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On the bounds of degree-based topological indices of the Cartesian product of F-sum of connected graphs

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Abstract

Topological indices are the mathematical tools that correlate the chemical structure with various physical properties, chemical reactivity or biological activity numerically. A topological index is a function having a set of graphs as its domain and a set of real numbers as its range. In QSAR/QSPR study, a prediction about the bioactivity of chemical compounds is made on the basis of physico-chemical properties and topological indices such as Zagreb, Randić and multiple Zagreb indices. In this paper, we determine the lower and upper bounds of Zagreb indices, the atom-bond connectivity (ABC) index, multiple Zagreb indices, the geometric-arithmetic (GA) index, the forgotten topological index and the Narumi-Katayama index for the Cartesian product of F-sum of connected graphs by using combinatorial inequalities.

MSC: 92E10; 05C90

Keywords: F-sum; Cartesian product; ABC-index; Zagreb index; GA-index; F-index

1 Introduction and preliminary results

We consider G as a simple, connected and finite graph with a vertex set $V(G) = \{u_1, u_2, u_3, \dots, u_n\}$, an edge set $E(G) = \{e_1, e_2, e_3, \dots, e_m\}$, the *order* of $G = |V(G)| = n$ and the size of $G = |E(G)| = m$. An *edge* $e \in E(G)$ with end vertices u_i and u_j is denoted by u_iu_j. The number of edges having u as an end vertex is called the *degree* of u in G and is denoted by $\deg_G(u)$. The minimum and maximum degrees of graph G are denoted by δ_G and Δ_G, respectively. P_n and C_n are used for *path* and *cycle* with order n, respectively.

The branch of chemistry in which we discuss and predict the chemical structure by using mathematical tools without referring to quantum mechanics is called *mathematical chemistry* [1, 2]. The branch of mathematical chemistry which applies graph theory to mathematical modeling of chemical phenomena is known as *chemical graph theory* [2]. This theory has a remarkable role in the development of chemical sciences.

The *Zagreb* indices are the first degree-based structure descriptors [3, 4]. The terms $\sum_{v\in V(G)}[\deg_G(v)]^2$, $\sum_{uv\in E(G)}\deg_G(u)\deg_G(v)$ and $\sum_{v\in V(G)}[\deg_G(v)]^3$ first appeared in the topological formula for total π-energy of conjugated molecules that was derived in 1972 by Gutman and Trinajstić [3]. Ten years later, Balaban et al. included

$$M_1(G) = \sum_{v\in V(G)} \left[\deg_G(v)\right]^2 = \sum_{uv\in E(G)} \left[\deg_G(u) + \deg_G(v)\right]$$

and

$$M_2(G) = \sum_{uv\in E(G)} \deg_G(u)\deg_G(v)$$

among topological indices and named them 'Zagreb group indices' [5], which was later on abbreviated to 'Zagreb indices', and now $M_1(G)$ and $M_2(G)$ are called the first and second *Zagreb* indices. Afterwards these indices were used as branching indices [6]. Later on, the Zagreb indices found applications in QSPR and QSAR studies [1, 7]. These indices have been used to study molecular complexity, chirality, *ZE*-isomorphism and hetero-systems. Chemical applications and mathematical properties of Zagreb indices can be studied from [8–10].

Narumi and Katayama studied the degree product of a graph G for the first time in 1984. The *Narumi-Katayama* index proposed by Narumi and Katayama [11] is defined as follows:

$$\mathrm{NK}(G) = \prod_{v\in V(G)} \deg_G(v).$$

Estrada et al. [12] introduced the atom-bond connectivity index defined as follows:

$$\mathrm{ABC}(G) = \sum_{uv\in E(G)} \sqrt{\frac{\deg_G(u) + \deg_G(v) - 2}{\deg_G(u)\deg_G(v)}}.$$

It has been applied up till now to study the stability of alkanes and the strain energy of cycloalkanes [12, 13]. The ABC-index can be used for modeling thermodynamic properties of organic chemical compounds. The ABC-index happens to be the only topological index for which theoretical, quantum-theory-based, foundation and justification have been found.

The first geometric-arithmetic connectivity index or simply geometric-arithmetic (GA) index of a connected graph G was introduced by Vukičević et al. in 2009 [14] and is defined as follows:

$$\mathrm{GA}(G) = \sum_{uv\in E(G)} \frac{2\sqrt{\deg_G(u)\deg_G(v)}}{\deg_G(u) + \deg_G(v)}.$$

The augmented Zagreb index proposed by Furtula et al. in 2010 [15] is defined as follows:

$$\mathrm{AZI}(G) = \sum_{uv\in E(G)} \left[\frac{\deg_G(u)\deg_G(v)}{\deg_G(u) + \deg_G(v) - 2}\right]^3.$$

This graph invariant is a valuable predictive index in the study of the heat of formation in octanes and heptanes [15].

The first multiple Zagreb index was introduced by Ghorbani and Azimi in 2012 [16] and defined as follows:

$$\mathrm{PM}_1(G) = \prod_{uv\in V(G)} \bigl[\deg_G(u) + \deg_G(v)\bigr] = \prod_{v\in V(G)} \bigl[\deg_G(v)\bigr]^2.$$

Clearly, the first multiple Zagreb index is the square of Narumi-Katayama index.

The third Zagreb index was introduced by Shirdel in 2013 [17] and defined as follows:

$$M_3(G) = \sum_{uv\in E(G)} \bigl[\deg_G(u) + \deg_G(v)\bigr]^2.$$

Furtula and Gutman showed that the term $\sum_{v\in V(G)}[\deg_G(v)]^3$ has a very promising application potential [18]. They called it the forgotten topological index or shortly the F-index, and it is defined as follows:

$$F(G) = \sum_{v\in V(G)} \bigl[\deg_G(v)\bigr]^3 = \sum_{uv\in E(G)} \bigl[\bigl(\deg_G(u)\bigr)^2 + \bigl(\deg_G(v)\bigr)^2\bigr].$$

They proved that the linear combination $M_1 + \lambda F$ yields a highly accurate mathematical model of certain physico-chemical properties of alkanes [18].

Clearly, this index is a combination of second and third *Zagreb* indices, i.e.,

$$F(G) = M_3(G) - 2M_2(G).$$

The Cartesian product is an important method to construct a bigger graph and plays an important role in the design and analysis of networks [19]. The Cartesian product of the graphs G and H, denoted by $G\Box H$, is a graph with a vertex set $V(G\Box H) = V(G) \times V(H)$ and $(u_1, v_1)(u_2, v_2) \in E(G\Box H)$ whenever $[u_1 = u_2$ and $v_1v_2 \in E(H)]$ or $[u_1u_2 \in E(G)$ and $v_1 = v_2]$.

Now we state distinct properties of the Cartesian product of graphs in form of the following lemma.

