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On rational bounds for the gamma function

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Abstract

In the article, we prove that the double inequality

$$\frac{x^2 + p_0}{x + p_0} < \Gamma(x + 1) < \frac{x^2 + 9/5}{x + 9/5}$$

holds for all $x \in (0, 1)$, we present the best possible constants λ and μ such that

$$\frac{\lambda(x^2 + 9/5)}{x + 9/5} \leq \Gamma(x + 1) \leq \frac{\mu(x^2 + p_0)}{x + p_0}$$

for all $x \in (0, 1)$, and we find the value of x^* in the interval $(0, 1)$ such that $\Gamma(x + 1) > (x^2 + 1/\gamma)/(x + 1/\gamma)$ for $x \in (0, x^*)$ and $\Gamma(x + 1) < (x^2 + 1/\gamma)/(x + 1/\gamma)$ for $x \in (x^*, 1)$, where $\Gamma(x)$ is the classical gamma function, $\gamma = \lim_{n \to \infty}(\sum_{k=1}^{n} 1/k - \log n) = 0.577\ldots$ is Euler-Mascheroni constant and $p_0 = \gamma/(1 - \gamma) = 1.365\ldots$.

MSC: 41A60; 33B15; 26D07

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1 Introduction

For $x > 0$, the classical Euler gamma function $\Gamma(x)$ and its logarithmic derivative, the so-called psi function $\psi(x)$ [1] are defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt, \qquad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)},$$

respectively.

A real-valued function f is said to be completely monotonic [2] on an interval I if f has derivatives of all orders on I and $(-1)^n f^{(n)}(x) \geq 0$ for all $n \geq 0$ and $x \in I$. The well-known Bernstein theorem [3] states that a function f on $[0, \infty)$ is completely monotonic if and only if there exists a bounded and non-decreasing function $\omega(t)$ such that $f(x) = \int_0^\infty e^{-xt} \, d\omega(t)$ converges for all $x \in [0, \infty)$.

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Recently, the gamma function have attracted the attention of many researchers. In particular, many remarkable inequalities and properties for $\Gamma(x)$ can be found in the literature [4–14].

Due to $\Gamma(x + 1) = x\Gamma(x)$ and $\Gamma(n + 1) = n!$, we will only need to focus our attention on $\Gamma(x + 1)$ with $x \in (0, 1)$. Gautschi [15] proved that the double inequality

$$n^{1-s} < \frac{\Gamma(n+1)}{\Gamma(n+s)} < e^{(1-s)\psi(n+1)} \tag{1.1}$$

holds for all $s \in (0, 1)$ and $n \in \mathbb{N}$.

Inequality (1.1) was generalized and improved by Kershaw [16] as follows:

$$\left(x + \frac{s}{2}\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < e^{(1-s)\psi[x+(1+s)/2]}$$

for all $x > 0$ and $s \in (0, 1)$.

Elezović, Giordano and Pečarić [17] established the double inequality

$$\left(\frac{1}{2} + \sqrt{\frac{1}{4} + x}\right)^{1-x} x^x < \Gamma(x+1) < 2^{1-x} x^x \tag{1.2}$$

for the gamma function being valid for all $x \in (0, 1)$, and asked for 'other bounds for the gamma function in terms of elementary functions'.

Ivády [18] provided the bounds for gamma function in terms of very simple rational functions as follows:

$$\frac{x^2 + 1}{x + 1} < \Gamma(x+1) < \frac{x^2 + 2}{x + 2} \tag{1.3}$$

for all $x \in (0, 1)$. Inequality (1.3) can be regarded as a simple estimation of the value of the gamma function.

In [19], Zhao, Guo and Qi proved that the function

$$x \to Q(x) = \frac{\log\Gamma(x+1)}{\log(x^2 + 1) - \log(x + 1)}$$

is strictly increasing on $(0, 1)$. The monotonicity of $Q(x)$ on the interval $(0, 1)$ and the facts that $Q(0^+) = \gamma$ and $Q(1^-) = 2(1 - \gamma)$ lead to the conclusion that

$$\left(\frac{x^2 + 1}{x + 1}\right)^{2(1-\gamma)} < \Gamma(x+1) < \left(\frac{x^2 + 1}{x + 1}\right)^{\gamma} \tag{1.4}$$

for all $x \in (0, 1)$, where $\gamma = \lim_{n\to\infty}(\sum_{k=1}^{n} 1/k - \log n) = 0.577\ldots$ is the Euler-Mascheroni constant.

