} > 0 \quad (i = 1,2),$$

and it follows that

$$\frac{k^{(i)}(m,(-1)^i y)}{(y-(-1)^i\eta)\ln^{1-\lambda_2}[(y-(-1)^i\eta)(1+(-1)^i\cos\beta)]} \quad (i = 1,2)$$

are strict decreasing and strictly convex in $(\frac{3}{2},\infty)$. By Hermite-Hadamard's inequality (see [22]) and (12) we find

$$\begin{aligned}\omega(\lambda_2,m) &< \frac{\ln^{\lambda_1} A_{\xi,\alpha}(m)}{1-\cos\beta}\int_{3/2}^{\infty}\frac{k^{(1)}(m,-y)}{(y+\eta)\ln^{1-\lambda_2}[(y+\eta)(1-\cos\beta)]}\,dy \\ &\quad + \frac{\ln^{\lambda_1} A_{\xi,\alpha}(m)}{1+\cos\beta}\int_{3/2}^{\infty}\frac{k^{(2)}(m,y)}{(y-\eta)\ln^{1-\lambda_2}[(y-\eta)(1+\cos\beta)]}\,dy.\end{aligned}$$

In view of (5), it follows that $(\frac{3}{2}\pm\eta)(1\mp\cos\beta)\geq 1$. Setting $u=\frac{\ln[(y+\eta)(1-\cos\beta)]}{\ln A_{\xi,\alpha}(m)}$ ($u=\frac{\ln[(y-\eta)(1+\cos\beta)]}{\ln A_{\xi,\alpha}(m)}$) in the first (second) integral, by simplifications we obtain

$$\begin{aligned}\omega(\lambda_2,m) &< \left(\frac{1}{1-\cos\beta}+\frac{1}{1+\cos\beta}\right)\int_0^{\infty}\frac{u^{\lambda_2-1}\,du}{(1+u)^{\lambda}} \\ &= 2B(\lambda_1,\lambda_2)\csc^2\beta = k_\beta(\lambda_1).\end{aligned}$$

By monotonicity and (12) we still have

$$\begin{aligned}\omega(\lambda_2,m) &> \frac{\ln^{\lambda_1} A_{\xi,\alpha}(m)}{1-\cos\beta}\int_{2}^{\infty}\frac{k^{(1)}(m,-y)}{(y+\eta)\ln^{1-\lambda_2}[(y+\eta)(1-\cos\beta)]}\,dy \\ &\quad + \frac{\ln^{\lambda_1} A_{\xi,\alpha}(m)}{1+\cos\beta}\int_{2}^{\infty}\frac{k^{(2)}(m,y)}{(y-\eta)\ln^{1-\lambda_2}[(y-\eta)(1+\cos\beta)]}\,dy \\ &\geq \left(\frac{1}{1-\cos\beta}+\frac{1}{1+\cos\beta}\right)\int_{\frac{\ln[(2+|\eta|)(1+|\cos\beta|)]}{\ln A_{\xi,\alpha}(m)}}^{\infty}\frac{u^{\lambda_2-1}\,du}{(1+u)^{\lambda}} \\ &= k_\beta(\lambda_1)-2\csc^2\beta\int_0^{\frac{\ln[(2+|\eta|)(1+|\cos\beta|)]}{\ln A_{\xi,\alpha}(m)}}\frac{u^{\lambda_2-1}\,du}{(1+u)^{\lambda}} = k_\beta(\lambda_1)\bigl(1-\theta(\lambda_2,m)\bigr)>0,\end{aligned}$$

where $\theta(\lambda_2,m)$ is indicated by (11). It follows that $\theta(\lambda_2,m)<1$ and

$$\begin{aligned}0<\theta(\lambda_2,m) &< \frac{1}{B(\lambda_1,\lambda_2)}\int_0^{\frac{\ln[(2+|\eta|)(1+|\cos\beta|)]}{\ln A_{\xi,\alpha}(m)}}u^{\lambda_2-1}\,du \\ &= \frac{1}{\lambda_2 B(\lambda_1,\lambda_2)}\left(\frac{\ln[(2+|\eta|)(1+|\cos\beta|)]}{\ln A_{\xi,\alpha}(m)}\right)^{\lambda_2}.\end{aligned}$$

Hence, (10) and (11) are valid. □

In the same way, we still have the following:

Lemma 2 *For* $\lambda_1\leq 1$, *we have the following inequalities*:

$$k_\alpha(\lambda_1)\bigl(1-\tilde{\theta}(\lambda_1,n)\bigr)<\varpi(\lambda_1,n)<k_\alpha(\lambda_1),\quad |n|\in\mathbb{N}\setminus\{1\}, \tag{13}$$

where

$$\tilde{\theta}(\lambda_1,n):=\frac{1}{B(\lambda_1,\lambda_2)}\int_0^{\frac{\ln[(2+|\xi|)(1+|\cos\alpha|)]}{\ln B_{\eta,\beta}(n)}}\frac{u^{\lambda_1-1}}{(1+u)^{\lambda}}\,du = O\left(\frac{1}{\ln^{\lambda_1} B_{\eta,\beta}(n)}\right)\in(0,1). \tag{14}$$

Lemma 3 *If* $\rho > 0$, $\gamma \in [\arccos\sqrt{\frac{1}{3}}, \pi - \arccos\sqrt{\frac{1}{3}}]$ $(\gamma = \alpha, \beta)$, *and*

$$\frac{1}{1-\cos\gamma} - \frac{3}{2} \le \varsigma \le \frac{-1}{1+\cos\gamma} + \frac{3}{2} \quad (\varsigma = \xi, \eta),$$

then for $(\varsigma, \gamma) = (\xi, \alpha)$ *(or* (η, β)*), we have*

$$H_\rho(\varsigma,\gamma) := \sum_{|n|=2}^{\infty} \frac{\ln^{-1-\rho}[|n-\varsigma| + (n-\varsigma)\cos\gamma]}{|n-\varsigma| + (n-\varsigma)\cos\gamma} = \frac{1}{\rho}\bigl(2\csc^2\gamma + o(1)\bigr) \quad \bigl(\rho \to 0^+\bigr). \tag{15}$$