Lemma 1 *Let G_1 and G_2 be graphs of orders n_1, n_2 and sizes m_1, m_2, respectively. Then we have*:

(a) $|V(G_1\Box G_2)| = |V(G_1)||V(G_2)|$ *and* $|E(G_1\Box G_2)| = |V(G_2)||E(G_1)| + |V(G_1)||E(G_2)|$,

(b) $\deg_{G_1\Box G_2}(u, v) = \deg_{G_1}(u) + \deg_{G_2}(v)$.

For a connected graph G, define four related graphs $S(G)$, $R(G)$, $Q(G)$ and $T(G)$ as follows:

(1) $S(G)$ is the graph obtained by inserting an additional vertex in each edge of G, i.e., replacing each edge of G by a path of length 2. The graph $S(G)$ is also known as a subdivision graph of G.

(2) $R(G)$ is the graph obtained by adding a new vertex corresponding to each edge of G, then joining each new vertex to the end vertices of the corresponding edge.

(3) $Q(G)$ is the graph obtained by inserting a new vertex into each edge of G, then joining with edges those pairs of new vertices on adjacent edges of G.

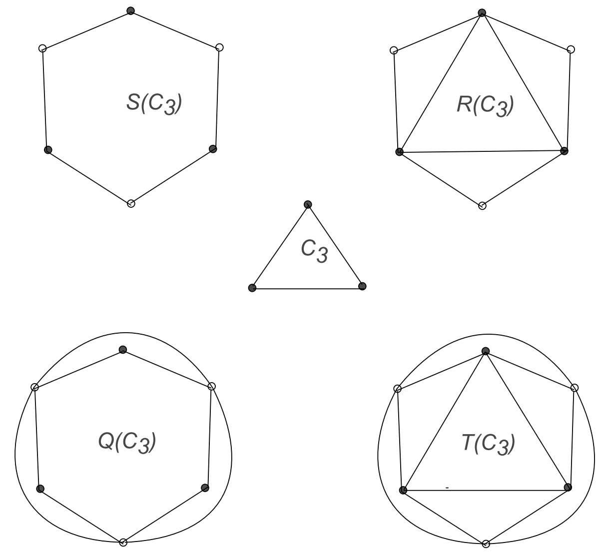

Figure 1 The graphs G, $S(G)$, $R(G)$, $Q(G)$ and $T(G)$.

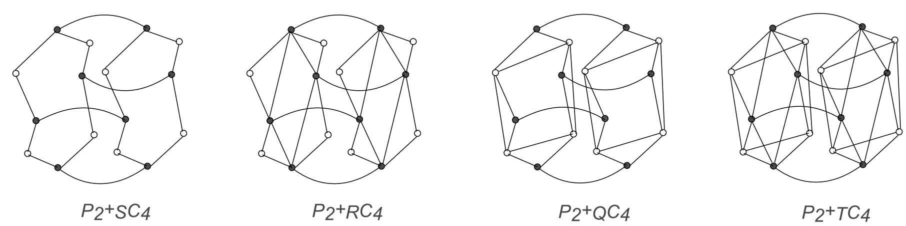

Figure 2 The graphs $P_2 +_S C_4$, $P_2 +_R C_4$, $P_2 +_Q C_4$ and $P_2 +_T C_4$.

(4) $T(G)$ has as its vertices, the edges and vertices of G. Adjacency in $T(G)$ is defined as adjacency or incidence for the corresponding elements of G. The graph $T(G)$ is called the total graph of G.

The four operations, $S(G)$, $R(G)$, $Q(G)$ and $T(G)$ on a graph G are illustrated in Figure 1.

Eliasi and Taeri [20] introduced four new operations that are based on $S(G)$, $R(G)$, $Q(G)$, $T(G)$ as follows.

Let F be one of the symbols S, R, Q or T. The F-sum, denoted by $G +_F H$, of graphs G and H having orders n_1 and n_2, respectively, is a graph with the set of vertices $V(G +_F H) = (V(G) \cup E(G)) \times V(H)$ and $(u_1, v_1)(u_2, v_2) \in E(G +_F H)$ if and only if $[u_1 = u_2 \in V(G)$ and $v_1v_2 \in E(H)]$ or $[v_1 = v_2 \in V(H)$ and $u_1u_2 \in E(F(G))]$, where $F \in \{S, R, Q, T\}$. $G +_F H$ consists of n_2 copies of the graph $F(G)$, and we label these copies by vertices of H. The vertices in each copy have two types, the vertices in $V(G)$ (black vertices) and the vertices in $E(G)$ (white vertices). Now we join only black vertices with the same name in $F(G)$ in which their corresponding labels are adjacent in H. The graphs $P_2 +_F C_4$ are shown in Figure 2.

Eliasi and Taeri [20] computed the expression for the Wiener index of four graph operations which are based on these graphs $S(G)$, $R(G)$, $Q(G)$, and $T(G)$ in terms of $W(F(G))$ and $W(H)$. Deng et al. [21] computed the first and second Zagreb indices for the graph operations $S(G)$, $R(G)$, $Q(G)$ and $T(G)$. Akhter and Imran computed bounds for the general sum-connectivity index of F-sums of graphs [22]. Some explicit computing formulas for different topological indices of some important graphs can be found in [7, 23–27].

To avoid computational complications, it is important to express the formulas for the product of F-sum of graphs in terms of their factor graphs. So, we presented bounds for the first Zagreb index, the ABC-index, the third Zagreb index, the augmented Zagreb index, the F-index, the first multiple Zagreb index and the GA-index for the Cartesian product of F-sum of graphs in form of its factor graphs.

2 Main results and discussions

This section is meant for determination of bounds for the first Zagreb, the third Zagreb, the augmented Zagreb, the first multiple Zagreb, ABC and GA indices of the Cartesian product of F-sum of graphs in terms of their factor graphs. Bounds for the F-index and the Narumi-Katayama index are also discussed. The following lemmas are useful for determination of these bounds.

In the following lemma, we compute the size of F-sum of graphs for $F = S$.

Lemma 2 *If $G = G_1 +_S G_2$, then the size of G is $n_1m_2 + 2n_2m_1$, where $|V(G_1)| = n_1$, $|V(G_2)| = n_2$, $|E(G_1)| = m_1$ and $|E(G_2)| = m_2$.*

Proof We know that $S(G_1)$ is a subdivision of G_1, therefore the size of $S(G_1)$ is $2|E(G_1)| = 2m_1$.

Hence $|E(G)| = |V(G_1)||E(G_2)| + 2|V(G_2)||E(G_1)| = n_1m_2 + 2n_2m_1$. □

In the following lemma, we compute the size of F-sum of graphs for $F = Q$.