Let

$$L_1(x) = \frac{x^2 + 1}{x + 1}, \qquad L_2(x) = \left(\frac{x^2 + 1}{x + 1}\right)^{2(1-\gamma)}, \tag{1.5}$$

$$U_1(x) = \frac{x^2 + 2}{x + 2}, \qquad U_2(x) = \left(\frac{x^2 + 1}{x + 1}\right)^{\gamma}. \tag{1.6}$$

Then we clearly see that

$$L_1(x) < L_2(x) \tag{1.7}$$

for all $x \in (0,1)$, and numerical computations show that

$$\begin{aligned}
U_1(1/4) &= 0.916\ldots > U_2(1/4) = 0.910\ldots, \\
U_1(1/8) &= 0.948\ldots > U_2(1/8) = 0.942\ldots.
\end{aligned} \tag{1.8}$$

Motivated by (1.3)-(1.8), it is natural to ask what the better parameters p and q on the interval $(1,2)$ are such that the double inequality

$$\frac{x^2 + p}{x + p} < \Gamma(x+1) < \frac{x^2 + q}{x + q}$$

holds for all $x \in (0,1)$. The main purpose of the article is to deal with this questions. Some complicated computations are carried out using the Mathematica computer algebra system.

2 Lemmas

In order to establish our main results we need several lemmas, which we present in this section.

Lemma 2.1 (See [20, Theorem 1.25]) *Let $-\infty < a < b < \infty$, $f,g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) \neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are the functions*

$$\frac{f(x) - f(a)}{g(x) - g(a)}, \qquad \frac{f(x) - f(b)}{g(x) - g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2.2 (See [21, Lemma 7]) *Let $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$ with $n > m$, $a_i \geq 0$ for all $0 \leq i \leq n$, $a_n a_m > 0$ and*

$$P_n(t) = -\sum_{i=0}^{m} a_i t^i + \sum_{i=m+1}^{n} a_i t^i.$$

Then there exists $t_0 \in (0,\infty)$ such that $P_n(t_0) = 0$, $P_n(t) < 0$ for $t \in (0,t_0)$ and $P_n(t) > 0$ for $t \in (t_0,\infty)$.

Lemma 2.3 (See [22, Corollary 3.1]) *The inequality*

$$\psi(x+1) < \frac{1}{92}\log\left(x^2 + x + \frac{4}{3}\right) + \frac{45}{92}\log\left(x^2 + x + \frac{14}{45}\right)$$

holds for all $x > 0$.

Lemma 2.4 (See [23, Corollary 3.3(ii)]) *The double inequality*

$$\frac{(x + \frac{1}{2})(x^2 + x + \frac{23}{21})}{x^4 + 2x^3 + \frac{17}{7}x^2 + \frac{10}{7}x + \frac{12}{35}} < \psi'(x + 1)$$

$$< \frac{(x + \frac{1}{2})(x^2 + x + \frac{\pi^2}{15(\pi^2 - 9)})}{x^4 + 2x^3 + \frac{7\pi^2 - 60}{5(\pi^2 - 9)}x^2 + \frac{2\pi^2 - 15}{5(\pi^2 - 9)}x + \frac{1}{5(\pi^2 - 9)}}$$

holds for all $x > 0$.

Lemma 2.5 *The inequalities*

$$\frac{1}{x + \frac{1}{2}} - \frac{1}{12(x + \frac{1}{2})^3} \leq \psi'(x + 1) \leq \frac{1}{x + \frac{1}{2}}, \tag{2.1}$$

$$-\frac{1}{(x + \frac{1}{2})^2} \leq \psi''(x + 1) \leq -\frac{1}{(x + \frac{1}{2})^2} + \frac{1}{4(x + \frac{1}{2})^4}, \tag{2.2}$$

$$\frac{2}{(x + \frac{1}{2})^3} - \frac{1}{(x + \frac{1}{2})^5} \leq \psi'''(x + 1) \leq \frac{2}{(x + \frac{1}{2})^3} \tag{2.3}$$

hold for all $x > -1/2$.