Proof By Hermite-Hadamard's inequality we find

$$\begin{aligned} H_\rho(\varsigma,\gamma) &= \sum_{n=-2}^{-\infty} \frac{\ln^{-1-\rho}[(n-\varsigma)(\cos\gamma - 1)]}{(n-\varsigma)(\cos\gamma-1)} + \sum_{n=2}^{\infty} \frac{\ln^{-1-\rho}[(n-\varsigma)(\cos\gamma+1)]}{(n-\varsigma)(\cos\gamma+1)} \\ &= \sum_{n=2}^{\infty}\left\{ \frac{\ln^{-1-\rho}[(n+\varsigma)(1-\cos\gamma)]}{(n-\varsigma)(1-\cos\gamma)} + \frac{\ln^{-1-\rho}[(n-\varsigma)(\cos\gamma+1)]}{(n-\varsigma)(\cos\gamma+1)} \right\} \\ &\le \int_{\frac{3}{2}}^{\infty} \left\{ \frac{\ln^{-1-\rho}[(y+\varsigma)(1-\cos\gamma)]}{(y-\varsigma)(1-\cos\gamma)} + \frac{\ln^{-1-\rho}[(y-\varsigma)(\cos\gamma+1)]}{(y-\varsigma)(\cos\gamma+1)} \right\} dy \\ &= \frac{1}{\rho}\left\{ \frac{\ln^{-\rho}[(\frac{3}{2}+\varsigma)(1-\cos\gamma)]}{1-\cos\gamma} + \frac{\ln^{-\rho}[(\frac{3}{2}-\varsigma)(1+\cos\gamma)]}{1+\cos\gamma} \right\} \\ &= \frac{1}{\rho}\left(\frac{1}{1-\cos\gamma} + \frac{1}{1+\cos\gamma} + o_1(1) \right) \quad \bigl(\rho \to 0^+\bigr). \end{aligned}$$

We still can find that

$$\begin{aligned} H_\rho(\varsigma,\gamma) &= \sum_{n=2}^{\infty}\left\{ \frac{\ln^{-1-\rho}[(n+\varsigma)(1-\cos\gamma)]}{(n-\varsigma)(1-\cos\gamma)} + \frac{\ln^{-1-\rho}[(n-\varsigma)(\cos\gamma+1)]}{(n-\varsigma)(\cos\gamma+1)} \right\} \\ &\ge \int_{2}^{\infty} \left\{ \frac{\ln^{-1-\rho}[(y+\varsigma)(1-\cos\gamma)]}{(y-\varsigma)(1-\cos\gamma)} + \frac{\ln^{-1-\rho}[(y-\varsigma)(\cos\gamma+1)]}{(y-\varsigma)(\cos\gamma+1)} \right\} dy \\ &= \frac{1}{\rho}\left\{ \frac{\ln^{-\rho}[(2+\varsigma)(1-\cos\gamma)]}{1-\cos\gamma} + \frac{\ln^{-\rho}[(2-\varsigma)(1+\cos\gamma)]}{1+\cos\gamma} \right\} \\ &= \frac{1}{\rho}\left(\frac{1}{1-\cos\gamma} + \frac{1}{1+\cos\gamma} + o_2(1) \right) \quad \bigl(\rho \to 0^+\bigr). \end{aligned}$$

Hence, we have (15). □

3 Main results and the reverses

We also set

$$K(\lambda_1) := h_\beta^{1/p}(\lambda_1) h_\alpha^{1/q}(\lambda_1) = 2B(\lambda_1,\lambda_2)\csc^{2/p}\beta\csc^{2/q}\alpha. \tag{16}$$

Theorem 1 *Suppose that* $p > 1$, $\lambda_1, \lambda_2 \le 1$, $a_m, b_n \ge 0$ $(|m|, |n| \in \mathbb{N} \setminus \{1\})$, *and*

$$0 < \sum_{|m|=2}^{\infty} \frac{\ln^{p(1-\lambda_1)-1} A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}} a_m^p < \infty, \qquad 0 < \sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} B_{\eta,\beta}(n)}{(B_{\eta,\beta}(n))^{1-q}} b_n^q < \infty.$$

We have the following equivalent inequalities:

$$I := \sum_{|n|=2}^{\infty} \sum_{|m|=2}^{\infty} k(m,n) a_m b_n$$

$$< K(\lambda_1) \left[\sum_{|m|=2}^{\infty} \frac{\ln^{p(1-\lambda_1)-1} A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}} a_m^p \right]^{\frac{1}{p}} \left[\sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} B_{\eta,\beta}(n)}{(B_{\eta,\beta}(n))^{1-q}} b_n^q \right]^{\frac{1}{q}}, \tag{17}$$

$$J := \left[\sum_{|n|=2}^{\infty} \frac{\ln^{p\lambda_2-1} B_{\eta,\beta}(n)}{B_{\eta,\beta}(n)} \left(\sum_{|m|=2}^{\infty} k(m,n) a_m \right)^p \right]^{\frac{1}{p}}$$

$$< K(\lambda_1) \left[\sum_{|m|=2}^{\infty} \frac{\ln^{p(1-\lambda_1)-1} A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}} a_m^p \right]^{\frac{1}{p}}. \tag{18}$$

In particular, (i) *for* $\xi, \eta \in [-\frac{1}{2}, \frac{1}{2}]$ $(\alpha = \beta = \frac{\pi}{2})$, *we have the following equivalent inequalities*:

$$\sum_{|n|=2}^{\infty} \sum_{|m|=2}^{\infty} \frac{a_m b_n}{\ln^{\lambda}(|m-\xi||n-\eta|)}$$

$$< 2B(\lambda_1, \lambda_2) \left[\sum_{|m|=2}^{\infty} \frac{\ln^{p(1-\lambda_1)-1} |m-\xi|}{|m-\xi|^{1-p}} a_m^p \right]^{\frac{1}{p}} \left[\sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} |n-\eta|}{|n-\eta|^{1-q}} b_n^q \right]^{\frac{1}{q}}, \tag{19}$$

$$\left\{ \sum_{|n|=2}^{\infty} \frac{\ln^{p\lambda_2-1} |n-\eta|}{|n-\eta|} \left[\sum_{|m|=2}^{\infty} \frac{a_m}{\ln^{\lambda}(|m-\xi||n-\eta|)} \right]^p \right\}^{\frac{1}{p}}$$

$$< 2B(\lambda_1, \lambda_2) \left[\sum_{|m|=2}^{\infty} \frac{\ln^{p(1-\lambda_1)-1} |m-\xi|}{|m-\xi|^{1-p}} a_m^p \right]^{\frac{1}{p}}. \tag{20}$$