Lemma 3 *If $G = G_1 +_Q G_2$, then the size of G is $n_1m_2 + \frac{n_2m_1(m_1+3)}{2}$, where $|V(G_1)| = n_1$, $|V(G_2)| = n_2$, $|E(G_1)| = m_1$ and $|E(G_2)| = m_2$.*

Proof By using combinations, the size of $Q(G_1) = 2m_1 +^{m_1} C_2 = \frac{m_1(m_1+3)}{2}$.

Hence $|E(G)| = |V(G_1)||E(G_2)| + |V(G_2)||E[Q(G_1)]| = n_1m_2 + \frac{n_2m_1(m_1+3)}{2}$. □

In the following lemma, we compute the size of F-sum of graphs for $F = R$.

Lemma 4 *If $G = G_1 +_R G_2$, then the size of G is $n_1m_2 + 3n_2m_1$, where $|V(G_1)| = n_1$, $|V(G_2)| = n_2$, $|E(G_1)| = m_1$ and $|E(G_2)| = m_2$.*

Proof We know that the size of $R(G_1)$ is equal to three times the size of G_1, therefore the size of $R(G_1)$ is $3|E(G_1)| = 3m_1$.

Hence $|E(G)| = |V(G_1)||E(G_2)| + 3|V(G_2)||E(G_1)| = n_1m_2 + 3n_2m_1$. □

In the following lemma, we compute the size of F-sum of graphs for $F = T$.

Lemma 5 *If $G = G_1 +_T G_2$, then the size of G is $n_1m_2 + \frac{n_2m_1(m_1+5)}{2}$, where $|V(G_1)| = n_1$, $|V(G_2)| = n_2$, $|E(G_1)| = m_1$ and $|E(G_2)| = m_2$.*

Proof By using combinations, the size of $T(G_1) = 3m_1 +^{m_1} C_2 = \frac{m_1(m_1+5)}{2}$.

Hence $|E(G)| = |V(G_1)||E(G_2)| + |V(G_2)||E[T(G_1)]| = n_1m_2 + \frac{n_2m_1(m_1+5)}{2}$. □

Let G_1, G_2, H_1, H_2 be simple, connected graphs such that $|V(G_1)| = n_1$, $|V(G_2)| = n_2$, $|V(H_1)| = n'_1$, $|V(H_2)| = n'_2$, $|E(G_1)| = m_1$, $|E(G_2)| = m_2$, $|E(H_1)| = m'_1$ and $|E(H_2)| = m'_2$.

In the following theorem the lower and upper bounds for the first Zagreb, the third Zagreb, the atom-bond connectivity (ABC), the augmented Zagreb, the first multiple Zagreb and geometric-arithmetic (GA) indices of the Cartesian product of F-sum of graphs in terms of their factor graphs for $F = S$ are determined.

Theorem 1 *Let $G = G_1 +_S H_1$ and $H = G_2 +_S H_2$, then*

(a) $2\alpha(\delta_G + \delta_H) \leq M_1(G\Box H) \leq 2\alpha(\Delta_G + \Delta_H)$,

(b) $\alpha\frac{\sqrt{2(\delta_G+\delta_H-1)}}{\Delta_G+\Delta_H} \leq \mathrm{ABC}(G\Box H) \leq \alpha\frac{\sqrt{2(\Delta_G+\Delta_H-1)}}{\delta_G+\delta_H}$,

(c) $4\alpha(\delta_G + \delta_H)^2 \leq M_3(G\Box H) \leq 4\alpha(\Delta_G + \Delta_H)^2$,

(d) $\frac{1}{8}\alpha[\frac{(\delta_G+\delta_H)^2}{\Delta_G+\Delta_H-1}]^3 \leq \mathrm{AZI}(G\Box H) \leq \frac{1}{8}\alpha[\frac{(\Delta_G+\Delta_H)^2}{\delta_G+\delta_H-1}]^3$,

(e) $2^\alpha(\delta_G + \delta_H)^\alpha \leq \mathrm{PM}_1(G\Box H) \leq 2^\alpha(\Delta_G + \Delta_H)^\alpha$,

(f) $\alpha(\frac{\delta_G+\delta_H}{\Delta_G+\Delta_H}) \leq \mathrm{GA}(G\Box H) \leq \alpha(\frac{\Delta_G+\Delta_H}{\delta_G+\delta_H})$,

where $\alpha = n_1(n'_1 + m'_1)(m_2n'_2 + 2n_2m'_2) + n_2(n'_2 + m'_2)(m_1n'_1 + 2n_1m'_1)$, $\delta_G + \delta_H = \delta_{G_1} + \delta_{G_2} + \delta_{H_1} + \delta_{H_2}$ *and* $\Delta_G + \Delta_H = \Delta_{G_1} + \Delta_{G_2} + \Delta_{H_1} + \Delta_{H_2}$.

Proof Let G and H be the graphs with vertex sets $\{u_1, u_2, \ldots, u_{n_1(n'_1+m'_1)}\}$ and $\{v_1, v_2, \ldots, v_{n_2(n'_2+m'_2)}\}$, respectively. Then

(a) By definition,

$$\begin{aligned}
M_1(G\Box H) &= \sum_{(u_i,v_j)(u_k,v_l)\in E(G\Box H)} \bigl[\deg_{G\Box H}(u_i,v_j) + \deg_{G\Box H}(u_k,v_l)\bigr] \\
&= \sum_{(u_i,v_j)(u_k,v_l)\in E(G\Box H), i\neq k} \bigl[\deg_{G\Box H}(u_i,v_j) + \deg_{G\Box H}(u_k,v_l)\bigr] \\
&\quad + \sum_{(u_i,v_j)(u_k,v_l)\in E(G\Box H), j\neq l} \bigl[\deg_{G\Box H}(u_i,v_j) + \deg_{G\Box H}(u_k,v_l)\bigr], \qquad (1) \\
M_1(G\Box H) &= \sum_{u_i\in V(G)} \sum_{v_jv_l\in E(H)} \bigl[\deg_{G\Box H}(u_i,v_j) + \deg_{G\Box H}(u_i,v_l)\bigr] \\
&\quad + \sum_{v_j\in V(H)} \sum_{u_iu_k\in E(G)} \bigl[\deg_{G\Box H}(u_i,v_j) + \deg_{G\Box H}(u_k,v_j)\bigr].
\end{aligned}$$

By using Lemma 1, part (b), we obtain

$$\deg_{G\Box H}(u_i,v_j) + \deg_{G\Box H}(u_k,v_l) = \deg_G(u_i) + \deg_H(v_j) + \deg_G(u_k) + \deg_H(v_l).$$