Proof Let $x > -1/2$, and $R_1(x)$ and $R_2(x)$ be defined by

$$R_1(x) = \psi(x + 1) - \log\left(x + \frac{1}{2}\right), \tag{2.4}$$

$$R_2(x) = -\psi(x + 1) + \log\left(x + \frac{1}{2}\right) + \frac{1}{24(x + \frac{1}{2})^2}, \tag{2.5}$$

respectively. Then making use of the well-known formulas

$$\psi(x) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1 - e^{-t}}\right) dt, \qquad \log x = \int_0^\infty \frac{e^{-t} - e^{-xt}}{t} dt$$

we get

$$\begin{aligned} R_1(x) &= \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-(x+1)t}}{1 - e^{-t}}\right) dt - \int_0^\infty \frac{e^{-t} - e^{-(x+1/2)t}}{t} dt \\ &= \int_0^\infty \left(\frac{1}{t} - \frac{e^{-t/2}}{1 - e^{-t}}\right) e^{-(x+1/2)t} dt \\ &= \int_0^\infty \left(\frac{1}{t} - \frac{1}{2\sinh(t/2)}\right) e^{-(x+1/2)t} dt \\ &= \int_0^\infty \frac{\sinh(t/2) - t/2}{t\sinh(t/2)} e^{-(x+1/2)t} dt, \end{aligned} \tag{2.6}$$

$$\begin{aligned} R_2(x) &= -\int_0^\infty \left(\frac{1}{t} - \frac{1}{2\sinh(t/2)}\right) e^{-(x+1/2)t} dt + \frac{1}{24}\int_0^\infty te^{-(x+1/2)t} dt \\ &= \int_0^\infty \left(\frac{(t^2 - 24)\sinh(t/2) + 12t}{24t\sinh(t/2)}\right) e^{-(x+1/2)t} dt, \end{aligned} \tag{2.7}$$

where $\sinh(t) = (e^t - e^{-t})/2$ is the hyperbolic sine function.

Note that

$$6t\sinh(t/2)\frac{(t^2-24)\sinh(t/2)+12t}{24t\sinh(t/2)} = \sum_{n=3}^{\infty} \frac{2(n-2)(2n+1)}{(2n-3)!}\left(\frac{t}{2}\right)^{2n-1} > 0, \tag{2.8}$$

$$\sinh(t/2) - \frac{t}{2} > 0 \tag{2.9}$$

for $t > 0$.

It follows from (2.6)-(2.9) and the Bernstein theorem for complete monotonicity property that the two functions $R_1(x)$ and $R_2(x)$ are completely monotonic on the interval $(-1/2, \infty)$.

Therefore, Lemma 2.5 follows easily from (2.4), (2.5) and the complete monotonicity of $R_1(x)$ and $R_2(x)$ on the interval $(-1/2, \infty)$ together with the facts that

$$\left[\log\left(x+\frac{1}{2}\right)\right]^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(x+\frac{1}{2})^n}, \qquad \left[\frac{1}{(x+\frac{1}{2})^2}\right]^{(n)} = \frac{(-1)^n(n+1)!}{(x+\frac{1}{2})^{n+2}}. \qquad \square$$

Lemma 2.6 *The double inequality*

$$\left(\frac{x^2+1}{x+1}\right)^{2/(q+1)} < \frac{x^2+q}{x+q} < \left(\frac{x^2+1}{x+1}\right)^{1/q} \tag{2.10}$$

holds for all $x \in (0,1)$ and $q > 1$.

Proof Let $x \in (0,1)$, $q > 1$, and $H_1(x)$ and $H_2(x)$ be defined by

$$H_1(x) = \log(x^2+q) - \log(x+q), \qquad H_2(x) = \log(x^2+1) - \log(x+1), \tag{2.11}$$

respectively. Then simple computations lead to

$$\lim_{x\to 0^+} \frac{H_1(x)}{H_2(x)} = \frac{1}{q}, \qquad \lim_{x\to 1^-}\frac{H_1(x)}{H_2(x)} = \frac{2}{q+1}, \tag{2.12}$$

$$H_1(0^+) = H_2(0^+) = 0, \tag{2.13}$$

$$H_1(1^-) = H_2(1^-) = 0, \tag{2.14}$$

$$\frac{H_1'(x)}{H_2'(x)} = \frac{(x+1)(x^2+1)(x^2+2qx-q)}{(x+q)(x^2+q)(x^2+2x-1)},$$

