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On p-radial Blaschke and harmonic Blaschke additions

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Abstract

In the paper, we first improve the radial Blaschke and harmonic Blaschke additions and introduce the p-radial Blaschke and p-harmonic Blaschke additions. Following this, Dresher type inequalities for the radial Blaschke-Minkowski homomorphisms with respect to p-radial Blaschke and p-harmonic Blaschke additions are established.

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1 Notation and preliminaries

The setting for this paper is an n-dimensional Euclidean space $\mathbb{R}^n$. We reserve the letter u for unit vectors, and the letter B is reserved for the unit ball centered at the origin. The surface of B is S^{n-1}. The volume of the unit n-ball is denoted by ω_n. We use $V(K)$ for the n-dimensional volume of a body K. Associated with a compact subset K of $\mathbb{R}^n$, which is star-shaped with respect to the origin, is its radial function $\rho(K,\cdot): S^{n-1} \to \mathbb{R}$ defined for $u \in S^{n-1}$ by

$$\rho(K,u) = \max\{\lambda \geq 0 : \lambda u \in K\}.$$

If $\rho(K,\cdot)$ is positive and continuous, K will be called a star body. Let $\mathcal{S}^n$ denote the set of star bodies in $\mathbb{R}^n$. Let $\tilde{\delta}$ denote the radial Hausdorff metric, i.e., if $K, L \in \mathcal{S}^n$, then $\tilde{\delta}(K,L) = |\rho(K,u) - \rho(L,u)|_\infty$, where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions $C(S^{n-1})$.

1.1 Dual mixed volumes

The radial Minkowski linear combination, $\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r$ is defined by

$$\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r = \{\lambda_1 x_1 \tilde{+} \cdots \tilde{+} \lambda_r x_r : x_i \in K_i, i = 1, \ldots, r\} \tag{1.1}$$

for $K_1, \ldots, K_r \in \mathcal{S}^n$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$. It has the following important property (see [1]):

$$\rho(\lambda K \tilde{+} \mu L, \cdot) = \lambda\rho(K,\cdot) + \mu\rho(L,\cdot) \tag{1.2}$$

for $K, L \in \mathcal{S}^n$ and $\lambda, \mu \geq 0$. For $K_1, \ldots, K_r \in \mathcal{S}^n$ and $\lambda_1, \ldots, \lambda_r \geq 0$, the volume of the radial Minkowski linear combination $\lambda_1 K_1 \widetilde{+} \cdots \widetilde{+} \lambda_r K_r$ is a homogeneous polynomial of degree n in the λ_i,

$$V(\lambda_1 K_1 \widetilde{+} \cdots \widetilde{+} \lambda_r K_r) = \sum_{i_1,\ldots,i_n=1}^{r} \widetilde{V}(K_{i_1}, \ldots, K_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n}. \tag{1.3}$$

If we require the coefficients of the polynomial in (1.3) to be symmetric in their arguments, then they are uniquely determined. The coefficient $\widetilde{V}(K_{i_1}, \ldots, K_{i_n})$ is nonnegative and depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$. It is called the dual mixed volume of $K_{i_1}, \ldots, K_{i_n}$.

If $K_1, \ldots, K_n \in \mathcal{S}^n$, then the dual mixed volume $\widetilde{V}(K_1, \ldots, K_n)$ can be represented in the form (see [2])

$$\widetilde{V}(K_1, \ldots, K_n) = \frac{1}{n} \int_{S^{n-1}} \rho(K_1, u) \cdots \rho(K_n, u)\, dS(u). \tag{1.4}$$

If $K_1 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = L$, then the dual mixed volume is written as $\widetilde{V}_i(K, L)$. If $L = B$, then the dual mixed volume $\widetilde{V}_i(K, L) = \widetilde{V}_i(K, B)$ is written as $\widetilde{W}_i(K)$. For $K, L \in \mathcal{S}^n$, the ith dual mixed volume of K and L, $\widetilde{V}_i(K, L)$ can be extended to all $i \in \mathbb{R}$ by

$$\widetilde{V}_i(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-i} \rho(L, u)^i \, dS(u), \tag{1.5}$$

where $i \in \mathbb{R}$. Thus, if $K \in \mathcal{S}^n$, then

$$\widetilde{W}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-i} \, dS(u). \tag{1.6}$$

1.2 Mixed intersection bodies

For $K \in \mathcal{S}^n$, there is a unique star body $\mathbf{I}K$ whose radial function satisfies, for $u \in S^{n-1}$,

$$\rho(\mathbf{I}K, u) = v(K \cap E_u),$$

where v is $(n-1)$-dimensional dual volume. It is called the intersection body of K. The volume of the intersection body of K is given by (see [1])

$$V(\mathbf{I}K) = \frac{1}{n} \int_{S^{n-1}} v(K \cap E_u)^n \, dS(u).$$

The mixed intersection body of $K_1, \ldots, K_{n-1} \in \mathcal{S}^n$, denoted by $\mathbf{I}(K_1, \ldots, K_{n-1})$, is defined by

$$\rho\big(\mathbf{I}(K_1, \ldots, K_{n-1}), u\big) = \tilde{v}(K_1 \cap E_u, \ldots, K_{n-1} \cap E_u),$$

where $\tilde{v}$ is the $(n-1)$-dimensional dual mixed volume. If $K_1 = \cdots = K_{n-i-1} = K, K_{n-i} = \cdots = K_{n-1} = L$, then $\mathbf{I}(K_1, \ldots, K_{n-1})$ is written as $\mathbf{I}_i(K, L)$. If $L = B$, then $\mathbf{I}_i(K, L)$ is written as $\mathbf{I}_i K$ and called the ith intersection body of K. For $\mathbf{I}_0 K$, we simply write $\mathbf{I}K$.