Since, for any vertex $u \in V(G)$, $\deg_G(u) \leq \Delta_G$ and $\deg_G(u) \geq \delta_G$, therefore, by using these facts, we obtain

$$\deg_{G\Box H}(u_i,v_j) + \deg_{G\Box H}(u_k,v_l) \leq \Delta_G + \Delta_H + \Delta_G + \Delta_H,$$

which implies the inequality

$$\deg_{G\Box H}(u_i,v_j) + \deg_{G\Box H}(u_k,v_l) \leq 2(\Delta_G + \Delta_H). \qquad (2)$$

By using inequality (2) in equation (1), we obtain

$$\begin{aligned} M_1(G\Box H) &= \sum_{(u_i,v_j)(u_k,v_l)\in E(G\Box H),i\neq k} \big[\deg_{G\Box H}(u_i,v_j)+\deg_{G\Box H}(u_k,v_l)\big] \\ &\quad + \sum_{(u_i,v_j)(u_k,v_l)\in E(G\Box H),j\neq l} \big[\deg_{G\Box H}(u_i,v_j)+\deg_{G\Box H}(u_k,v_l)\big] \\ &= \sum_{u_i\in V(G)}\sum_{v_jv_l\in E(H)} \big[\deg_{G\Box H}(u_i,v_j)+\deg_{G\Box H}(u_i,v_l)\big] \\ &\quad + \sum_{v_j\in V(H)}\sum_{u_iu_k\in E(G)} \big[\deg_{G\Box H}(u_i,v_j)+\deg_{G\Box H}(u_k,v_j)\big] \\ &\leq \big|V(G)\big|\big|E(H)\big|2(\Delta_G+\Delta_H)+\big|E(G)\big|\big|V(H)\big|2(\Delta_G+\Delta_H). \end{aligned}$$

Since $|V(G)| = n_1(n_1' + m_1')$, $|V(H)| = n_2(n_2' + m_2')$, $|E(G)| = m_1n_1' + 2n_1m_1'$, $|E(H)| = m_2n_2' + 2n_2m_2'$, $\Delta_G = \Delta_{G_1} + \Delta_{H_1}$ and $\Delta_H = \Delta_{G_2} + \Delta_{H_2}$, therefore we obtain

$$\begin{aligned} M_1(G\Box H) &\leq 2\big[n_1\big(n_1'+m_1'\big)\big(m_2n_2'+2n_2m_2'\big)+n_2\big(n_2'+m_2'\big)\big(m_1n_1'+2n_1m_1'\big)\big] \\ &\quad \times (\Delta_{G_1}+\Delta_{H_1}+\Delta_{G_2}+\Delta_{H_2}). \end{aligned} \tag{3}$$

By using similar arguments with $\deg_G(u) \geq \delta_G$, we obtain

$$\begin{aligned} M_1(G\Box H) &\geq 2\big[n_1\big(n_1'+m_1'\big)\big(m_2n_2'+2n_2m_2'\big)+n_2\big(n_2'+m_2'\big)\big(m_1n_1'+2n_1m_1'\big)\big] \\ &\quad \times (\delta_{G_1}+\delta_{H_1}+\delta_{G_2}+\delta_{H_2}). \end{aligned} \tag{4}$$

Hence part (a) of the theorem is proved by substituting $n_1(n_1' + m_1')(m_2n_2' + 2n_2m_2') + n_2(n_2' + m_2')(m_1n_1' + 2n_1m_1') = \alpha$ in inequalities (3) and (4).

(b) By definition,

$$\mathrm{ABC}(G\Box H) = \sum_{(u_i,v_j)(u_k,v_l)\in E(G\Box H)} \sqrt{\frac{\deg_{G\Box H}(u_i,v_j)+\deg_{G\Box H}(u_k,v_l)-2}{\deg_{G\Box H}(u_i,v_j)\deg_{G\Box H}(u_k,v_l)}}.$$

$$\begin{aligned} \mathrm{ABC}(G\Box H) &= \sum_{u_i\in V(G)}\sum_{v_jv_l\in E(H)} \sqrt{\frac{\deg_{G\Box H}(u_i,v_j)+\deg_{G\Box H}(u_i,v_l)-2}{\deg_{G\Box H}(u_i,v_j)\deg_{G\Box H}(u_i,v_l)}} \\ &\quad + \sum_{v_j\in V(H)}\sum_{u_iu_k\in E(G)} \sqrt{\frac{\deg_{G\Box H}(u_i,v_j)+\deg_{G\Box H}(u_k,v_j)-2}{\deg_{G\Box H}(u_i,v_j)\deg_{G\Box H}(u_k,v_j)}}. \end{aligned} \tag{5}$$

By using Lemma 1, part (b), we obtain

$$\deg_{G\Box H}(u_i,v_j)\deg_{G\Box H}(u_k,v_l) = \big[\deg_G(u_i)+\deg_H(v_j)\big]\big[\deg_G(u_k)+\deg_H(v_l)\big].$$

Since, for any vertex $u \in V(G)$, $\deg_G(u) \leq \Delta_G$ and $\deg_G(u) \geq \delta_G$, therefore, by using these facts, we obtain

$$\deg_{G\Box H}(u_i,v_j)\deg_{G\Box H}(u_k,v_l) \leq (\Delta_G+\Delta_H)(\Delta_G+\Delta_H),$$

which implies the inequality

$$\deg_{G\square H}(u_i,v_j)\deg_{G\square H}(u_k,v_l)\le(\Delta_G+\Delta_H)^2. \tag{6}$$

By using inequalities (2) and (6) in equation (5), we obtain

$$\begin{aligned}
\mathrm{ABC}(G\square H) &= \sum_{(u_i,v_j)(u_k,v_l)\in E(G\square H),i\neq k}\sqrt{\frac{\deg_{G\square H}(u_i,v_j)+\deg_{G\square H}(u_k,v_l)-2}{\deg_{G\square H}(u_i,v_j)\deg_{G\square H}(u_k,v_l)}}\\
&\quad+\sum_{(u_i,v_j)(u_k,v_l)\in E(G\square H),j\neq l}\sqrt{\frac{\deg_{G\square H}(u_i,v_j)+\deg_{G\square H}(u_k,v_l)-2}{\deg_{G\square H}(u_i,v_j)\deg_{G\square H}(u_k,v_l)}}\\
&=\sum_{u_i\in V(G)}\sum_{v_jv_l\in E(H)}\sqrt{\frac{\deg_{G\square H}(u_i,v_j)+\deg_{G\square H}(u_i,v_l)-2}{\deg_{G\square H}(u_i,v_j)\deg_{G\square H}(u_i,v_l)}}\\
&\quad+\sum_{v_j\in V(H)}\sum_{u_iu_k\in E(G)}\sqrt{\frac{\deg_{G\square H}(u_i,v_j)+\deg_{G\square H}(u_i,v_l)-2}{\deg_{G\square H}(u_i,v_j)\deg_{G\square H}(u_i,v_l)}}\\
&\le\left|V(G)\right|\left|E(H)\right|\sqrt{\frac{2(\Delta_G+\Delta_H)-2}{(\Delta_G+\Delta_H)^2}}+\left|E(G)\right|\left|V(H)\right|\sqrt{\frac{2(\Delta_G+\Delta_H)-2}{(\Delta_G+\Delta_H)^2}}.
\end{aligned}$$