(ii) *For* $\alpha, \beta \in [\arccos \frac{1}{3}, \pi - \arccos \frac{1}{3}]$ $(\xi = \eta = 0)$, *we have the following equivalent inequalities*:

$$\sum_{|n|=2}^{\infty} \sum_{|m|=2}^{\infty} \frac{a_m b_n}{\ln^{\lambda}[(|m| + m\cos\alpha)(|n| + n\cos\beta)]}$$

$$< K(\lambda_1) \left[\sum_{|m|=2}^{\infty} \frac{\ln^{p(1-\lambda_1)-1}(|m| + m\cos\alpha)}{(|m| + m\cos\alpha)^{1-p}} a_m^p \right]^{\frac{1}{p}}$$

$$\times \left[\sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1}(|n| + n\cos\beta)}{(|n| + n\cos\beta)^{1-q}} b_n^q \right]^{\frac{1}{q}}, \tag{21}$$

$$\left\{ \sum_{|n|=2}^{\infty} \frac{\ln^{p\lambda_2-1}(|n| + n\cos\beta)}{|n| + n\cos\beta} \left[\sum_{|m|=2}^{\infty} \frac{a_m}{\ln^{\lambda}[(|m| + m\cos\alpha)(|n| + n\cos\beta)]} \right]^p \right\}^{\frac{1}{p}}$$

$$< K(\lambda_1) \left[\sum_{|m|=2}^{\infty} \frac{\ln^{p(1-\lambda_1)-1}(|m| + m\cos\alpha)}{(|m| + m\cos\alpha)^{1-p}} a_m^p \right]^{\frac{1}{p}}. \tag{22}$$

Proof By Hölder's inequality with weight (*cf.* [22]) and (9) we find

$$\begin{aligned}\left(\sum_{|m|=2}^{\infty} k(m,n)a_m\right)^p &= \left\{\sum_{|m|=2}^{\infty} k(m,n)\left[\frac{(A_{\xi,\alpha}(m))^{\frac{1}{q}} \ln^{\frac{1-\lambda_1}{q}} A_{\xi,\alpha}(m)}{\ln^{\frac{1-\lambda_2}{p}} B_{\eta,\beta}(n)} a_m\right]\right. \\ &\quad \left.\times \left[\frac{\ln^{\frac{1-\lambda_2}{p}} B_{\eta,\beta}(n)}{(A_{\xi,\alpha}(m))^{\frac{1}{q}} \ln^{\frac{1-\lambda_1}{q}} A_{\xi,\alpha}(m)}\right]\right\}^p \\ &\le \sum_{|m|=2}^{\infty} k(m,n)\frac{(A_{\xi,\alpha}(m))^{\frac{p}{q}} \ln^{\frac{(1-\lambda_1)p}{q}} A_{\xi,\alpha}(m)}{\ln^{1-\lambda_2} B_{\eta,\beta}(n)} a_m^p \\ &\quad \times \left[\sum_{|m|=2}^{\infty} k(m,n)\frac{\ln^{\frac{(1-\lambda_2)q}{p}} B_{\eta,\beta}(n)}{A_{\xi,\alpha}(m)\ln^{1-\lambda_1} A_{\xi,\alpha}(m)}\right]^{p-1} \\ &= \frac{(\varpi(\lambda_1,n))^{p-1} B_{\eta,\beta}(n)}{\ln^{p\lambda_2-1} B_{\eta,\beta}(n)} \sum_{|m|=2}^{\infty} k(m,n)\frac{(A_{\xi,\alpha}(m))^{\frac{p}{q}} \ln^{\frac{(1-\lambda_1)p}{q}} A_{\xi,\alpha}(m)}{B_{\eta,\beta}(n)\ln^{1-\lambda_2} B_{\eta,\beta}(n)} a_m^p.\end{aligned}$$

By (13) it follows that

$$\begin{aligned}J &< k_\alpha^{1/q}(\lambda_1)\left[\sum_{|n|=2}^{\infty}\sum_{|m|=2}^{\infty} k(m,n)\frac{(A_{\xi,\alpha}(m))^{\frac{p}{q}} \ln^{\frac{(1-\lambda_1)p}{q}} A_{\xi,\alpha}(m)}{B_{\eta,\beta}(n)\ln^{1-\lambda_2} B_{\eta,\beta}(n)} a_m^p\right]^{\frac{1}{p}} \\ &= k_\alpha^{1/q}(\lambda_1)\left[\sum_{|m|=2}^{\infty}\sum_{|n|=2}^{\infty} k(m,n)\frac{(A_{\xi,\alpha}(m))^{\frac{p}{q}} \ln^{\frac{(1-\lambda_1)p}{q}} A_{\xi,\alpha}(m)}{B_{\eta,\beta}(n)\ln^{1-\lambda_2} B_{\eta,\beta}(n)} a_m^p\right]^{\frac{1}{p}} \\ &= k_\alpha^{1/q}(\lambda_1)\left[\sum_{|m|=2}^{\infty} \omega(\lambda_2,m)\frac{n^{p(1-\lambda_1)-1} A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}} a_m^p\right]^{\frac{1}{p}}. \qquad (23)\end{aligned}$$

By (10) and (16) we have (18).

Using Hölder's inequality again, we have

$$\begin{aligned}I &= \sum_{|n|=2}^{\infty}\left[\frac{(B_{\eta,\beta}(n))^{\frac{-1}{p}}}{\ln^{\frac{1}{p}-\lambda_2} B_{\eta,\beta}(n)}\sum_{|m|=2}^{\infty} k(m,n)a_m\right]\left[\frac{\ln^{\frac{1}{p}-\lambda_2} B_{\eta,\beta}(n)}{(B_{\eta,\beta}(n))^{\frac{-1}{p}}} b_n\right] \\ &\le J\left[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1} B_{\eta,\beta}(n)}{(B_{\eta,\beta}(n))^{1-q}} b_n^q\right]^{\frac{1}{q}}, \qquad (24)\end{aligned}$$

and then by (18) we have (17).