2 Improvement of the radial Blaschke addition

Let us recall the concept of radial Blaschke addition defined by Lutwak [1]. Suppose that K and L are star bodies in $\mathbb{R}^n$, the radial Blaschke addition denoted by $K\widehat{+}L$ is a star body whose radial function is

$$\rho(K\widehat{+}L,\cdot)^{n-1} = \rho(K,\cdot)^{n-1} + \rho(L,\cdot)^{n-1}. \tag{2.1}$$

The dual Knesser-Süss inequality for the radial Blaschke addition was established by Lutwak [1]. If $K, L \in \mathcal{S}^n$, then

$$V(K\widehat{+}L)^{(n-1)/n} \leq V(K)^{(n-1)/n} + V(L)^{(n-1)/n}, \tag{2.2}$$

with equality if and only if K and L are dilates.

In the section, we give a generalized concept of the radial Blaschke addition.

Definition 2.1 If $K, L \in \mathcal{S}^n$, $0 \leq p < n-1$ and $\lambda, \mu > 0$ (not both zero), the p-radial Blaschke linear combination of K and L denoted by $\lambda \diamond K\widehat{+}_p\mu \diamond L$ is a star body whose radial function is defined by

$$\rho(\lambda \diamond K\widehat{+}_p\mu \diamond L,\cdot)^{n-p-1} = \lambda\rho(K,\cdot)^{n-p-1} + \mu\rho(L,\cdot)^{n-p-1}. \tag{2.3}$$

From (2.3), it is easy to see that

$$\lambda \diamond K = \lambda^{1/(n-p-1)}K.$$

When $\lambda = \mu = 1$, the p-radial Blaschke combination becomes the p-radial Blaschke addition $K\widehat{+}_pL$ and

$$\rho(K\widehat{+}_pL,\cdot)^{n-p-1} = \rho(K,\cdot)^{n-p-1} + \rho(L,\cdot)^{n-p-1}. \tag{2.4}$$

Obviously, when $p = 0$, (2.4) becomes (2.1).

In the following, we define the dual mixed quermassintegral with respect to the p-radial Blaschke addition. First, we show two propositions. The following proposition follows immediately from (2.3) with L'Hôpital's rule.

Proposition 2.2 *Let $0 \leq p < n-1$, $0 \leq i < n$ and $\varepsilon > 0$. If $K, L \in \mathcal{S}^n$, then*

$$\lim_{\varepsilon\to 0^+} \frac{\rho(K\widehat{+}_p\varepsilon \diamond L, u)^{n-i} - \rho(K,u)^{n-i}}{\varepsilon} = \frac{n-i}{n-p-1}\rho(K,u)^{p-i+1}\rho(L,u)^{n-p-1}. \tag{2.5}$$

The following proposition follows immediately from Proposition 2.2 and (1.6).

Proposition 2.3 *Let $0 \leq p < n-1$, $0 \leq i < n$ and $\varepsilon > 0$. If $K, L \in \mathcal{S}^n$, then*

$$\begin{aligned}&\frac{n-p-1}{n-i}\lim_{\varepsilon\to 0^+}\frac{\widetilde{W}_i(K\widehat{+}_p\varepsilon \diamond L, u) - \widetilde{W}_i(K)}{\varepsilon}\\ &\quad= \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{p-i+1}\rho(L,u)^{n-p-1}\,dS(u).\end{aligned} \tag{2.6}$$

Definition 2.4 For $0 \le p < n-1$, $0 \le i < n$ and $K, L \in \mathcal{S}^n$, the p-dual mixed quermassintegral of star bodies K and L, denoted by $\widetilde{W}_{p,i}(K,L)$, is defined by

$$\widetilde{W}_{p,i}(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{p-i+1}\rho(L,u)^{n-p-1}\,dS(u). \tag{2.7}$$

Obviously, when $K = L$, $\widetilde{W}_{p,i}(K,L)$ becomes the dual quermassintegral of star body K, i.e., $\widetilde{W}_{p,i}(K,K) = \widetilde{W}_i(K)$. Taking $i = 0$ in (2.7), $\widetilde{W}_{p,i}(K,L)$ becomes the p-dual mixed volume $\widetilde{V}_p(K,L)$ and

$$\widetilde{V}_p(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{p+1}\rho(L,u)^{n-p-1}\,dS(u). \tag{2.8}$$

From (2.7), combining Hölder's integral inequality (see [3]) gives the following.