Since $|V(G)|=n_1(n_1'+m_1')$, $|V(H)|=n_2(n_2'+m_2')$, $|E(G)|=m_1n_1'+2n_1m_1'$, $|E(H)|=m_2n_2'+2n_2m_2'$, $\Delta_G=\Delta_{G_1}+\Delta_{H_1}$ and $\Delta_H=\Delta_{G_2}+\Delta_{H_2}$, therefore we get

$$\begin{aligned}
\mathrm{ABC}(G\square H)&\le\left[n_1\left(n_1'+m_1'\right)\left(m_2n_2'+2n_2m_2'\right)+n_2\left(n_2'+m_2'\right)\left(m_1n_1'+2n_1m_1'\right)\right]\\
&\quad\times\frac{\sqrt{2(\Delta_{G_1}+\Delta_{G_2}+\Delta_{H_1}+\Delta_{H_2}-1)}}{\delta_{G_1}+\delta_{G_2}+\delta_{H_1}+\delta_{H_2}}.
\end{aligned}\tag{7}$$

By using similar arguments with $\deg_G(u)\ge\delta_G$, we obtain

$$\begin{aligned}
\mathrm{ABC}(G\square H)&\ge\left[n_1\left(n_1'+m_1'\right)\left(m_2n_2'+2n_2m_2'\right)+n_2\left(n_2'+m_2'\right)\left(m_1n_1'+2n_1m_1'\right)\right]\\
&\quad\times\frac{\sqrt{2(\delta_{G_1}+\delta_{G_2}+\delta_{H_1}+\delta_{H_2}-1)}}{\Delta_{G_1}+\Delta_{G_2}+\Delta_{H_1}+\Delta_{H_2}}.
\end{aligned}\tag{8}$$

Hence from inequalities (7) and (8), part (b) of the theorem is proved.

(c) By definition,

$$M_3(G\square H)=\sum_{(u_i,v_j)(u_k,v_l)\in E(G\square H)}\left[\deg_{G\square H}(u_i,v_j)+\deg_{G\square H}(u_k,v_l)\right]^2,$$

$$\begin{aligned}
M_3(G\square H)&=\sum_{u_i\in V(G)}\sum_{v_jv_l\in E(H)}\left[\deg_{G\square H}(u_i,v_j)+\deg_{G\square H}(u_i,v_l)\right]^2\\
&\quad+\sum_{v_j\in V(H)}\sum_{u_iu_k\in E(G)}\left[\deg_{G\square H}(u_i,v_j)+\deg_{G\square H}(u_k,v_j)\right]^2.
\end{aligned}\tag{9}$$

By using inequality (2) in equation (9) and adopting the same procedure as in part (a) of this theorem,

$$M_3(G\square H) \leq 4\big[n_1\big(n_1'+m_1'\big)\big(m_2n_2'+2n_2m_2'\big)+n_2\big(n_2'+m_2'\big)\big(m_1n_1'+2n_1m_1'\big)\big] \times (\Delta_{G_1}+\Delta_{H_1}+\Delta_{G_2}+\Delta_{H_2})^2 \tag{10}$$

and

$$M_3(G\square H) \geq 4\big[n_1\big(n_1'+m_1'\big)\big(m_2n_2'+2n_2m_2'\big)+n_2\big(n_2'+m_2'\big)\big(m_1n_1'+2n_1m_1'\big)\big] \times (\delta_{G_1}+\delta_{H_1}+\delta_{G_2}+\delta_{H_2})^2. \tag{11}$$

Inequalities (10) and (11) complete the proof of part (c) of the theorem.

(d) By definition,

$$\mathrm{AZI}(G\square H) = \sum_{(u_i,v_j)(u_k,v_l)\in E(G\square H)} \left[\frac{\deg_{G\square H}(u_i,v_j).\deg_{G\square H}(u_k,v_l)}{\deg_{G\square H}(u_i,v_j)+\deg_{G\square H}(u_k,v_l)-2}\right]^3.$$

$$\begin{aligned}\mathrm{AZI}(G\square H) = {} & \sum_{u_i\in V(G)}\sum_{v_jv_l\in E(H)} \left[\frac{\deg_{G\square H}(u_i,v_j)\deg_{G\square H}(u_k,v_l)}{\deg_{G\square H}(u_i,v_j)+\deg_{G\square H}(u_k,v_l)-2}\right]^3 \\ & + \sum_{v_j\in V(H)}\sum_{u_iu_k\in E(G)} \left[\frac{\deg_{G\square H}(u_i,v_j)\deg_{G\square H}(u_k,v_l)}{\deg_{G\square H}(u_i,v_j)+\deg_{G\square H}(u_k,v_l)-2}\right]^3.\end{aligned} \tag{12}$$

By using inequalities (2) and (6) in equation (12) and adopting the same procedure as in parts (a) and (b) of this theorem, we obtain

$$\mathrm{AZI}(G\square H) \leq \frac{1}{8}\big[n_1\big(n_1'+m_1'\big)\big(m_2n_2'+2n_2m_2'\big)+n_2\big(n_2'+m_2'\big)\big(m_1n_1'+2n_1m_1'\big)\big] \times \left[\frac{(\Delta_{G_1}+\Delta_{H_1}+\Delta_{G_2}+\Delta_{H_2})^2}{\delta_{G_1}+\delta_{H_1}+\delta_{G_2}+\delta_{H_2}-1}\right]^3 \tag{13}$$

and

$$\mathrm{AZI}(G\square H) \geq \frac{1}{8}\big[n_1\big(n_1'+m_1'\big)\big(m_2n_2'+2n_2m_2'\big)+n_2\big(n_2'+m_2'\big)\big(m_1n_1'+2n_1m_1'\big)\big] \times \left[\frac{(\delta_{G_1}+\delta_{H_1}+\delta_{G_2}+\delta_{H_2})^2}{\Delta_{G_1}+\Delta_{H_1}+\Delta_{G_2}+\Delta_{H_2}-1}\right]^3. \tag{14}$$

Inequalities (13) and (14) complete the proof of part (d) of the theorem.