On the other hand, assuming that (17) is valid, we set

$$b_n := \frac{\ln^{p\lambda_2-1} B_{\eta,\beta}(n)}{B_{\eta,\beta}(n)}\left(\sum_{|m|=2}^{\infty} k(m,n)a_m\right)^{p-1}, \quad |n| \in \mathbb{N}\setminus\{1\},$$

and find

$$J = \left[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1} B_{\eta,\beta}(n)}{(B_{\eta,\beta}(n))^{1-q}} b_n^q\right]^{\frac{1}{p}}.$$

By (23) it follows that $J < \infty$. If $J = 0$, then (18) is trivially valid; if $J > 0$, then we have

$$\begin{aligned} 0 &< \sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} B_{\eta,\beta}(n)}{(B_{\eta,\beta}(n))^{1-q}} b_n^q = J^p = I \\ &< K(\lambda_1) \left[\sum_{|m|=2}^{\infty} \frac{\ln^{p(1-\lambda_1)-1} A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}} a_m^p \right]^{\frac{1}{p}} \left[\sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} B_{\eta,\beta}(n)}{(B_{\eta,\beta}(n))^{1-q}} b_n^q \right]^{\frac{1}{q}}, \\ J &= \left[\sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} B_{\eta,\beta}(n)}{(B_{\eta,\beta}(n))^{1-q}} b_n^q \right]^{\frac{1}{p}} < K(\lambda_1) \left[\sum_{|m|=2}^{\infty} \frac{\ln^{p(1-\lambda_1)-1} A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}} a_m^p \right]^{\frac{1}{p}}. \end{aligned}$$

Hence (18) is valid, which is equivalent to (17). □

Theorem 2 *With regards to the assumptions of Theorem* 1, *the constant factor* $K(\lambda_1)$ *in* (17) *and* (18) *is the best possible.*

Proof For $0 < \varepsilon < q\lambda_2$, we set $\tilde{\lambda}_1 = \lambda_1 + \frac{\varepsilon}{q}$ (> 0), $\tilde{\lambda}_2 = \lambda_2 - \frac{\varepsilon}{q}$ ($\in (0,1)$), and

$$\begin{aligned} \tilde{a}_m &:= \frac{\ln^{\lambda_1 - \frac{\varepsilon}{p} - 1} A_{\xi,\alpha}(m)}{A_{\xi,\alpha}(m)} = \frac{\ln^{\tilde{\lambda}_1 - \varepsilon - 1} A_{\xi,\alpha}(m)}{A_{\xi,\alpha}(m)} \quad \big(|m| \in \mathbb{N} \setminus \{1\}\big), \\ \tilde{b}_n &:= \frac{\ln^{\lambda_2 - \frac{\varepsilon}{q} - 1} B_{\eta,\beta}(n)}{B_{\eta,\beta}(n)} = \frac{\ln^{\tilde{\lambda}_2 - 1} B_{\eta,\beta}(n)}{B_{\eta,\beta}(n)} \quad \big(|n| \in \mathbb{N} \setminus \{1\}\big). \end{aligned}$$

By (15) and (10) we find

$$\begin{aligned} \tilde{I}_1 &:= \left[\sum_{|m|=2}^{\infty} \frac{\ln^{p(1-\lambda_1)-1} A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}} \tilde{a}_m^p \right]^{\frac{1}{p}} \left[\sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} B_{\eta,\beta}(n)}{(B_{\eta,\beta}(n))^{1-q}} \tilde{b}_n^q \right]^{\frac{1}{q}} \\ &= \left[\sum_{|m|=2}^{\infty} \frac{\ln^{-1-\varepsilon} A_{\xi,\alpha}(m)}{A_{\xi,\alpha}(m)} \right]^{\frac{1}{p}} \left[\sum_{|n|=2}^{\infty} \frac{\ln^{-1-\varepsilon} B_{\eta,\beta}(n)}{B_{\eta,\beta}(n)} \right]^{\frac{1}{q}} \\ &= \frac{1}{\varepsilon} \big(2\csc^2\alpha + o(1)\big)^{\frac{1}{p}} \big(2\csc^2\beta + \tilde{o}(1)\big)^{\frac{1}{q}} \quad \big(\varepsilon \to 0^+\big), \\ \tilde{I} &:= \sum_{|n|=2}^{\infty} \sum_{|m|=2}^{\infty} k(m,n) \tilde{a}_m \tilde{b}_n \\ &= \sum_{|m|=2}^{\infty} \sum_{|n|=2}^{\infty} k(m,n) \frac{\ln^{\tilde{\lambda}_1 - \varepsilon - 1} A_{\xi,\alpha}(m)}{A_{\xi,\alpha}(m)} \frac{\ln^{\tilde{\lambda}_2 - 1} B_{\eta,\beta}(n)}{B_{\eta,\beta}(n)} \\ &= \sum_{|m|=2}^{\infty} \omega(\tilde{\lambda}_2, m) \frac{\ln^{-1-\varepsilon} A_{\xi,\alpha}(m)}{A_{\xi,\alpha}(m)} \geq k_\beta(\tilde{\lambda}_1) \sum_{|m|=2}^{\infty} \big(1 - \theta(\tilde{\lambda}_2, m)\big) \frac{\ln^{-1-\varepsilon} A_{\xi,\alpha}(m)}{A_{\xi,\alpha}(m)} \\ &= k_\beta(\tilde{\lambda}_1) \left[\sum_{|m|=2}^{\infty} \frac{\ln^{-1-\varepsilon} A_{\xi,\alpha}(m)}{A_{\xi,\alpha}(m)} - \sum_{|m|=2}^{\infty} \frac{O(\ln^{-1-(\frac{\varepsilon}{p}+\lambda_2)} A_{\xi,\alpha}(m))}{A_{\xi,\alpha}(m)} \right] \\ &= \frac{1}{\varepsilon} k_\beta(\tilde{\lambda}_1) \big(2\csc^2\alpha + o(1) - \varepsilon O(1)\big). \end{aligned}$$

If there exists a positive number $k \le K(\lambda_1)$, such that (17) is still valid when replacing $K(\lambda_1)$ by k, then in particular we have

$$\varepsilon\tilde{I} = \varepsilon \sum_{|n|=2}^{\infty}\sum_{|m|=2}^{\infty} k(m,n)\tilde{a}_m\tilde{b}_n < \varepsilon k\tilde{I}_1.$$

We obtain from the above results that

$$k_\beta\left(\lambda_1 + \frac{\varepsilon}{q}\right)\big(2\csc^2\alpha + o(1) - \varepsilon O(1)\big) < k\big(2\csc^2\alpha + o(1)\big)^{\frac{1}{p}}\big(2\csc^2\beta + \tilde{o}(1)\big)^{\frac{1}{q}},$$

and then

$$4B(\lambda_1,\lambda_2)\csc^2\beta\csc^2\alpha \le 2k\csc^{\frac{2}{p}}\alpha\csc^{\frac{2}{q}}\beta \quad (\varepsilon \to 0^+),$$

namely, $K(\lambda_1) = 2B(\lambda_1,\lambda_2)\csc^{\frac{2}{p}}\beta\csc^{\frac{2}{q}}\alpha \le k$. Hence, $k = K(\lambda_1)$ is the best value of (17).

