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An inequality for generalized complete elliptic integral

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Abstract
In this paper, we show an elegant inequality involving the ratio of generalized complete elliptic integrals of the first kind and generalize an interesting result of Alzer.

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1 Introduction

The generalized complete elliptic integral of the first kind is defined for $r \in (0,1)$ by

$$K_p(r) = \int_0^{\frac{\pi_p}{2}} \frac{d\theta}{(1 - r^p \sin_p^p \theta)^{1-\frac{1}{p}}} = \int_0^1 \frac{dt}{(1 - t^p)^{\frac{1}{p}}(1 - r^p t^p)^{1-\frac{1}{p}}},$$

where $\sin_p \theta$ is the generalized trigonometric function and

$$\pi_p = 2\int_0^1 \frac{dt}{(1 - t^p)^{\frac{1}{p}}} = \frac{2}{p} B\left(\frac{1}{p}, 1 - \frac{1}{p}\right).$$

The function $\sin_p \theta$ and the number π_p play important roles in expressing the solutions of inhomogeneous eigenvalue problem of p-Laplacian $-(|u'|^{p-2}u')' = \lambda |u|^{p-2}u$ with a boundary condition. These functions have some applications in the quasi-conformal theory, geometric function theory and the theory of Ramanujan modular equation. Báricz [1] established some Turán type inequalities for a Gauss hypergeometric function and for a generalized complete elliptic integral and showed a sharp bound for the generalized complete elliptic integral of the first kind in 2007. In 2012, Bhayo and Vuorinen [2] dealt with generalized elliptic integrals and generalized modular functions. Several new inequalities are given for these and related functions.

For more details on monotonicity, inequalities and convexity and concavity of these functions, the reader may refer to [3–5] and [6] and the references therein.

In 1990, Anderson et al. [7] presented the following inequality:

$$\frac{K(r)}{K(\sqrt{r})} > \frac{1}{1+r} \quad \text{for } r \in (0,1). \tag{1.1}$$

Inspired by this work, Alzer and Richards [6] gave the refinement of (1.1): for all $r \in (0,1)$, the following inequality

$$\frac{K(r)}{K(\sqrt{r})} > \frac{1}{1+\frac{r}{4}} \tag{1.2}$$

holds true.

It is natural how inequality (1.2) is generalized to $K_p(r)$. Our main result reads as follows.

Theorem 1.1 *For $r \in (0,1)$ and $p \in [1,2]$, we have*

$$\frac{1}{1+\lambda_p r} < \frac{K_p(r)}{K_p(\sqrt{r})} < \frac{1}{1+u_p r}, \tag{1.3}$$

where the constants $\lambda_p = \frac{1}{p}(1-\frac{1}{p})$ and $u_p = 0$ are the best possible.

2 Lemmas

Lemma 2.1 *The function $\Delta(x) = \frac{1+ax}{1+bx}$ $(1+bx \neq 0)$ is strictly increasing (decreasing) in $(0,\infty)$ if and only if $a-b>0$ $(a-b<0)$.*

Proof Simple computation yields

$$\frac{d}{dx}\left(\frac{1+ax}{1+bx}\right) = \frac{a-b}{(1+bx)^2}.$$

The proof is complete. □

Lemma 2.2 (Lemma 2.1 in [8]) *The psi function $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ is strictly concave on $(0,\infty)$ and satisfies the duplication formula*

$$\psi(2x) = \frac{1}{2}\psi(x) + \frac{1}{2}\psi\left(x+\frac{1}{2}\right) + \log 2 \tag{2.1}$$

for $x>0$.

Lemma 2.3 (Lemma 3 in [9]) *For $x>0$, we have*

$$\ln x - \frac{1}{x} < \psi(x) < \ln x - \frac{1}{2x}. \tag{2.2}$$

Lemma 2.4 *For $x>0$ and $p \in [1,2]$, we have*

$$\psi\left(x+\frac{1}{p}\right) + \psi\left(x+1-\frac{1}{p}\right) > \psi\left(x+\frac{1}{2}-\frac{1}{2p}\right) + \psi\left(x+\frac{1}{2}+\frac{1}{2p}\right). \tag{2.3}$$

Proof Using Lemma 2.3, we only need to prove the following inequality:

$$\ln\left[\frac{(x+\frac{1}{p})(x+1-\frac{1}{p})}{(x+\frac{1}{2}-\frac{1}{2p})(x+\frac{1}{2}+\frac{1}{2p})}\right] + \frac{1}{x+1-\frac{1}{p}} - \frac{1}{x+\frac{1}{p}} > 0.$$