$$\left[\frac{H_1'(x)}{H_2'(x)}\right]' = \frac{(q-1)\triangle(x,q)}{(x+q)^2(x^2+q)^2(x^2+2x-1)^2}, \tag{2.15}$$

where

$$\begin{aligned}
\triangle(x,q) &= \left[8x^5+4x^4+(1-x)^3\left(4x^2+3x+1\right)\right]q^2 + 4x^2\left[2x^4+4x^3+(1-x)^2\right]q \\
&\quad - x^4\left(x^4-4x^3-6x^2-4x+1\right) \\
&> \triangle(x,1) \\
&= (1-x)x^7+3x^7+10x^6+\left(2x^3-1\right)^2+24x^5+8x^4+2x^2(2x-1)^2 \\
&> 0.
\end{aligned} \tag{2.16}$$

From (2.15) and (2.16) we clearly see that $H_1'(x)/H_2'(x)$ is strictly increasing on $(0, \sqrt{2}-1) \cup (\sqrt{2}-1, 1)$. We assert that the function $H_1(x)/H_2(x)$ is strictly increasing on $(0,1)$. Indeed, if $x \in (0, \sqrt{2}-1)$, then $H_2'(x) \neq 0$, and Lemma 2.1 and (2.13) together with the monotonicity of $H_1'(x)/H_2'(x)$ on $(0, \sqrt{2}-1)$ lead to the conclusion that $H_1(x)/H_2(x)$ is strictly increasing on $(0, \sqrt{2}-1)$; if $x \in (\sqrt{2}-1, 1)$, then $H_2'(x) \neq 0$, and Lemma 2.1 and (2.14) together with the monotonicity of $H_1'(x)/H_2'(x)$ on $(\sqrt{2}-1, 1)$ lead to the conclusion that $H_1(x)/H_2(x)$ is strictly increasing on $(\sqrt{2}-1, 1)$.

Therefore, Lemma 2.6 follows easily from (2.11) and (2.12) together with the monotonicity of the function $H_1(x)/H_2(x)$ on $(0,1)$. □

Let $p > 0$, $x \in (0,1)$, and $f(p;x), f_1(p;x), f_2(p;x)$ and $f_3(p;x)$ be defined by

$$f(p,x) = \log \Gamma(x+1) - \log\left(\frac{x^2+p}{x+p}\right), \tag{2.17}$$

$$f_1(p,x) = \frac{\partial f(p,x)}{\partial x} = \psi(x+1) - \frac{2x}{x^2+p} + \frac{1}{x+p}, \tag{2.18}$$

$$f_2(p,x) = \frac{\partial^2 f(p,x)}{\partial x^2} = \psi'(x+1) + \frac{4x^2}{(x^2+p)^2} - \frac{2}{x^2+p} - \frac{1}{(x+p)^2}, \tag{2.19}$$

$$f_3(p,x) = \frac{\partial^3 f(p,x)}{\partial x^3} = \psi''(x+1) - \frac{16x^3}{(x^2+p)^3} + \frac{12x}{(x^2+p)^2} + \frac{2}{(x+p)^3}. \tag{2.20}$$

Lemma 2.7 *Let $f_2(p,x)$ be defined by* (2.19). *Then*

$$f_2(p, 1/3) < 0 \tag{2.21}$$

for $p \in [8/5, 9/5]$.

Proof From (2.19) and the second inequality in Lemma 2.4 we have

$$f_2(p, 1/3) = \left[\psi'(x+1) + \frac{4x^2}{(x^2+p)^2} - \frac{2}{x^2+p} - \frac{1}{(x+p)^2}\right]_{x=1/3}$$

$$< \left[\frac{(x+\frac{1}{2})(x^2+x+\frac{\pi^2}{15(\pi^2-9)})}{x^4 + 2x^3 + \frac{7\pi^2-60}{5(\pi^2-9)}x^2 + \frac{2\pi^2-15}{5(\pi^2-9)}x + \frac{1}{5(\pi^2-9)}}\right]_{x=1/3}$$

$$+ \left[\frac{4x^2}{(x^2+p)^2} - \frac{2}{x^2+p} - \frac{1}{(x+p)^2}\right]_{x=1/3}$$

$$= \frac{15(23\pi^2 - 180)}{2(152\pi^2 - 1{,}179)} - \frac{9(162p^3 + 171p^2 + 24p - 1)}{(3p+1)^2(9p+1)^2}. \tag{2.22}$$