(e) By definition,

$$\mathrm{PM}_1(G\square H) = \prod_{(u_i,v_j)(u_k,v_l)\in E(G\square H)} \big[\deg_{G\square H}(u_i,v_j)+\deg_{G\square H}(u_k,v_l)\big]. \tag{15}$$

By using inequality (2) in equation (15) and adopting the same procedure as in part (i) of this theorem,

$$\mathrm{PM}_1(G\square H) \leq \big[2(\Delta_G+\Delta_H)\big]^{n_1(n_1'+m_1')(m_2n_2'+2n_2m_2')+n_2(n_2'+m_2')(m_1n_1'+2n_1m_1')} \tag{16}$$

and

$$\mathrm{PM}_1(G\Box H) \geq \left[2(\delta_G+\delta_H)\right]^{n_1(n_1'+m_1')(m_2n_2'+2n_2m_2')+n_2(n_2'+m_2')(m_1n_1'+2n_1m_1')}. \tag{17}$$

Hence from inequalities (16) and (17), part (e) of the theorem is proved.

(f) By definition,

$$\mathrm{GA}(G\Box H) = \sum_{(u_i,v_j)(u_k,v_l)\in E(G\Box H)} \frac{2\sqrt{\deg_{G\Box H}(u_i,v_j)\deg_{G\Box H}(u_k,v_l)}}{\deg_{G\Box H}(u_i,v_j)+\deg_{G\Box H}(u_k,v_l)}. \tag{18}$$

By using inequalities (2) and (6) in equation (18) and adopting the same procedure as in part (b) of this theorem,

$$\begin{aligned}\mathrm{GA}(G\Box H) \leq{}& \left[n_1\left(n_1'+m_1'\right)\left(m_2n_2'+2n_2m_2'\right)+n_2\left(n_2'+m_2'\right)\left(m_1n_1'+2n_1m_1'\right)\right]\\ &\times\left(\frac{\Delta_{G_1}+\Delta_{G_2}+\Delta_{H_1}+\Delta_{H_2}}{\delta_{G_1}+\delta_{G_2}+\delta_{H_1}+\delta_{H_2}}\right)\end{aligned} \tag{19}$$

and

$$\begin{aligned}\mathrm{GA}(G\Box H) \geq{}& \left[n_1\left(n_1'+m_1'\right)\left(m_2n_2'+2n_2m_2'\right)+n_2\left(n_2'+m_2'\right)\left(m_1n_1'+2n_1m_1'\right)\right]\\ &\times\left(\frac{\delta_{G_1}+\delta_{G_2}+\delta_{H_1}+\delta_{H_2}}{\Delta_{G_1}+\Delta_{G_2}+\Delta_{H_1}+\Delta_{H_2}}\right).\end{aligned} \tag{20}$$

Hence from inequalities (19) and (20), part (f) of the theorem is proved. □

We determine the lower and upper bounds for the F-index and the Narumi-Katayama index of the Cartesian product of F-sum of graphs in terms of their factor graphs for $F = S$.

Corollary 1 *Let $G = G_1 +_S H_1$ and $H = G_2 +_S H_2$, then*

(a) $2\alpha(\delta_G+\delta_H)^2 \leq F(G\Box H) \leq 2\alpha(\Delta_G+\Delta_H)^2$,

(b) $[2(\delta_G+\delta_H)]^{\frac{\alpha}{2}} \leq \mathrm{NK}(G\Box H) \leq [2(\Delta_G+\Delta_H)]^{\frac{\alpha}{2}}$,

where $\alpha = n_1(n_1'+m_1')(m_2n_2'+2n_2m_2')+n_2(n_2'+m_2')(m_1n_1'+2n_1m_1')$, $\delta_G+\delta_H = \delta_{G_1}+\delta_{G_2}+\delta_{H_1}+\delta_{H_2}$ *and* $\Delta_G+\Delta_H = \Delta_{G_1}+\Delta_{G_2}+\Delta_{H_1}+\Delta_{H_2}$.

Proof (a) By using the relation $F(G) = M_3(G) - 2M_2(G)$, in Theorem 1, we obtain the required result.

(b) By using the relation $\mathrm{NK}(G) = \sqrt{\mathrm{PM}_1(G)}$, in Theorem 1, we obtain the required result. □

In the following theorem the lower and upper bounds for the first Zagreb, the third Zagreb, the atom-bond connectivity (ABC), the augmented Zagreb, the first multiple Zagreb and geometric-arithmetic (GA) indices of the Cartesian product of F-sum of graphs in terms of their factor graphs for $F = Q$ are determined.

Theorem 2 *Let $G = G_1 +_Q H_1$ and $H = G_2 +_Q H_2$, then*

(a) $2\beta(\delta_G+\delta_H) \leq M_1(G\Box H) \leq 2\beta(\Delta_G+\Delta_H)$,

(b) $\beta\frac{\sqrt{2(\delta_G+\delta_H-1)}}{\Delta_G+\Delta_H} \leq \mathrm{ABC}(G\Box H) \leq \beta\frac{\sqrt{2(\Delta_G+\Delta_H-1)}}{\delta_G+\delta_H}$,

(c) $4\beta(\delta_G+\delta_H)^2 \le M_3(G\square H) \le 4\beta(\Delta_G+\Delta_H)^2$,
(d) $\frac{1}{8}\beta[\frac{(\delta_G+\delta_H)^2}{\Delta_G+\Delta_H-1}]^3 \le \mathrm{AZI}(G\square H) \le \frac{1}{8}\beta[\frac{(\Delta_G+\Delta_H)^2}{\delta_G+\delta_H-1}]^3$,
(e) $2^\beta(\delta_G+\delta_H)^\beta \le \mathrm{PM}_1(G\square H) \le 2^\beta(\Delta_G+\Delta_H)^\beta$,
(f) $\beta(\frac{\delta_G+\delta_H}{\Delta_G+\Delta_H}) \le \mathrm{GA}(G\square H) \le \beta(\frac{\Delta_G+\Delta_H}{\delta_G+\delta_H})$,

where $\beta = \frac{1}{2}[n_1(n_1'+m_1')\{2n_2m_2'+n_2'm_2(m_2+3)\}+n_2(n_2'+m_2')\{2n_1m_1'+n_1'm_1(m_1+3)\}]$, $\delta_G+\delta_H=\delta_{G_1}+\delta_{G_2}+\delta_{H_1}+\delta_{H_2}$ *and* $\Delta_G+\Delta_H=\Delta_{G_1}+\Delta_{G_2}+\Delta_{H_1}+\Delta_{H_2}$.

Proof Let G and H be the graphs with vertex sets $\{u_1,u_2,\ldots,u_{n_1(n_1'+m_1')}\}$ and $\{v_1,v_2,\ldots,v_{n_2(n_2'+m_2')}\}$, respectively. The proof is similar to that of Theorem 1 using Lemma (3). □

Corollary 2 *Let* $G=G_1+_Q H_1$ *and* $H=G_2+_Q H_2$, *then*
(a) $2\beta(\delta_G+\delta_H)^2 \le F(G\square H) \le 2\beta(\Delta_G+\Delta_H)^2$,
(b) $[2(\delta_G+\delta_H)]^{\frac{\beta}{2}} \le \mathrm{NK}(G\square H) \le [2(\Delta_G+\Delta_H)]^{\frac{\beta}{2}}$,

where $\beta = \frac{1}{2}[n_1(n_1'+m_1')\{2n_2m_2'+n_2'm_2(m_2+3)\}+n_2(n_2'+m_2')\{2n_1m_1'+n_1'm_1(m_1+3)\}]$, $\delta_G+\delta_H=\delta_{G_1}+\delta_{G_2}+\delta_{H_1}+\delta_{H_2}$ *and* $\Delta_G+\Delta_H=\Delta_{G_1}+\Delta_{G_2}+\Delta_{H_1}+\Delta_{H_2}$.