Proposition 2.5 (Minkowski type inequality) *If $K, L \in \mathcal{S}^n$, $0 \le i < n$ and $0 \le p < n-1$, then*

$$\widetilde{W}_{p,i}(K,L)^{n-i} \le \widetilde{W}_i(K)^{p-i+1}\widetilde{W}_i(L)^{n-p-1}, \tag{2.9}$$

with equality if and only if K and L are dilates.

Taking $i = 0$ in (2.9), we have: If $K, L \in \mathcal{S}^n$ and $0 \le p < n-1$, then

$$\widetilde{V}_p(K,L)^n \le V(K)^{p+1}V(L)^{n-p-1}, \tag{2.10}$$

with equality if and only if K and L are dilates. In the following, we establish the Brunn-Minkowski inequality for the p-radial Blaschke addition.

Proposition 2.6 *If $K, L \in \mathcal{S}^n$, $0 \le i < n$ and $0 \le p < n-1$, then*

$$\widetilde{W}_i(K \widehat{+}_p L)^{(n-p-1)/(n-i)} \le \widetilde{W}_i(K)^{(n-p-1)/(n-i)} + \widetilde{W}_i(L)^{(n-p-1)/(n-i)}, \tag{2.11}$$

with equality if and only if K and L are dilates.

Proof From (2.3) and (2.7), it is easily seen that the p-dual mixed quermassintegral $\widehat{W}_{p,i}(K,L)$ is linear with respect to the p-radial Blaschke addition and together with inequality (2.9) shows that

$$\begin{aligned}\widetilde{W}_{p,i}(Q, K \widehat{+}_p L) &= \widetilde{W}_{p,i}(Q,K) + \widetilde{W}_{p,i}(Q,L) \\ &\le \widetilde{W}_i(Q)^{(p-i+1)/(n-i)}\bigl(\widetilde{W}_i(K)^{(n-p-1)/(n-i)} + \widetilde{W}_i(L)^{(n-p-1)/(n-i)}\bigr),\end{aligned} \tag{2.12}$$

with equality if and only if K and L are dilates of Q. Take $K \widehat{+}_p L$ for Q in (2.12), recall that $\widetilde{W}_{p,i}(Q,Q) = \widetilde{W}_i(Q)$, inequality (2.11) follows easy.

Taking $i = 0$ in (2.11), we obtain that if $K, L \in \mathcal{S}^n$ and $0 \le p < n-1$, then

$$V(K \widehat{+}_p L)^{(n-p-1)/n} \le V(K)^{(n-p-1)/n} + V(L)^{(n-p-1)/n},$$

with equality if and only if K and L are dilates. Taking $p = 0$ and $i = 0$ in (2.11), (2.11) becomes the well-known dual Knesser-Süss inequality (2.2). □

3 Improvement of the harmonic Blaschke addition

Let us recall the concept of harmonic Blaschke addition defined by Lutwak [4]. Suppose that K and L are star bodies in $\mathbb{R}^n$, the harmonic Blaschke addition denoted by $K\check{+}L$ is defined by

$$\frac{\rho(K\check{+}L,\cdot)^{n+1}}{V(K\check{+}L)} = \frac{\rho(K,\cdot)^{n+1}}{V(K)} + \frac{\rho(L,\cdot)^{n+1}}{V(L)}. \tag{3.1}$$

Lutwak's Brunn-Minkowski inequality for the harmonic Blaschke addition was established (see [4]). If $K, L \in \mathcal{S}^n$, then

$$V(K\check{+}L)^{1/n} \geq V(K)^{1/n} + V(L)^{1/n}, \tag{3.2}$$

with equality if and only if K and L are dilates.

In the section, we give an improved concept of the harmonic Blaschke addition.

Definition 3.1 For $0 \leq i < n$, $p < i - 1$ and $K, L \in \mathcal{S}^n$, we define the p-harmonic Blaschke addition of star bodies K and L denoted by $K\check{+}_pL$ and defined by

$$\frac{\rho(K\check{+}_pL,\cdot)^{n-p-1}}{\widetilde{W}_i(K\check{+}_pL)} = \frac{\rho(K,\cdot)^{n-p-1}}{\widetilde{W}_i(K)} + \frac{\rho(L,\cdot)^{n-p-1}}{\widetilde{W}_i(L)}. \tag{3.3}$$

The Brunn-Minkowski inequality for the p-harmonic Blaschke addition follows immediately from (1.6), (3.3) and Minkowski's integral inequality (see [3]).

Proposition 3.2 *If $K, L \in \mathcal{S}^n$, $0 \leq i < n$ and $p < i - 1$, then*

$$\widetilde{W}_i(K\check{+}_pL)^{-(p+1-i)/(n-i)} \leq \widetilde{W}_i(K)^{-(p+1-i)/(n-i)} + \widetilde{W}_i(L)^{-(p+1-i)/(n-i)}, \tag{3.4}$$

with equality if and only if K and L are dilates.

4 Radial Blaschke-Minkowski homomorphisms

Definition 4.1 ([5]) A map $\Psi : \mathcal{S}^n \to \mathcal{S}^n$ is called a radial Blaschke-Minkowski homomorphism if it satisfies the following conditions:

(a) Ψ is continuous.