Elaborated computations lead to

$$\left(\frac{15(23\pi^2 - 180)}{2(152\pi^2 - 1{,}179)} - \frac{9(162p^3 + 171p^2 + 24p - 1)}{(3p+1)^2(9p+1)^2}\right)'$$

$$= \frac{54(729p^4 + 1{,}215p^3 + 243p^2 - 27p - 8)}{(3p+1)^3(9p+1)^3} > 0 \tag{2.23}$$

for $p \in [8/5, 9/5]$.

From (2.22) and (2.23) we get

$$f_2(p, 1/3) < \left[\frac{15(23\pi^2 - 180)}{2(152\pi^2 - 1{,}179)} - \frac{9(162p^3 + 171p^2 + 24p - 1)}{(3p+1)^2(9p+1)^2} \right]_{p=9/5}$$

$$= \frac{15(23\pi^2 - 180)}{2(152\pi^2 - 1{,}179)} - \frac{2{,}167{,}065}{1{,}893{,}376} = -0.047\ldots < 0. \qquad \square$$

Lemma 2.8 *Let $f_2(p, x)$ be defined by* (2.19). *Then*

$$f_2(9/5, 7/50) > 0. \qquad (2.24)$$

Proof From (2.19) and the first inequality in Lemma 2.4 we have

$$f_2(9/5, 7/50) = \left[\psi'(x+1) + \frac{4x^2}{(x^2+p)^2} - \frac{2}{x^2+p} - \frac{1}{(x+p)^2} \right]_{p=9/5, x=7/50}$$

$$> \left[\frac{(x+\frac{1}{2})(x^2+x+\frac{23}{21})}{x^4 + 2x^3 + \frac{17}{7}x^2 + \frac{10}{7}x + \frac{12}{35}} + \frac{4x^2}{(x^2+p)^2} - \frac{2}{x^2+p} - \frac{1}{(x+p)^2} \right]_{p=9/5, x=7/50}$$

$$= \frac{84{,}826{,}873{,}256{,}410{,}100}{15{,}239{,}152{,}138{,}614{,}823{,}989} > 0. \qquad \square$$

Lemma 2.9 *Let $f_1(p, x)$ be defined by* (2.18). *Then*

$$f_1(9/5, x) < 0$$

for $x \in (7/50, 1/3)$.

Proof It follows from Lemma 2.3 and (2.18) that

$$f_1(9/5, x) < \frac{1}{92} \log\left(x^2 + x + \frac{4}{3} \right) + \frac{45}{92} \log\left(x^2 + x + \frac{14}{45} \right) - \frac{2x}{x^2 + 9/5} + \frac{1}{x + 9/5}. \qquad (2.25)$$

Elaborated computations lead to

$$\left[\frac{1}{92} \log\left(x^2 + x + \frac{4}{3} \right) + \frac{45}{92} \log\left(x^2 + x + \frac{14}{45} \right) - \frac{2x}{x^2 + 9/5} + \frac{1}{x + 9/5} \right]'$$

$$= \frac{h(x)}{2(5x+9)^2(5x^2+9)^2(3x^2+3x+4)(45x^2+45x+14)}, \qquad (2.26)$$

where

$$h(x) = 168{,}750x^9 + 1{,}029{,}375x^8 + 3{,}923{,}625x^7 + 7{,}884{,}000x^6 + 9{,}344{,}775x^5$$

$$+ 5{,}316{,}100x^4 + 203{,}355x^3 - 2{,}426{,}940x^2 - 544{,}401x + 118{,}017,$$

$$h(7/50) = -\frac{406{,}357{,}216{,}255{,}013}{156{,}250{,}000{,}000} < 0, \qquad (2.27)$$

$$h'(x) = 1{,}518{,}750x^8 + 8{,}235{,}000x^7 + 27{,}465{,}375x^6 + 47{,}304{,}000x^5 + 46{,}723{,}875x^4$$

$$+ 21{,}264{,}400x^3 + 610{,}065x^2 - 4{,}853{,}880x - 544{,}401, \qquad (2.28)$$

$$h'(1/3) = -\frac{120{,}000{,}368}{243} < 0. \qquad (2.29)$$

From Lemma 2.2, (2.28) and (2.29) we know that $h(x)$ is strictly decreasing on $(7/50, 1/3)$, then (2.27) leads to the conclusion that $h(x) < 0$ for $x \in (7/50, 1/3)$.