The constant factor $K(\lambda_1)$ in (18) is still the best possible. Otherwise, we would reach a contradiction by (24) that the constant factor in (17) is not the best possible. □

Theorem 3 *Suppose that* $0 < p < 1$, $\lambda_1, \lambda_2 \le 1$, $a_m, b_n \ge 0$ $(|m|, |n| \in \mathbb{N} \setminus \{1\})$, *and*

$$0 < \sum_{|m|=2}^{\infty} \frac{\ln^{p(1-\lambda_1)-1} A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}} a_m^p < \infty, \qquad 0 < \sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} B_{\eta,\beta}(n)}{(B_{\eta,\beta}(n))^{1-q}} b_n^q < \infty.$$

We have the following equivalent inequalities:

$$\begin{aligned} I &= \sum_{|n|=2}^{\infty}\sum_{|m|=2}^{\infty} k(m,n)a_m b_n \\ &> K(\lambda_1)\left[\sum_{|m|=2}^{\infty}\big(1-\theta(\lambda_2,m)\big)\frac{\ln^{p(1-\lambda_1)-1} A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}} a_m^p\right]^{\frac{1}{p}} \\ &\quad \times \left[\sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} B_{\eta,\beta}(n)}{(B_{\eta,\beta}(n))^{1-q}} b_n^q\right]^{\frac{1}{q}}, \end{aligned} \tag{25}$$

$$\begin{aligned} J &= \left[\sum_{|n|=2}^{\infty} \frac{\ln^{p\lambda_2-1} B_{\eta,\beta}(n)}{B_{\eta,\beta}(n)}\left(\sum_{|m|=2}^{\infty} k(m,n)a_m\right)^p\right]^{\frac{1}{p}} \\ &> K(\lambda_1)\left[\sum_{|m|=2}^{\infty}\big(1-\theta(\lambda_2,m)\big)\frac{\ln^{p(1-\lambda_1)-1} A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}} a_m^p\right]^{\frac{1}{p}}, \end{aligned} \tag{26}$$

$$\begin{aligned} L &:= \left[\sum_{|m|=2}^{\infty} \frac{\ln^{q\lambda_1-1} A_{\xi,\alpha}(m)}{(1-\theta(\lambda_2,m))^{q-1} A_{\xi,\alpha}(m)}\left(\sum_{|n|=2}^{\infty} k(m,n)b_n\right)^q\right]^{\frac{1}{q}} \\ &> K(\lambda_1)\left[\sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} B_{\eta,\beta}(n)}{(B_{\eta,\beta}(n))^{1-q}} b_n^q\right]^{\frac{1}{q}}, \end{aligned} \tag{27}$$

where the constant factor $K(\lambda_1)$ *in* (25), (26), *and* (27) *is the best possible*.

Proof By the reverse Hölder inequality with weight (*cf.* [22]), and (9), we find

$$\begin{aligned}\left(\sum_{|m|=2}^{\infty}k(m,n)a_m\right)^p &= \left\{\sum_{|m|=2}^{\infty}k(m,n)\left[\frac{(A_{\xi,\alpha}(m))^{\frac{1}{q}}\ln^{\frac{1-\lambda_1}{q}}A_{\xi,\alpha}(m)}{\ln^{\frac{1-\lambda_2}{p}}B_{\eta,\beta}(n)}a_m\right]\right.\\ &\quad\left.\times\left[\frac{\ln^{\frac{1-\lambda_2}{p}}B_{\eta,\beta}(n)}{(A_{\xi,\alpha}(m))^{\frac{1}{q}}\ln^{\frac{1-\lambda_1}{q}}A_{\xi,\alpha}(m)}\right]\right\}^p\\ &\geq \sum_{|m|=2}^{\infty}k(m,n)\frac{(A_{\xi,\alpha}(m))^{\frac{p}{q}}\ln^{\frac{(1-\lambda_1)p}{q}}A_{\xi,\alpha}(m)}{\ln^{1-\lambda_2}B_{\eta,\beta}(n)}a_m^p\\ &\quad\times\left[\sum_{|m|=2}^{\infty}k(m,n)\frac{\ln^{\frac{(1-\lambda_2)q}{p}}B_{\eta,\beta}(n)}{A_{\xi,\alpha}(m)\ln^{1-\lambda_1}A_{\xi,\alpha}(m)}\right]^{p-1}\\ &= \frac{(\varpi(\lambda_1,n))^{p-1}B_{\eta,\beta}(n)}{\ln^{p\lambda_2-1}B_{\eta,\beta}(n)}\sum_{|m|=2}^{\infty}k(m,n)\frac{(A_{\xi,\alpha}(m))^{\frac{p}{q}}\ln^{\frac{(1-\lambda_1)p}{q}}A_{\xi,\alpha}(m)}{B_{\eta,\beta}(n)\ln^{1-\lambda_2}B_{\eta,\beta}(n)}a_m^p.\end{aligned}$$

Since $p-1<0$, by (13) it follows that

$$\begin{aligned}J &> k_\alpha^{1/q}(\lambda_1)\left[\sum_{|n|=2}^{\infty}\sum_{|m|=2}^{\infty}k(m,n)\frac{(A_{\xi,\alpha}(m))^{\frac{p}{q}}\ln^{\frac{(1-\lambda_1)p}{q}}A_{\xi,\alpha}(m)}{B_{\eta,\beta}(n)\ln^{1-\lambda_2}B_{\eta,\beta}(n)}a_m^p\right]^{\frac{1}{p}}\\ &= k_\alpha^{1/q}(\lambda_1)\left[\sum_{|m|=2}^{\infty}\omega(\lambda_2,m)\frac{n^{p(1-\lambda_1)-1}A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}}a_m^p\right]^{\frac{1}{p}}.\end{aligned}\tag{28}$$

By (10) and (16) we have (26).