(b) For all $K, L \in \mathcal{S}^n$,

$$\Psi(K\ddot{+}L) = \Psi(K)\widetilde{+}\Psi(L).$$

(c) For all $K, L \in \mathcal{S}^n$ and every $\vartheta \in SO(n)$,

$$\Psi(\vartheta K) = \vartheta\Psi(K),$$

where $SO(n)$ is the group of rotations in n dimensions.

Radial Blaschke-Minkowski homomorphisms are important examples of star body valued valuations. Their natural duals, Blaschke-Minkowski homomorphisms, are an important notion in the theory of convex body valued valuations (see, e.g., [6–12] and [13–20]). In 2006, Schuster [5] established the following Brunn-Minkowski inequality for radial Blaschke-Minkowski homomorphisms of star bodies. If K and L are star bodies in $\mathbb{R}^n$, then

$$V\big(\Psi(K\tilde{+}L)\big)^{1/n(n-1)} \le V(\Psi K)^{1/n(n-1)} + V(\Psi L)^{1/n(n-1)}, \tag{4.1}$$

with equality if and only if K and L are dilates.

If K and L are star bodies in $\mathbb{R}^n$, $p \neq 0$ and $\lambda, \mu \ge 0$, then $\lambda \cdot K\tilde{+}_p\mu \cdot L$ is the star body whose radial function is given by (see, e.g., [21])

$$\rho(\lambda \cdot K\tilde{+}_p\mu \cdot L, \cdot)^p = \lambda\rho(K, \cdot)^p + \mu\rho(L, \cdot)^p. \tag{4.2}$$

The addition $\tilde{+}_p$ is called L_p-radial addition. The L_p dual Brunn-Minkowski inequality states: If $K, L \in \mathcal{S}^n$ and $0 < p \le n$, then

$$V(K\tilde{+}_pL)^{p/n} \le V(K)^{p/n} + V(L)^{p/n},$$

with equality when $p \neq n$ if and only if K and L are dilates. The inequality is reversed when $p > n$ or $p < 0$ (see [21]).

In 2013, an L_p Brunn-Minkowski inequality for radial Blaschke-Minkowski homomorphisms was established in [22]: If K and L are star bodies in $\mathbb{R}^n$ and $0 < p < n-1$, then

$$V\big(\Psi(K\tilde{+}_pL)\big)^{p/n(n-1)} \le V(\Psi K)^{p/n(n-1)} + V(\Psi L)^{p/n(n-1)}, \tag{4.3}$$

with equality if and only if K and L are dilates. Taking $p = 1$, (4.3) reduces to (4.1).

Theorem 4.2 (see [5]) *Let $\Psi : \mathcal{S}^n \to \mathcal{S}^n$ be a radial Blaschke-Minkowski homomorphism. There is a continuous operator $\Psi : \underbrace{\mathcal{S}^n \times \cdots \times \mathcal{S}^n}_{n-1} \to \mathcal{S}^n$ symmetric in its arguments such that, for $K_1, \ldots, K_m \in \mathcal{S}^n$ and $\lambda_1, \ldots, \lambda_m \ge 0$,*

$$\Psi(\lambda_1K_1\tilde{+}\cdots\tilde{+}\lambda_mK_m) = \sum_{i_1,\ldots,i_{n-1}} \lambda_{i_1}\cdots\lambda_{i_{n-1}}\Psi(K_{i_1}, \ldots, K_{i_{n-1}}). \tag{4.4}$$

Clearly, Theorem 4.2 generalizes the notion of radial Blaschke-Minkowski homomorphisms. We call $\Psi : \mathcal{S}^n \times \cdots \times \mathcal{S}^n \to \mathcal{S}^n$ a mixed radial Blaschke-Minkowski homomorphism induced by Ψ. Mixed radial Blaschke-Minkowski homomorphisms were first studied in more detail in [23, 24]. If $K_1 = \cdots = K_{n-i-1} = K, K_{n-i} = \cdots = K_{n-1} = L$, we write $\Psi_i(K,L)$ for $\Psi(\underbrace{K,\ldots,K}_{n-i-1},\underbrace{L,\ldots,L}_{i})$. If $K_1 = \cdots = K_{n-i-1} = K, K_{n-i} = \cdots = K_{n-1} = B$, we write Ψ_iK for $\Psi(\underbrace{K,\ldots,K}_{n-i-1},\underbrace{B,\ldots,B}_{i})$ and call Ψ_iK the mixed Blaschke-Minkowski homomorphism of order i of K. Ψ_0K is written simply as ΨK.

Lemma 4.3 (see [5]) *A map $\Psi : \mathcal{S}^n \to \mathcal{S}^n$ is a radial Blaschke-Minkowski homomorphism if and only if there is a measure $\mu \in \mathcal{M}_+(S^{n-1}, \hat{e})$ such that*

$$\rho(\Psi K, \cdot) = \rho(K, \cdot)^{n-1} * \mu, \tag{4.5}$$

where $\mathcal{M}_+(S^{n-1}, \hat{e})$ denotes the set of nonnegative zonal measures on S^{n-1}.