Therefore,

$$
\begin{aligned}
f_1(9/5, x) < & \left[\frac{1}{92} \log\left(x^2 + x + \frac{4}{3} \right) + \frac{45}{92} \log\left(x^2 + x + \frac{14}{45} \right) \right. \\
& \left. - \frac{2x}{x^2 + 9/5} + \frac{1}{x + 9/5} \right]_{x=7/50} \\
= & \frac{1}{92} \log \frac{11{,}197}{7{,}500} + \frac{45}{92} \log \frac{10{,}591}{22{,}500} + \frac{159{,}550}{441{,}253} = -0.0026\ldots < 0
\end{aligned}
$$

for $x \in (7/50, 1/3)$ follows from (2.25) and (2.26) together with $h(x) < 0$ for $x \in (7/50, 1/3)$. $\qquad\square$

Lemma 2.10 *Let $p \in [3/2, 2]$ and $f_3(p, x)$ be defined by (2.20). Then there exists $\eta(p) \in (0, 1)$ such that $f_3(p, x) < 0$ for $x \in (0, \eta(p))$ and $f_3(p, x) > 0$ for $x \in (\eta(p), 1)$.*

Proof Let

$$
g(p, x) = \left(x^2 + p\right)^3 f_3(p, x) = \left(x^2 + p\right)^3 \psi''(x+1) + \frac{2(x^2 + p)^3}{(x + p)^3} - 4x^3 + 12px. \tag{2.30}
$$

Then simple computations lead to

$$
g\left(p, 0^+\right) = p^3 \psi''(1) + 2, \qquad g\left(p, 1^-\right) = (p+1)^3 \psi''(2) + 12p - 2, \tag{2.31}
$$

$$
\begin{aligned}
\frac{\partial g(p, x)}{\partial x} = & \left(x^2 + p\right)^3 \psi'''(x+1) + 6x\left(x^2 + p\right)^2 \psi''(x+1) \\
& - \frac{6(x^2 + p)^3}{(x + p)^4} + \frac{12x(x^2 + p)^2}{(x + p)^3} - 12x^2 + 12p.
\end{aligned} \tag{2.32}
$$

It follows from the first inequalities in (2.2) and (2.3) together with the identity $\psi^{(n)}(x+1) = \psi^{(n)}(x) + (-1)^n n!/x^{n+1}$ that

$$
\psi''(x+1) = \psi''(x+2) - \frac{2}{(x+1)^3} \geq -\frac{1}{(x+3/2)^2} - \frac{2}{(x+1)^3}, \tag{2.33}
$$

$$
\psi'''(x+1) = \psi'''(x+2) + \frac{6}{(x+1)^4} \geq \frac{2}{(x+3/2)^3} - \frac{1}{(x+3/2)^5} + \frac{6}{(x+1)^4}. \tag{2.34}
$$

From (2.32)-(2.34) we have

$$
\begin{aligned}
\frac{\partial g(p, x)}{\partial x} \geq & \left(x^2 + p\right)^3 \left[\frac{2}{(x+3/2)^3} - \frac{1}{(x+3/2)^5} + \frac{6}{(x+1)^4} \right] \\
& + 6x\left(x^2 + p\right)^2 \left[-\frac{1}{(x+3/2)^2} - \frac{2}{(x+1)^3} \right] \\
& - \frac{6(x^2 + p)^3}{(x + p)^4} + \frac{12x(x^2 + p)^2}{(x + p)^3} - 12x^2 + 12p \\
= & \frac{2g_1(p, x)}{(2x+3)^5 (x+1)^4 (x+p)^4},
\end{aligned} \tag{2.35}
$$

where

$$g_1(p,x) = \sum_{k=0}^{6} b_k x^k - \sum_{k=7}^{16} b_k x^k \tag{2.36}$$

with

$$b_0 = \left(785p^4 + 1{,}458p^2 - 729\right)p^3,$$

$$b_1 = 2\left(1{,}375p^4 + 679p^3 + 5{,}346p^2 + 2{,}916p - 1{,}944\right)p^3,$$