Using the reverse Hölder's inequality again, we have

$$\begin{aligned}I &= \sum_{|n|=2}^{\infty}\left[\frac{\ln^{\lambda_2-\frac{1}{p}}B_{\eta,\beta}(n)}{(B_{\eta,\beta}(n))^{\frac{1}{p}}}\sum_{|m|=2}^{\infty}k(m,n)a_m\right]\left[\frac{\ln^{\frac{1}{p}-\lambda_2}B_{\eta,\beta}(n)}{(B_{\eta,\beta}(n))^{\frac{-1}{p}}}b_n\right]\\ &\geq J\left[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}B_{\eta,\beta}(n)}{(B_{\eta,\beta}(n))^{1-q}}b_n^q\right]^{\frac{1}{q}},\end{aligned}\tag{29}$$

and then by using (26) we have (25).

On the other hand, assuming that (25) is valid, we set

$$b_n := \frac{\ln^{p\lambda_2-1}B_{\eta,\beta}(n)}{B_{\eta,\beta}(n)}\left(\sum_{|m|=2}^{\infty}k(m,n)a_m\right)^{p-1},\quad |n|\in\mathbb{N}\setminus\{1\},$$

and find

$$J = \left[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}B_{\eta,\beta}(n)}{(B_{\eta,\beta}(n))^{1-q}}b_n^q\right]^{\frac{1}{p}}.$$

By (28) it follows that $J > 0$. If $J = \infty$, then (26) is trivially valid; if $0 < J < \infty$, then we have

$$\begin{aligned}
&\sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} B_{\eta,\beta}(n)}{(B_{\eta,\beta}(n))^{1-q}} b_n^q \\
&= J^p = I \\
&> K(\lambda_1)\left[\sum_{|m|=2}^{\infty}\big(1-\theta(\lambda_2,m)\big)\frac{\ln^{p(1-\lambda_1)-1} A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}} a_m^p\right]^{\frac{1}{p}}\left[\sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} B_{\eta,\beta}(n)}{(B_{\eta,\beta}(n))^{1-q}} b_n^q\right]^{\frac{1}{q}},
\end{aligned}$$

$$J = \left[\sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} B_{\eta,\beta}(n)}{(B_{\eta,\beta}(n))^{1-q}} b_n^q\right]^{\frac{1}{p}} > K(\lambda_1)\left[\sum_{|m|=2}^{\infty}\big(1-\theta(\lambda_2,m)\big)\frac{\ln^{p(1-\lambda_1)-1} A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}} a_m^p\right]^{\frac{1}{p}}.$$

Hence (26) is valid, which is equivalent to (25).

By the reverse Hölder inequality with weight (*cf.* [22]), and (9) we find

$$\begin{aligned}
\left(\sum_{|n|=2}^{\infty} k(m,n) b_n\right)^q &\le \left[\sum_{|n|=2}^{\infty} k(m,n)\frac{\ln^{(1-\lambda_1)p/q} A_{\xi,\alpha}(m)}{B_{\eta,\beta}(n)\ln^{1-\lambda_2} B_{\eta,\beta}(n)}\right]^{q-1} \\
&\quad\times \sum_{|n|=2}^{\infty} k(m,n)\frac{(B_{\eta,\beta}(n))^{q/p}\ln^{(1-\lambda_2)q/p} B_{\eta,\beta}(n)}{\ln^{1-\lambda_1} A_{\xi,\alpha}(m)} b_n^q \\
&= \frac{(\omega(\lambda_2,m))^{q-1} A_{\xi,\alpha}(m)}{\ln^{q\lambda_1-1} A_{\xi,\alpha}(m)} \sum_{|n|=2}^{\infty} k(m,n)\frac{(B_{\eta,\beta}(n))^{\frac{q}{p}}\ln^{\frac{(1-\lambda_2)q}{p}} B_{\eta,\beta}(n)}{A_{\xi,\alpha}(m)\ln^{1-\lambda_1} A_{\xi,\alpha}(m)} b_n^q.
\end{aligned}$$

Since $q < 0$, by (10) it follows that

$$\begin{aligned}
L &> k_\beta^{1/p}(\lambda_1)\left[\sum_{|n|=2}^{\infty}\sum_{|m|=2}^{\infty} k(m,n)\frac{(B_{\eta,\beta}(n))^{\frac{q}{p}}\ln^{\frac{(1-\lambda_2)q}{p}} B_{\eta,\beta}(n)}{A_{\xi,\alpha}(m)\ln^{1-\lambda_1} A_{\xi,\alpha}(m)} b_n^q\right]^{\frac{1}{p}} \\
&= k_\beta^{1/p}(\lambda_1)\left[\sum_{|n|=2}^{\infty} \varpi(\lambda_1,n)\frac{\ln^{q(1-\lambda_2)-1} B_{\eta,\beta}(n)}{(B_{\eta,\beta}(n))^{1-q}} b_n^q\right]^{\frac{1}{p}}.
\end{aligned}$$

By (13) and (16) we have (27).

In the same way, we find

$$\begin{aligned}
I &= \sum_{|m|=2}^{\infty}\left[\big(1-\theta(\lambda_2,m)\big)^{\frac{1}{p}}\frac{\ln^{\frac{1}{q}-\lambda_1} A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{-1/q}} a_m\right] \\
&\quad\times\left[\frac{\ln^{\frac{-1}{q}+\lambda_1} A_{\xi,\alpha}(m)}{(1-\theta(\lambda_2,m))^{\frac{1}{p}}(A_{\xi,\alpha}(m))^{1/q}}\sum_{|n|=2}^{\infty} k(m,n) b_n\right] \\
&\ge \left[\sum_{|m|=2}^{\infty}\big(1-\theta(\lambda_2,m)\big)\frac{\ln^{p(1-\lambda_1)-1} A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}} a_m^p\right]^{\frac{1}{p}} L, \qquad (30)
\end{aligned}$$

and then we can prove that (27) and (25) are equivalent.