For $p \in [1,2]$, we easily obtain $\frac{1}{p} \geq 1 - \frac{1}{p}$. So, we have

$$\frac{1}{x+1-\frac{1}{p}} - \frac{1}{x-\frac{1}{p}} > 0.$$

On the other hand,

$$\begin{aligned}
&\frac{(x+\frac{1}{p})(x+1-\frac{1}{p})}{(x+\frac{1}{2}-\frac{1}{2p})(x+\frac{1}{2}+\frac{1}{2p})} \geq 1 \\
&\quad\Leftrightarrow \quad \frac{1}{p}\left(1-\frac{1}{p}\right) \geq \frac{1}{4}\left(1-\frac{1}{p}\right)\left(1+\frac{1}{p}\right) \\
&\quad\Leftrightarrow \quad 3\left(\frac{1}{p}\right)^2 - 4\frac{1}{p} + 1 \leq 0.
\end{aligned}$$

So, we complete the proof. □

Lemma 2.5 *We have*

$$\lim_{r\to 1} \frac{K_p(r)}{K_p(\sqrt[p]{r})} = 1. \tag{2.4}$$

Proof Applying the asymptotic formula ([10], equality (2))

$$F(a,b;a+b;x) \sim -\frac{1}{B(a,b)} \log(1-x) \quad (x \to 1) \tag{2.5}$$

and expression [10]

$$K_p(r) = \frac{\pi_p}{2} F\left(\frac{1}{p}, 1-\frac{1}{p}; 1, r^p\right), \tag{2.6}$$

where $F(a;b;c;z)$ and $B(x,y)$ denote a classical hypergeometric function and a beta function, respectively, we obtain

$$\lim_{r\to 1} \frac{F(a,b;a+b;r^p)}{F(a,b;a+b;r)} = 1. \tag{2.7}$$

Putting $a = \frac{1}{p}$ and $b = 1 - \frac{1}{p}$, we complete the proof. □

3 Proof of Theorem 1.1

Define

$$f_p(r) = \frac{2}{\pi_p}\left[1 + \frac{1}{p}\left(1-\frac{1}{p}\right)r\right] K_p(r)$$

and

$$g_p(r) = \frac{2}{\pi_p} K_p(\sqrt[p]{r}).$$

By applying (2.6), we get

$$f_p(r) = \sum_{n=0}^{\infty}(1+\lambda_p r)r^{pn}$$

and

$$g_p(r) = \sum_{n=0}^{\infty}(a_{2n}+a_{2n+1}r)r^{2n},$$

where $a_n = \frac{(\frac{1}{p})_n(1-\frac{1}{p})_n}{(n!)^2}$ and $(r)_n = r(r+1)\cdots(r+n-1)$.

Because of $1 \leq p \leq 2$, we have

$$\frac{f_p(r)}{g_p(r)} \geq \frac{\sum_{n=0}^{\infty}(1+\lambda_p r)a_n r^{pn}}{\sum_{n=0}^{\infty}(a_{2n}+a_{2n+1}r)r^{pn}}.$$

Let

$$\theta_{p,n}(r) = \frac{(1+\lambda_p r)a_n}{a_{2n}+a_{2n+1}r}.$$

Simple computation results in

$$\begin{aligned}\theta_{p,n}(r) &= \frac{(1+\frac{1}{p}(1-\frac{1}{p})r)}{1+\frac{(\frac{1}{p}+2n)(1-\frac{1}{p}+2n)}{(2n+1)^2}}\cdot\frac{a_n}{a_{2n}} \\ &\geq \frac{(1+\frac{1}{p}(1-\frac{1}{p})r)}{1+\frac{(\frac{1}{p}+2n)(1-\frac{1}{p}+2n)}{(2n+1)^2}}\cdot\frac{a_n}{a_{2n}}\end{aligned}$$

by using Lemma 2.1.

(In fact, we easily know

$$\begin{aligned}&\frac{1}{p}\left(1-\frac{1}{p}\right) \leq \frac{1}{4} \leq \frac{(\frac{1}{p}+2n)(1-\frac{1}{p}+2n)}{(2n+1)^2} \\ &\quad\Leftrightarrow\quad 4n^2+4n+1 \leq 4\left(\frac{1}{p}+2n\right)\left(1-\frac{1}{p}+2n\right).\end{aligned}$$

Next, considering $1 \leq p \leq 2$, we only need to prove $12n^2+4n-1 \leq 0$. It is obvious.)