For the mixed radial Blaschke-Minkowski homomorphism induced by Ψ, Schuster [5] proved that

$$\rho\big(\Psi(K_1, \ldots, K_{n-1}), \cdot\big) = \rho(K_1, \cdot) \cdots \rho(K_{n-1}, \cdot) * \mu.$$

Obviously, a special case is the following:

$$\rho(\Psi_i K, \cdot) = \rho(K, \cdot)^{n-1-i} * \mu,$$

where i are integers. We now extend the integers i to real numbers, define the Blaschke-Minkowski homomorphism of order p of K.

Definition 4.4 Let $K \in \mathcal{S}^n$, the Blaschke-Minkowski homomorphism of order p of K, denoted by $\Psi_p K$, is defined for all $p \in \mathbb{R}$ by

$$\rho(\Psi_p K, \cdot) = \rho(K, \cdot)^{n-1-p} * \mu. \tag{4.6}$$

This extended definition will be required to prove our main results.

5 Inequalities for the radial Blaschke-Minkowski homomorphism

Theorem 5.1 *Let $K, L \in \mathcal{S}^n$. If $0 \le p < n-1$ and $i \le n-1 \le j \le n$, then*

$$\left(\frac{\widetilde{W}_i(\Psi_p(K \widehat{+}_p L))}{\widetilde{W}_j(\Psi_p(K \widehat{+}_p L))}\right)^{1/(j-i)} \le \left(\frac{\widetilde{W}_i(\Psi_p K)}{\widetilde{W}_j(\Psi_p K)}\right)^{1/(j-i)} + \left(\frac{\widetilde{W}_i(\Psi_p L)}{\widetilde{W}_j(\Psi_p L)}\right)^{1/(j-i)}, \tag{5.1}$$

with equality if and only if $\Psi_p K$ and $\Psi_p L$ are dilates.

Remark 5.2 Taking $j = n$ in (5.1) and noting that $\widetilde{W}_n(K) = \int_{S^{n-1}} dS(u) = n\omega_n$, (5.1) becomes the following inequality: If $K, L \in \mathcal{S}^n$, $0 \le p < n-1$ and $i \le n-1$, then

$$\widetilde{W}_i\big(\Psi_p(K \widehat{+}_p L)\big)^{1/(n-i)} \le \widetilde{W}_i(\Psi_p K)^{1/(n-i)} + \widetilde{W}_i(\Psi_p L)^{1/(n-i)}, \tag{5.2}$$

with equality if and only if $\Psi_p K$ and $\Psi_p L$ are dilates. Taking $p = 0$ in (5.1), (5.1) becomes the following inequality: If $K, L \in \mathcal{S}^n$ and $i \le n-1 \le j \le n$, then

$$\left(\frac{\widetilde{W}_i(\Psi(K \widehat{+} L))}{\widetilde{W}_j(\Psi(K \widehat{+} L))}\right)^{1/(j-i)} \le \left(\frac{\widetilde{W}_i(\Psi K)}{\widetilde{W}_j(\Psi K)}\right)^{1/(j-i)} + \left(\frac{\widetilde{W}_i(\Psi L)}{\widetilde{W}_j(\Psi L)}\right)^{1/(j-i)}, \tag{5.3}$$

with equality if and only if ΨK and ΨL are dilates.

Theorem 5.3 *Let $K, L \in \mathcal{S}^n$. If $0 \le i < n$, $p < i-1$ and $k, j \in \mathbb{R}$ satisfy $j \le n-1 \le k \le n$, then*

$$\frac{1}{\widetilde{W}_i(K \check{+}_p L)} \left(\frac{\widetilde{W}_j(\Psi_p(K \check{+}_p L))}{\widetilde{W}_k(\Psi_p(K \check{+}_p L))} \right)^{1/(k-j)} \le \frac{1}{\widetilde{W}_i(K)} \left(\frac{\widetilde{W}_j(\Psi_p K)}{\widetilde{W}_k(\Psi_p K)} \right)^{1/(k-j)} + \frac{1}{\widetilde{W}_i(L)} \left(\frac{\widetilde{W}_j(\Psi_p L)}{\widetilde{W}_k(\Psi_p L)} \right)^{1/(k-j)}, \tag{5.4}$$

with equality if and only if $\Psi_p K$ and $\Psi_p L$ are dilates.

Remark 5.4 Taking $k = n$ in (5.4) and noting that $\widetilde{W}_n(K) = \int_{S^{n-1}} dS(u) = n\omega_n$, (5.4) becomes the following inequality: If $K, L \in \mathcal{S}^n$, $0 \le i < n$, $p < i-1$ and $j \le n-1$, then

$$\frac{\widetilde{W}_j(\Psi_p(K \check{+}_p L))^{1/(n-j)}}{\widetilde{W}_i(K \check{+}_p L)} \le \frac{\widetilde{W}_j(\Psi_p K)^{1/(n-j)}}{\widetilde{W}_i(K)} + \frac{\widetilde{W}_j(\Psi_p L)^{1/(n-j)}}{\widetilde{W}_i(L)}, \tag{5.5}$$

with equality if and only if $\Psi_p K$ and $\Psi_p L$ are dilates. Taking $i = 0$, $j = 0$ and $k = n$ in (5.4), we have: If $K, L \in \mathcal{S}^n$ and $p < -1$, then

$$\frac{V(\Psi_p(K \check{+}_p L))^{1/n}}{V(K \check{+}_p L)} \le \frac{V(\Psi_p K)^{1/n}}{V(K)} + \frac{V(\Psi_p L)^{1/n}}{V(L)}, \tag{5.6}$$

with equality if and only if $\Psi_p K$ and $\Psi_p L$ are dilates.