For $0 < \varepsilon < \min\{p\lambda_1, |q|\lambda_1\}$, we set $\tilde{\lambda}_1 = \lambda_1 - \frac{\varepsilon}{p}$ $(\in (0,1))$, $\tilde{\lambda}_2 = \lambda_2 + \frac{\varepsilon}{p}$ (> 0), and

$$\tilde{a}_m := \frac{\ln^{\lambda_1 - \frac{\varepsilon}{p} - 1} A_{\xi,\alpha}(m)}{A_{\xi,\alpha}(m)} = \frac{\ln^{\tilde{\lambda}_1 - 1} A_{\xi,\alpha}(m)}{A_{\xi,\alpha}(m)} \quad \bigl(|m| \in \mathbb{N} \setminus \{1\}\bigr),$$

$$\tilde{b}_n := \frac{\ln^{\lambda_2 - \frac{\varepsilon}{q} - 1} B_{\eta,\beta}(n)}{B_{\eta,\beta}(n)} = \frac{\ln^{\tilde{\lambda}_2 - \varepsilon - 1} B_{\eta,\beta}(n)}{B_{\eta,\beta}(n)} \quad \bigl(|n| \in \mathbb{N} \setminus \{1\}\bigr).$$

By (15) and (13) we find

$$\begin{aligned}
\tilde{I}_2 &:= \left[\sum_{|m|=2}^{\infty} \bigl(1 - \theta(\lambda_2, m)\bigr) \frac{\ln^{p(1-\lambda_1)-1} A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}} \tilde{a}_m^p\right]^{\frac{1}{p}} \left[\sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} B_{\eta,\beta}(n)}{(B_{\eta,\beta}(n))^{1-q}} \tilde{b}_n^q\right]^{\frac{1}{q}} \\
&= \left[\sum_{|m|=2}^{\infty} \frac{\ln^{-1-\varepsilon} A_{\xi,\alpha}(m)}{A_{\xi,\alpha}(m)} - \sum_{|m|=2}^{\infty} \frac{O(\ln^{-1-(\lambda_2+\varepsilon)} A_{\xi,\alpha}(m))}{A_{\xi,\alpha}(m)}\right]^{\frac{1}{p}} \left[\sum_{|n|=2}^{\infty} \frac{\ln^{-1-\varepsilon} B_{\eta,\beta}(n)}{B_{\eta,\beta}(n)}\right]^{\frac{1}{q}} \\
&= \frac{1}{\varepsilon}\bigl(2\csc^2\alpha + o(1) - \varepsilon O(1)\bigr)^{\frac{1}{p}} \bigl(2\csc^2\beta + \tilde{o}(1)\bigr)^{\frac{1}{q}} \quad \bigl(\varepsilon \to 0^+\bigr), \\
\tilde{I} &= \sum_{|n|=2}^{\infty} \sum_{|m|=2}^{\infty} k(m,n)\tilde{a}_m \tilde{b}_n = \sum_{|m|=2}^{\infty} \sum_{|n|=2}^{\infty} k(m,n) \frac{\ln^{\tilde{\lambda}_1 - 1} A_{\xi,\alpha}(m)}{A_{\xi,\alpha}(m)} \frac{\ln^{\tilde{\lambda}_2 - \varepsilon - 1} B_{\eta,\beta}(n)}{B_{\eta,\beta}(n)} \\
&= \sum_{|n|=2}^{\infty} \varpi(\tilde{\lambda}_1, n) \frac{\ln^{-1-\varepsilon} B_{\eta,\beta}(n)}{B_{\eta,\beta}(n)} \leq k_\alpha(\tilde{\lambda}_1) \sum_{|n|=2}^{\infty} \frac{\ln^{-1-\varepsilon} B_{\eta,\beta}(n)}{B_{\eta,\beta}(n)} \\
&= \frac{1}{\varepsilon} k_\alpha(\tilde{\lambda}_1)\bigl(2\csc^2\beta + o(1)\bigr).
\end{aligned}$$

If there exists a positive number $k \geq K(\lambda_1)$, such that (25) is still valid when replacing $K(\lambda_1)$ by k, then in particular we have

$$\varepsilon \tilde{I} = \varepsilon \sum_{|m|=2}^{\infty} \sum_{|n|=2}^{\infty} k(m,n)\tilde{a}_m \tilde{b}_n > \varepsilon k \tilde{I}_2.$$

We obtain from the above results that

$$k_\beta\left(\lambda_1 + \frac{\varepsilon}{q}\right)\bigl(2\csc^2\alpha + o(1)\bigr) > k\bigl(2\csc^2\alpha + o(1) - \varepsilon O(1)\bigr)^{\frac{1}{p}} \bigl(2\csc^2\beta + \tilde{o}(1)\bigr)^{\frac{1}{q}},$$

and then

$$4B(\lambda_1, \lambda_2)\csc^2\beta \csc^2\alpha \geq 2k \csc^{\frac{2}{p}}\alpha \csc^{\frac{2}{q}}\beta \quad \bigl(\varepsilon \to 0^+\bigr),$$

namely, $K(\lambda_1) = 2B(\lambda_1, \lambda_2)\csc^{\frac{2}{p}}\beta \csc^{\frac{2}{q}}\alpha \geq k$. Hence, $k = K(\lambda_1)$ is the best value of (25).

The constant factor $K(\lambda_1)$ in (26) is still the best possible. Otherwise, we would reach a contradiction by (30) that the constant factor in (25) is not the best possible.

In the same way, by (30) we can proved that the constant factor $K(\lambda_1)$ in (27) is still the best possible. □

4 Operator expressions

Setting $\varphi(m) := \frac{\ln^{p(1-\lambda_1)-1} A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}}$ ($|m| \in \mathbb{N} \setminus \{1\}$), and $\psi(n) := \frac{\ln^{q(1-\lambda_2)-1} B_{\eta,\beta}(n)}{(B_{\eta,\beta}(n))^{1-q}}$, wherefrom $\psi^{1-p}(n) = \frac{\ln^{p\lambda_2-1} B_{\eta,\beta}(n)}{B_{\eta,\beta}(n)}$ ($|n| \in \mathbb{N} \setminus \{1\}$), we define the real weighted normed function spaces as follows:

$$l_{p,\varphi} := \left\{ a = \{a_m\}_{|m|=2}^{\infty}; \|a\|_{p,\varphi} = \left(\sum_{|m|=2}^{\infty} \varphi(m)|a_m|^p \right)^{\frac{1}{p}} < \infty \right\},$$

$$l_{q,\psi} := \left\{ b = \{b_n\}_{|n|=2}^{\infty}; \|b\|_{q,\psi} = \left(\sum_{|n|=2}^{\infty} \psi(n)|b_n|^q \right)^{\frac{1}{q}} < \infty \right\},$$

$$l_{p,\psi^{1-p}} := \left\{ c = \{c_n\}_{|n|=2}^{\infty}; \|c\|_{p,\psi^{1-p}} = \left(\sum_{|n|=2}^{\infty} \psi^{1-p}(n)|c_n|^p \right)^{\frac{1}{p}} < \infty \right\}.$$

For $a = \{a_m\}_{|m|=2}^{\infty} \in l_{p,\varphi}$, putting $c_n = \sum_{|m|=2}^{\infty} k(m,n)a_m$ and $c = \{c_n\}_{|n|=2}^{\infty}$, it follows by (18) that $\|c\|_{p,\psi^{1-p}} < K(\lambda_1)\|a\|_{p,\varphi}$, namely $c \in l_{p,\psi^{1-p}}$.