Proof (a) By using the relation $F(G)=M_3(G)-2M_2(G)$, in Theorem 2, we obtain the required result.

(b) By using the relation $\mathrm{NK}(G)=\sqrt{\mathrm{PM}_1(G)}$, in Theorem 2, we obtain the required result. □

In the following theorem we determine the lower and upper bounds for the first Zagreb, ABC, the third Zagreb, the augmented Zagreb, the first multiple Zagreb and GA indices of the Cartesian product of F-sum of graphs in terms of their factor graphs for $F=R$.

Theorem 3 *Let* $G=G_1+_R H_1$ *and* $H=G_2+_R H_2$, *then*
(a) $2\gamma(\delta_G+\delta_H) \le M_1(G\square H) \le 2\gamma(\Delta_G+\Delta_H)$,
(b) $\gamma\frac{\sqrt{2(\delta_G+\delta_H-1)}}{\Delta_G+\Delta_H} \le \mathrm{ABC}(G\square H) \le \gamma\frac{\sqrt{2(\Delta_G+\Delta_H-1)}}{\delta_G+\delta_H}$.
(c) $4\gamma(\delta_G+\delta_H)^2 \le M_3(G\square H) \le 4\gamma(\Delta_G+\Delta_H)^2$,
(d) $\frac{1}{8}\gamma[\frac{(\delta_G+\delta_H)^2}{\Delta_G+\Delta_H-1}]^3 \le \mathrm{AZI}(G\square H) \le \frac{1}{8}\gamma[\frac{(\Delta_G+\Delta_H)^2}{\delta_G+\delta_H-1}]^3$,
(e) $2^\gamma(\delta_G+\delta_H)^\gamma \le \mathrm{PM}_1(G\square H) \le 2^\gamma(\Delta_G+\Delta_H)^\gamma$,
(f) $\gamma(\frac{\delta_G+\delta_H}{\Delta_G+\Delta_H}) \le \mathrm{GA}(G\square H) \le \gamma(\frac{\Delta_G+\Delta_H}{\delta_G+\delta_H})$,

where $\gamma = n_1(n_1'+m_1')(m_2n_2'+3n_2m_2')+n_2(n_2'+m_2')(m_1n_1'+3n_1m_1')$, $\delta_G+\delta_H=\delta_{G_1}+\delta_{G_2}+\delta_{H_1}+\delta_{H_2}$ *and* $\Delta_G+\Delta_H=\Delta_{G_1}+\Delta_{G_2}+\Delta_{H_1}+\Delta_{H_2}$.

Proof Let G and H be the graphs with vertex sets $\{u_1,u_2,\ldots,u_{n_1(n_1'+m_1')}\}$ and $\{v_1,v_2,\ldots,v_{n_2(n_2'+m_2')}\}$, respectively. The proof is similar to that of Theorem 1 using Lemma (4). □

We determine the lower and upper bounds for the F-index and the Narumi-Katayama index of the Cartesian product of F-sum of graphs in terms of their factor graphs for $F=R$.

Corollary 3 *Let* $G=G_1+_R H_1$ *and* $H=G_2+_R H_2$, *then*
(a) $2\gamma(\delta_G+\delta_H)^2 \le F(G\square H) \le 2\gamma(\Delta_G+\Delta_H)^2$,
(b) $[2(\delta_G+\delta_H)]^{\frac{\gamma}{2}} \le \mathrm{NK}(G\square H) \le [2(\Delta_G+\Delta_H)]^{\frac{\gamma}{2}}$,

where $\gamma = n_1(n_1'+m_1')(m_2n_2'+3n_2m_2')+n_2(n_2'+m_2')(m_1n_1'+3n_1m_1')$, $\delta_G+\delta_H=\delta_{G_1}+\delta_{G_2}+\delta_{H_1}+\delta_{H_2}$ *and* $\Delta_G+\Delta_H=\Delta_{G_1}+\Delta_{G_2}+\Delta_{H_1}+\Delta_{H_2}$.

Proof (a) Using the relation $F(G) = M_3(G) - 2M_2(G)$, in Theorem 3, we get the required result.

(b) Using the relation $\mathrm{NK}(G) = \sqrt{\mathrm{PM}_1(G)}$, in Theorem 3, we get the required result. □

In the following theorem we determine the lower and upper bounds for the first Zagreb, ABC, the third Zagreb, the augmented Zagreb, the first multiple Zagreb and GA indices of the Cartesian product of F-sum of graphs in terms of their factor graphs for $F = T$.

Theorem 4 *Let $G = G_1 +_T H_1$ and $H = G_2 +_T H_2$, then*

(a) $2\eta(\delta_G + \delta_H) \leq M_1(G\Box H) \leq 2\eta(\Delta_G + \Delta_H)$,

(b) $\eta\frac{\sqrt{2(\delta_G+\delta_H-1)}}{\Delta_G+\Delta_H} \leq \mathrm{ABC}(G\Box H) \leq \eta\frac{\sqrt{2(\Delta_G+\Delta_H-1)}}{\delta_G+\delta_H}$,

(c) $4\eta(\delta_G + \delta_H)^2 \leq M_3(G\Box H) \leq 4\eta(\Delta_G + \Delta_H)^2$,

(d) $\frac{1}{8}\eta[\frac{(\delta_G+\delta_H)^2}{\Delta_G+\Delta_H-1}]^3 \leq \mathrm{AZI}(G\Box H) \leq \frac{1}{8}\eta[\frac{(\Delta_G+\Delta_H)^2}{\delta_G+\delta_H-1}]^3$,

(e) $2^\eta(\delta_G + \delta_H)^\eta \leq \mathrm{PM}_1(G\Box H) \leq 2^\eta(\Delta_G + \Delta_H)^\eta$,

(f) $\eta(\frac{\delta_G+\delta_H}{\Delta_G+\Delta_H}) \leq \mathrm{GA}(G\Box H) \leq \eta(\frac{\Delta_G+\Delta_H}{\delta_G+\delta_H})$,

where $\eta = \frac{1}{2}[n_1(n_1' + m_1')\{2n_2m_2' + n_2'm_2(m_2 + 5)\} + n_2(n_2' + m_2')\{2n_1m_1' + n_1'm_1(m_1 + 5)\}]$, $\delta_G + \delta_H = \delta_{G_1} + \delta_{G_2} + \delta_{H_1} + \delta_{H_2}$ *and* $\Delta_G + \Delta_H = \Delta_{G_1} + \Delta_{G_2} + \Delta_{H_1} + \Delta_{H_2}$.