6 Dresher's inequalities for *p*-radial Blaschke and harmonic Blaschke additions

An extension of Beckenbach's inequality (see [3], p. 27) was obtained by Dresher [25] by means of moment-space techniques.

Lemma 6.1 (Dresher's inequality) *If $p \ge 1 \ge r \ge 0$, $f, g \ge 0$ and ϕ is a distribution function, then*

$$\left(\frac{\int (f+g)^p \, d\phi}{\int (f+g)^r \, d\phi} \right)^{1/(p-r)} \le \left(\frac{\int f^p \, d\phi}{\int f^r \, d\phi} \right)^{1/(p-r)} + \left(\frac{\int g^p \, d\phi}{\int g^r \, d\phi} \right)^{1/(p-r)}, \tag{6.1}$$

with equality if and only if the functions f and g are proportional.

We are now in a position to prove Theorem 5.1. The following statement is just a slight reformulation of it.

Theorem 6.2 *Let $K, L \in \mathcal{S}^n$. If $0 \le p < n-1$ and $s, t \in \mathbb{R}$ satisfy $s \ge 1 \ge t \ge 0$, then*

$$\left(\frac{\widetilde{W}_{n-s}(\Psi_p(K \widehat{+}_p L))}{\widetilde{W}_{n-t}(\Psi_p(K \widehat{+}_p L))} \right)^{1/(s-t)} \le \left(\frac{\widetilde{W}_{n-s}(\Psi_p K)}{\widetilde{W}_{n-t}(\Psi_p K)} \right)^{1/(s-t)} + \left(\frac{\widetilde{W}_{n-s}(\Psi_p L)}{\widetilde{W}_{n-t}(\Psi_p L)} \right)^{1/(s-t)}, \tag{6.2}$$

with equality if and only if $\Psi_p K$ and $\Psi_p L$ are dilates.

Proof From (2.4), we obtain

$$\rho(K \widehat{+}_p L, \cdot)^{n-p-1} * \mu = \rho(K, \cdot)^{n-p-1} * \mu + \rho(L, \cdot)^{n-p-1} * \mu,$$

where μ is the generating measure of Ψ from Lemma 4.3. Hence, from (4.6), we obtain

$$\rho\big(\Psi_p(K\widehat{+}_pL),\cdot\big) = \rho(\Psi_pK,\cdot) + \rho(\Psi_pL,\cdot).$$

Therefore, by (1.6), we have

$$\widetilde{W}_{n-s}\big(\Psi_p(K\widehat{+}_pL)\big) = \frac{1}{n}\int_{S^{n-1}} \big(\rho(\Psi_pK,u) + \rho(\Psi_pL,u)\big)^s\,dS(u) \tag{6.3}$$

and

$$\widetilde{W}_{n-t}\big(\Psi_p(K\widehat{+}_pL)\big) = \frac{1}{n}\int_{S^{n-1}} \big(\rho(\Psi_pK,u) + \rho(\Psi_pL,u)\big)^t\,dS(u). \tag{6.4}$$

From (6.3), (6.4) and Lemma 6.1, we obtain

$$\begin{aligned}
&\left(\frac{\widetilde{W}_{n-s}(\Psi_p(K\widehat{+}_pL))}{\widetilde{W}_{n-t}(\Psi_p(K\widehat{+}_pL))}\right)^{1/(s-t)} \\
&\quad= \left(\frac{\int_{S^{n-1}}(\rho(\Psi_pK,u) + \rho(\Psi_pL,u))^s\,dS(u)}{\int_{S^{n-1}}(\rho(\Psi_pK,u) + \rho(\Psi_pL,u))^t\,dS(u)}\right)^{1/(s-t)} \\
&\quad\le \left(\frac{\int_{S^{n-1}}\rho(\Psi_pK,u)^s\,dS(u)}{\int_{S^{n-1}}\rho(\Psi_pK,u)^t\,dS(u)}\right)^{1/(s-t)} + \left(\frac{\int_{S^{n-1}}\rho(\Psi_pL,u)^s\,dS(u)}{\int_{S^{n-1}}\rho(\Psi_pL,u)^t\,dS(u)}\right)^{1/(s-t)} \\
&\quad= \left(\frac{\widetilde{W}_{n-s}(\Psi_pK)}{\widetilde{W}_{n-t}(\Psi_pK)}\right)^{1/(s-t)} + \left(\frac{\widetilde{W}_{n-s}(\Psi_pL)}{\widetilde{W}_{n-t}(\Psi_pL)}\right)^{1/(s-t)}.
\end{aligned}$$

Equality holds if and only if the functions $\rho(\Psi_pK,u)$ and $\rho(\Psi_pL,u)$ are proportional.