Definition 2 Define the Mulholland-type operator $T : l_{p,\varphi} \to l_{p,\psi^{1-p}}$ as follows: For $a_m \geq 0$, $a = \{a_m\}_{|m|=2}^{\infty} \in l_{p,\varphi}$, there exists a unique representation $Ta = c \in l_{p,\psi^{1-p}}$. We also define the following formal inner product of Ta and $b = \{b_n\}_{|n|=2}^{\infty} \in l_{q,\psi}$ ($b_n \geq 0$):

$$(Ta, b) := \sum_{|n|=2}^{\infty} \sum_{|m|=2}^{\infty} k(m,n)a_m b_n. \tag{31}$$

Hence, we may rewrite (17) and (18) in the following operator expressions:

$$(Ta, b) < K(\lambda_1)\|a\|_{p,\varphi}\|b\|_{q,\psi}, \tag{32}$$

$$\|Ta\|_{p,\psi^{1-p}} < K(\lambda_1)\|a\|_{p,\varphi}. \tag{33}$$

It follows that the operator T is bounded by

$$\|T\| := \sup_{a(\neq\theta)\in l_{p,\varphi}} \frac{\|Ta\|_{p,\psi^{1-p}}}{\|a\|_{p,\varphi}} \leq K(\lambda_1). \tag{34}$$

Since the constant factor $K(\lambda_1)$ in (18) is the best possible, we have

$$\|T\| = K(\lambda_1) = 2B(\lambda_1, \lambda_2) \csc^{2/p} \beta \csc^{2/q} \alpha. \tag{35}$$

Remark 1

(i) For $\xi = \eta = 0$ in (19), we have the following new inequality:

$$\sum_{|n|=2}^{\infty} \sum_{|m|=2}^{\infty} \frac{a_m b_n}{\ln^{\lambda} |mn|} < 2B(\lambda_1, \lambda_2) \left[\sum_{|m|=2}^{\infty} \frac{\ln^{p(1-\lambda_1)-1} |m|}{|m|^{1-p}} a_m^p \right]^{\frac{1}{p}} \left[\sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} |n|}{|n|^{1-q}} b_n^q \right]^{\frac{1}{q}}. \tag{36}$$

It follows that (19) is a more accurate inequalitythan (36); so is (17).

(ii) If $a_{-m} = a_m$, $b_{-n} = b_n$ $(m, n \in \mathbb{N} \setminus \{1\})$, $\xi, \eta \in [0, \frac{1}{2}]$, then (19) reduces to the following inequality:

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\left[\frac{1}{\ln^{\lambda}[(m-\xi)(n-\eta)]}+\frac{1}{\ln^{\lambda}[(m-\xi)(n+\eta)]}+\frac{1}{\ln^{\lambda}[(m+\xi)(n-\eta)]}+\frac{1}{\ln^{\lambda}[(m+\xi)(n+\eta)]}\right]a_m b_n$$
$$< 2B(\lambda_1,\lambda_2)\left\{\sum_{m=2}^{\infty}\left[\frac{\ln^{p(1-\lambda_1)-1}(m-\xi)}{(m-\xi)^{1-p}}+\frac{\ln^{p(1-\lambda_1)-1}(m+\xi)}{(m+\xi)^{1-p}}\right]a_m^p\right\}^{\frac{1}{p}}$$
$$\times\left\{\sum_{n=2}^{\infty}\left[\frac{\ln^{q(1-\lambda_2)-1}(n-\eta)}{(n-\eta)^{1-q}}+\frac{\ln^{q(1-\lambda_2)-1}(n+\eta)}{(n+\eta)^{1-q}}\right]b_n^q\right\}^{\frac{1}{q}}. \tag{37}$$

(iii) If $\lambda = 1$, $\lambda_1 = \frac{1}{q}$, $\lambda_2 = \frac{1}{p}$, $\xi = \eta \in [0, \frac{1}{2}]$, then (37) reduces to

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\left[\frac{1}{\ln(m-\xi)(n-\xi)}+\frac{1}{\ln(m-\xi)(n+\xi)}+\frac{1}{\ln(m+\xi)(n-\xi)}+\frac{1}{\ln(m+\xi)(n+\xi)}\right]a_m b_n$$
$$< \frac{2\pi}{\sin(\frac{\pi}{p})}\left\{\sum_{m=2}^{\infty}\left[\frac{1}{(m-\xi)^{1-p}}+\frac{1}{(m+\xi)^{1-p}}\right]a_m^p\right\}^{\frac{1}{p}}$$
$$\times\left\{\sum_{n=2}^{\infty}\left[\frac{1}{(n-\xi)^{1-q}}+\frac{1}{(n+\xi)^{1-q}}\right]b_n^q\right\}^{\frac{1}{q}}. \tag{38}$$

For $\xi = 0$, (38) reduces to (3). Hence, (17) is a more accurate extension of (3).

5 Conclusions

In this paper, by introducing independent parameters and applying the weight coefficients and Hermite-Hadamard's inequality we give a more accurate Mulholland-type inequality in the whole plane with a best possible constant factor in Theorems 1-2. Furthermore, the equivalent forms, the reverses in Theorem 3, a few particular cases, and the operator expressions are considered. The method of real analysis is very important, which is the key to prove the equivalent inequalities with the best possible constant factor. The lemmas and theorems provide an extensive account of this type inequalities.

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Competing interests
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Authors' contributions
BY carried out the mathematical studies, participated in the sequence alignment, and drafted the manuscript. YZ and QC participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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