Proof Let G and H be the graphs with vertex sets $\{u_1, u_2, \ldots, u_{n_1(n_1'+m_1')}\}$ and $\{v_1, v_2, \ldots, v_{n_2(n_2'+m_2')}\}$, respectively. The proof is similar to that of Theorem 1 using Lemma (5). □

We determine the lower and upper bounds for the F-index and the Narumi-Katayama index of the Cartesian product of F-sum of graphs in terms of their factor graphs for $F = T$.

Corollary 4 *Let $G = G_1 +_T H_1$ and $H = G_2 +_T H_2$, then*

(a) $2\eta(\delta_G + \delta_H)^2 \leq F(G\Box H) \leq 2\eta(\Delta_G + \Delta_H)^2$,

(b) $[2(\delta_G + \delta_H)]^{\frac{\eta}{2}} \leq \mathrm{NK}(G\Box H) \leq [2(\Delta_G + \Delta_H)]^{\frac{\eta}{2}}$,

where $\eta = \frac{1}{2}[n_1(n_1' + m_1')\{2n_2m_2' + n_2'm_2(m_2 + 5)\} + n_2(n_2' + m_2')\{2n_1m_1' + n_1'm_1(m_1 + 5)\}]$, $\delta_G + \delta_H = \delta_{G_1} + \delta_{G_2} + \delta_{H_1} + \delta_{H_2}$ *and* $\Delta_G + \Delta_H = \Delta_{G_1} + \Delta_{G_2} + \Delta_{H_1} + \Delta_{H_2}$.

Proof (a) Using the relation $F(G) = M_3(G) - 2M_2(G)$, in Theorem 4, we get the required result.

(b) Using the relation $\mathrm{NK}(G) = \sqrt{\mathrm{PM}_1(G)}$, in Theorem 4, we get the required result. □

In the following theorem we determine the lower and upper bounds for the first Zagreb, ABC, the third Zagreb, the augmented Zagreb, the first multiple Zagreb and *GA* indices of the Cartesian product of F-sum of graphs in terms of their factor graphs for $F = S$ and $F = R$.

Theorem 5 *Let $G = G_1 +_S H_1$ and $H = G_2 +_R H_2$, then*

(a) $2\xi(\delta_G + \delta_H) \leq M_1(G\Box H) \leq 2\xi(\Delta_G\Delta_H)$,

(b) $\xi\frac{\sqrt{2(\delta_G\delta_H-1)}}{\Delta_G+\Delta_H} \leq \mathrm{ABC}(G\Box H) \leq \xi\frac{\sqrt{2(\Delta_G+\Delta_H-1)}}{\delta_G+\delta_H}$,

(c) $4\xi(\delta_G + \delta_H)^2 \leq M_3(G\Box H) \leq 4\xi(\Delta_G + \Delta_H)^2$,

(d) $\frac{1}{8}\xi[\frac{(\delta_G\delta_H)^2}{\Delta_G+\Delta_H-1}]^3 \leq \mathrm{AZI}(G\Box H) \leq \frac{1}{8}\xi[\frac{(\Delta_G+\Delta_H)^2}{\delta_G+\delta_H-1}]^3$,

(e) $[2(\delta_G + \delta_H)]^\xi \leq \mathrm{PM}_1(G\Box H) \leq [2(\Delta_G + \Delta_H)]^\xi$,

(f) $\xi\frac{\delta_G+\delta_H}{\Delta_G+\Delta_H} \leq \mathrm{GA}(G\Box H) \leq \xi\frac{\Delta_G+\Delta_H}{\delta_G+\delta_H}$, *where* $\xi = n_1(n'_1 + m'_1)(m_2n'_2 + 2n_2m'_2) + n_2(n'_2 + m'_2)(m_1n'_1 + 3n_1m'_1)$, $\delta_G + \delta_H = \delta_{G_1} + \delta_{G_2} + \delta_{H_1} + \delta_{H_2}$ *and* $\Delta_G + \Delta_H = \Delta_{G_1} + \Delta_{G_2} + \Delta_{H_1} + \Delta_{H_2}$.

Proof Let G and H be the graphs with vertex sets $\{u_1, u_2, \ldots, u_{n_1(n'_1+m'_1)}\}$ and $\{v_1, v_2, \ldots, v_{n_2(n'_2+m'_2)}\}$, respectively. The proof is similar to that of 1 with $|E(G)| = m_1n'_1 + 2n_1m'_1$ and $|E(H)| = m_2n'_2 + 3n_2m'_2$. □

We determine the lower and upper bounds for the F-index and the Narumi-Katayama index of the Cartesian product of F-sum of graphs in terms of their factor graphs for $F = S$ and $F = R$.

Corollary 5 *Let* $G = G_1 +_S H_1$ *and* $H = G_2 +_R H_2$, *then*

(a) $2\xi(\delta_G\delta_H)^2 \leq F(G\Box H) \leq 2\xi(\Delta_G + \Delta_H)^2$,

(b) $[2(\delta_G + \delta_H)]^{\frac{\xi}{2}} \leq \mathrm{NK}(G\Box H) \leq [2(\Delta_G + \Delta_H)]^{\frac{\xi}{2}}$,

where $\xi = n_1(n'_1 + m'_1)(m_2n'_2 + 3n_2m'_2) + n_2(n'_2 + m'_2)(m_1n'_1 + 2n_1m'_1)$, $\delta_G + \delta_H = \delta_{G_1} + \delta_{G_2} + \delta_{H_1} + \delta_{H_2}$ *and* $\Delta_G + \Delta_H = \Delta_{G_1} + \Delta_{G_2} + \Delta_{H_1} + \Delta_{H_2}$.

Proof (a) Using the relation $F(G) = M_3(G) - 2M_2(G)$, in Theorem 5, we get the required result.

(b) Using the relation $\mathrm{NK}(G) = \sqrt{\mathrm{PM}_1(G)}$, in Theorem 5, we get the required result. □

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The idea to obtain the lower and upper bounds of Zagreb indices, multiple Zagreb indices, the ABC-index, the GA-index, the forgotten topological index and the Narumi-Katayama index for the Cartesian product of F-sum of connected graphs was proposed by MI and HMAS. After several discussions, SB, HMAS and MKS obtained some sharp lower and upper bounds. MI and HMAS checked these results and suggested improving them. The first draft was prepared by SB and was verified and improved by MI and MKS. The final version was prepared by HMAS and SB. All authors read and approved the final manuscript.

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