Taking $s = n - i$ and $t = n - j$ in Theorem 6.2, Theorem 6.2 becomes Theorem 5.1 stated in Section 5. If $\Psi : \underbrace{\mathcal{S}^n \times \cdots \times \mathcal{S}^n}_{n-1} \to \mathcal{S}^n$ is the mixed intersection operator $\mathbf{I} : \underbrace{\mathcal{S}^n \times \cdots \times \mathcal{S}^n}_{n-1} \to \mathcal{S}^n$ in (6.2) and $n - s = i$ and $n - t = j$, we obtain the following result: If $K, L \in \mathcal{S}^n$, $0 \le p < n - 1$ and $i \le n - 1 \le j \le n$, then

$$\left(\frac{\widetilde{W}_i(\mathbf{I}_p(K\widehat{+}_pL))}{\widetilde{W}_j(\mathbf{I}_p(K\widehat{+}_pL))}\right)^{1/(j-i)} \le \left(\frac{\widetilde{W}_i(\mathbf{I}_pK)}{\widetilde{W}_j(\mathbf{I}_pK)}\right)^{1/(j-i)} + \left(\frac{\widetilde{W}_i(\mathbf{I}_pL)}{\widetilde{W}_j(\mathbf{I}_pL)}\right)^{1/(j-i)}, \tag{6.5}$$

with equality if and only if $\mathbf{I}_pK$ and $\mathbf{I}_pL$ are dilates. Taking $j = n$ in (6.5) and noting that $\widetilde{W}_n(K) = \int_{S^{n-1}} dS(u) = n\omega_n$, (6.5) becomes the following inequality: If $K, L \in \mathcal{S}^n$, $0 \le p < n - 1$ and $i \le n - 1$, then

$$\widetilde{W}_i\big(\mathbf{I}_p(K\widehat{+}_pL)\big)^{1/(n-i)} \le \widetilde{W}_i(\mathbf{I}_pK)^{1/(n-i)} + \widetilde{W}_i(\mathbf{I}_pL)^{1/(n-i)},$$

with equality if and only if $\mathbf{I}_pK$ and $\mathbf{I}_pL$ are dilates. □

We are now in a position to prove Theorem 5.3. The following statement is just a slight reformulation of it.

Theorem 6.3 *Let $K, L \in \mathcal{S}^n$. If $0 \le i < n$, $p < i-1$ and $s, t \in \mathbb{R}$ satisfy $s \ge 1 \ge t \ge 0$, then*

$$\begin{aligned} &\frac{1}{\widetilde{W}_i(K \check{+}_p L)}\left(\frac{\widetilde{W}_{n-s}(\Psi_p(K \check{+}_p L))}{\widetilde{W}_{n-t}(\Psi_p(K \check{+}_p L))}\right)^{1/(s-t)} \\ &\quad \le \frac{1}{\widetilde{W}_i(K)}\left(\frac{\widetilde{W}_{n-s}(\Psi_p K)}{\widetilde{W}_{n-t}(\Psi_p K)}\right)^{1/(s-t)} + \frac{1}{\widetilde{W}_i(L)}\left(\frac{\widetilde{W}_{n-s}(\Psi_p L)}{\widetilde{W}_{n-t}(\Psi_p L)}\right)^{1/(s-t)}, \end{aligned} \tag{6.6}$$

with equality if and only if $\Psi_p K$ and $\Psi_p L$ are dilates.

Proof From (3.3), we obtain

$$\frac{\rho(K \check{+}_p L, \cdot)^{n-p-1} * \mu}{\widetilde{W}_i(K \check{+}_p L)} = \frac{\rho(K, \cdot)^{n-p-1} * \mu}{\widetilde{W}_i(K)} + \frac{\rho(L, \cdot)^{n-p-1} * \mu}{\widetilde{W}_i(L)}.$$

Hence, from (4.6), we obtain

$$\frac{\rho(\Psi_p(K \check{+}_p L), \cdot)}{\widetilde{W}_i(K \check{+}_p L)} = \frac{\rho(\Psi_p K, \cdot)}{\widetilde{W}_i(K)} + \frac{\rho(\Psi_p L, \cdot)}{\widetilde{W}_i(L)}.$$

By (1.6), we have

$$\frac{\widetilde{W}_{n-s}(\Psi_p(K \check{+}_p L))}{\widetilde{W}_i(K \check{+}_p L)^s} = \frac{1}{n}\int_{S^{n-1}}\left(\frac{\rho(\Psi_p K, u)}{\widetilde{W}_i(K)} + \frac{\rho(\Psi_p L, u)}{\widetilde{W}_i(L)}\right)^s dS(u) \tag{6.7}$$

and

$$\frac{\widetilde{W}_{n-t}(\Psi_p(K \check{+}_p L))}{\widetilde{W}_i(K \check{+}_p L)^t} = \frac{1}{n}\int_{S^{n-1}}\left(\frac{\rho(\Psi_p K, u)}{\widetilde{W}_i(K)} + \frac{\rho(\Psi_p L, u)}{\widetilde{W}_i(L)}\right)^t dS(u). \tag{6.8}$$