Setting

$$Q_p(x) = \frac{\Gamma(\frac{1}{p}+x)\Gamma(1-\frac{1}{p}+x)\Gamma^2(2x+1)}{\Gamma^2(x+1)\Gamma(\frac{1}{p}+2x)\Gamma(1-\frac{1}{p}+2x)},$$

we have

$$\begin{aligned}\frac{Q_p'(x)}{Q_p(x)} &= \psi\left(\frac{1}{p}+x\right)+\psi\left(1-\frac{1}{p}+x\right)+4\psi(2x+1)-2\psi(x+1) \\ &\quad -2\psi\left(2x+\frac{1}{2}\right)-2\psi\left(2x+1-\frac{1}{p}\right).\end{aligned}$$

Using Lemma 2.2, we easily get

$$\begin{aligned}\frac{Q_p'(x)}{Q_p(x)} &= \psi\left(x+\frac{1}{p}\right)+\psi\left(x+1-\frac{1}{p}\right)+2\psi\left(x+\frac{1}{2}\right)-\psi\left(x+\frac{1}{2p}\right)\\ &\quad -\psi\left(x+\frac{1}{2}+\frac{1}{2p}\right)-\psi\left(x+\frac{1}{2}-\frac{1}{2p}\right)-\psi\left(x+1-\frac{1}{2p}\right).\end{aligned}$$

Applying Lemma 2.2 again, we have

$$\frac{1}{2}\left[\psi\left(x+\frac{1}{2p}\right)+\psi\left(x+1-\frac{1}{2p}\right)\right]\le\psi\left(x+\frac{1}{2}\right). \tag{3.1}$$

Hence, we have

$$\frac{Q_p'(x)}{Q_p(x)}>0.$$

It follows that the function $Q_p(x)$ is increasing in $x\in(0,\infty)$. So, we have

$$\frac{a_{2n}}{a_{2n+1}}=Q_p(n)>Q_p(1)=\frac{4}{(\frac{1}{p}+1)(2-\frac{1}{p})}\ge\frac{16}{9},$$

where we apply

$$\left(\frac{1}{p}+1\right)\left(2-\frac{1}{p}\right)\le\left(\frac{\frac{1}{p}+1+2-\frac{1}{p}}{2}\right)^2=\left(\frac{3}{2}\right)^2=\frac{9}{4}.$$

Hence, we obtain

$$\begin{aligned}Q_{p,n}(r)&\ge\frac{16}{9}\frac{1+\frac{1}{p}(1-\frac{1}{p})}{1+\frac{(\frac{1}{p}+2n)(1-\frac{1}{p}+2n)}{(2n+1)^2}}\\ &\ge\frac{16}{9}\frac{1+\frac{1}{4}}{1+(\frac{4n+1}{4n+2})^2}\\ &=1+\frac{32n^2+104n+35}{9(32n^2+24n+5)}\\ &>1,\end{aligned}$$

and $f_p(r)>g_p(r)$.

On the other hand, since the function $K_p(r)$ is strictly increasing on $r\in(0,1)$, we have

$$\frac{K_p(r)}{K_p(\sqrt[p]{r})}<1.$$

Hence, we rewrite formula (1.3) as

$$u<u_p(r)=\frac{\frac{K_p(\sqrt[p]{r})}{K_p(r)}-1}{r}<\lambda.$$

Simple calculation leads to

$$u_p(r) = \frac{1}{r}\left(\frac{\sum_{n=0}^{\infty} \frac{(\frac{1}{p})_n(1-\frac{1}{p})_n}{(1)_n}\frac{r^n}{n!}}{\sum_{n=0}^{\infty} \frac{(\frac{1}{p})_n(1-\frac{1}{p})_n}{(1)_n}\frac{r^{pn}}{n!}} - 1\right)$$

and

$$\lim_{r\to 0} u_p(r) = \frac{1}{p}\left(1 - \frac{1}{p}\right),$$
$$\lim_{r\to 1} u_p(r) = 0.$$

The proof is complete.

4 Conclusions

We show an elegant inequality involving the ratio of generalized complete elliptic integrals of the first kind and generalize an interesting result of Alzer.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

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