From (6.7), (6.8) and Lemma 6.1, we obtain

$$\begin{aligned} &\frac{1}{\widetilde{W}_i(K \check{+}_p L)}\left(\frac{\widetilde{W}_{n-s}(\Psi_p(K \check{+}_p L))}{\widetilde{W}_{n-t}(\Psi_p(K \check{+}_p L))}\right)^{1/(s-t)} \\ &\quad = \left(\frac{\int_{S^{n-1}}(\frac{\rho(\Psi_p K,u)}{\widetilde{W}_i(K)} + \frac{\rho(\Psi_p L,u)}{\widetilde{W}_i(L)})^s\, dS(u)}{\int_{S^{n-1}}(\frac{\rho(\Psi_p K,u)}{\widetilde{W}_i(K)} + \frac{\rho(\Psi_p L,u)}{\widetilde{W}_i(L)})^t\, dS(u)}\right)^{1/(s-t)} \\ &\quad \le \left(\frac{\int_{S^{n-1}}(\frac{\rho(\Psi_p K,u)}{\widetilde{W}_i(K)})^s\, dS(u)}{\int_{S^{n-1}}(\frac{\rho(\Psi_p K,u)}{\widetilde{W}_i(K)})^t\, dS(u)}\right)^{1/(s-t)} + \left(\frac{\int_{S^{n-1}}(\frac{\rho(\Psi_p L,u)}{\widetilde{W}_i(L)})^s\, dS(u)}{\int_{S^{n-1}}(\frac{\rho(\Psi_p L,u)}{\widetilde{W}_i(L)})^t\, dS(u)}\right)^{1/(s-t)} \\ &\quad = \frac{1}{\widetilde{W}_i(K)}\left(\frac{\widetilde{W}_{n-s}(\Psi_p K)}{\widetilde{W}_{n-t}(\Psi_p K)}\right)^{1/(s-t)} + \frac{1}{\widetilde{W}_i(L)}\left(\frac{\widetilde{W}_{n-s}(\Psi_p L)}{\widetilde{W}_{n-t}(\Psi_p L)}\right)^{1/(s-t)}, \end{aligned}$$

with equality if and only if $\Psi_p K$ and $\Psi_p L$ are dilates.

Taking $s = n-j$ and $t = n-k$ in Theorem 6.3, Theorem 6.3 becomes Theorem 5.3 stated in Section 5. If $\Psi : \underbrace{\mathcal{S}^n \times \cdots \times \mathcal{S}^n}_{n-1} \to \mathcal{S}^n$ is the mixed intersection operator $\mathbf{I}$: $\underbrace{\mathcal{S}^n \times \cdots \times \mathcal{S}^n}_{n-1} \to \mathcal{S}^n$ in (6.6) and $j = n-s$ and $k = n-t$, we obtain the following result:

If $K, L \in \mathcal{S}^n$, $0 \le i < n$, $p < i-1$ and $j \le n-1 \le k \le n$, then

$$\frac{1}{\widetilde{W}_i(K \check{+}_p L)} \left(\frac{\widetilde{W}_j(\mathbf{I}_p(K \check{+}_p L))}{\widetilde{W}_k(\mathbf{I}_p(K \check{+}_p L))} \right)^{1/(k-j)} \le \frac{1}{\widetilde{W}_i(K)} \left(\frac{\widetilde{W}_j(\mathbf{I}_p K)}{\widetilde{W}_k(\mathbf{I}_p K)} \right)^{1/(k-j)} + \frac{1}{\widetilde{W}_i(L)} \left(\frac{\widetilde{W}_j(\mathbf{I}_p L)}{\widetilde{W}_k(\mathbf{I}_p L)} \right)^{1/(k-j)}, \tag{6.9}$$

with equality if and only if $\mathbf{I}_p K$ and $\mathbf{I}_p L$ are dilates. Taking $k = n$ in (6.9) and noting that $\widetilde{W}_n(K) = \int_{S^{n-1}} dS(u) = n\omega_n$, (6.9) becomes the following inequality: If $K, L \in \mathcal{S}^n$, $0 \le i < n$, $p < i-1$ and $j \le n-1$, then

$$\frac{\widetilde{W}_j(\mathbf{I}_p(K \check{+}_p L))^{1/(n-j)}}{\widetilde{W}_i(K \check{+}_p L)} \le \frac{\widetilde{W}_j(\mathbf{I}_p K)^{1/(n-j)}}{\widetilde{W}_i(K)} + \frac{\widetilde{W}_j(\mathbf{I}_p L)^{1/(n-j)}}{\widetilde{W}_i(L)}, \tag{6.10}$$

with equality if and only if $\mathbf{I}_p K$ and $\mathbf{I}_p L$ are dilates. □

7 Conclusions

In the present study, we first revised and improved the concepts of radial Blaschke addition and harmonic Blaschke addition in an L_p space. Following this, we established Dresher's inequalities (Brunn-Minkowski type) for the radial Blaschke-Minkowski homomorphisms with respect to the p-radial addition and the p-harmonic Blaschke addition.

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Authors' contributions
C-JZ provided the questions and gave the proof for the main results. He read and approved the manuscript.

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