$$b_2 = \left(3{,}992p^5 + 5{,}093p^4 + 32{,}250p^3 + 41{,}310p^2 + 2{,}106p - 729\right)p^2,$$

$$b_3 = 2\left(1{,}544p^5 + 3{,}955p^4 + 27{,}270p^3 + 60{,}214p^2 + 30{,}186p + 1{,}701\right)p^2,$$

$$b_4 = \left(1{,}352p^6 + 6{,}520p^5 + 56{,}049p^4 + 187{,}562p^3 + 184{,}880p^2 + 38{,}070p + 2{,}187\right)p,$$

$$b_5 = 2\left(160p^6 + 1{,}510p^5 + 16{,}743p^4 + 78{,}961p^3 + 128{,}815p^2 + 54{,}729p + 5{,}832\right)p,$$

$$b_6 = 32p^7 + 752p^6 + 8{,}088p^5 + 44{,}529p^4 + 135{,}746p^3 + 123{,}972p^2 + 19{,}926p - 729,$$

$$b_7 = -80p^6 + 3{,}000p^5 + 47{,}018p^4 + 106{,}872p^3 + 27{,}156p^2 + 9{,}720p + 5{,}346,$$

$$b_8 = 2{,}904p^5 + 60{,}104p^4 + 245{,}744p^3 + 259{,}818p^2 + 99{,}246p + 17{,}334,$$

$$b_9 = 864p^5 + 31{,}572p^4 + 204{,}464p^3 + 360{,}816p^2 + 195{,}790p + 34{,}398,$$

$$b_{10} = 96p^5 + 8{,}896p^4 + 96{,}576p^3 + 272{,}304p^2 + 216{,}076p + 46{,}726,$$

$$b_{11} = 1{,}264p^4 + 26{,}944p^3 + 126{,}120p^2 + 151{,}960p + 44{,}884,$$

$$b_{12} = 64p^4 + 4{,}096p^3 + 35{,}712p^2 + 69{,}752p + 30{,}400,$$

$$b_{13} = 256p^3 + 5{,}664p^2 + 20{,}320p + 14{,}180,$$

$$b_{14} = 384p^2 + 3{,}424p + 4{,}336,$$

$$b_{15} = 256p + 784,$$

$$b_{16} = 16,$$

$$g_1(p,1) = -199{,}181 - 732{,}767p - 813{,}801p^2 - 48{,}835p^3 + 408{,}665p^4$$
$$+ 189{,}699p^5 + 24{,}733p^6 + 12{,}319p^7, \tag{2.37}$$

$$g_1(3/2,1) = \frac{90{,}546{,}875}{128} > 0. \tag{2.38}$$

From Lemma 2.2, (2.37) and (2.38) we clearly see that

$$g_1(p,1) > 0 \tag{2.39}$$

for $p \in [3/2,1]$.

Making use of Lemma 2.2 again, and (2.36) and (2.39) together with the facts that $b_k > 0$ for $p \in [3/2,1]$ and $k = 0,1,2,\ldots,16$ we know that $g_1(p,x) > 0$ for $p \in [3/2,1]$ and $x \in (0,1)$. Then inequality (2.35) leads to the conclusion that the function $x \to g(p,x)$ is strictly increasing on $(0,1)$ for $p \in [3/2,2]$.

From (2.2) and the identity $\psi^{(n)}(x) = \psi^{(n)}(x+1) + (-1)^{n+1} n!/x^{n+1}$ we get

$$-\frac{1}{(x+1/2)^2} \leq \psi''(x+1) = \psi''(x+2) - \frac{2}{(x+1)^3}$$

$$\leq -\frac{1}{(x+3/2)^2} + \frac{1}{4(x+3/2)^4} - \frac{2}{(x+1)^3}. \tag{2.40}$$

Taking $x = 1$ in the first inequality of (2.40) and $x = 0$ in the second inequality of (2.40), one has

$$\psi''(2) \geq -\frac{4}{9}, \qquad \psi''(1) \leq -\frac{194}{81}. \tag{2.41}$$

It follows from (2.31) and (2.41) that

$$g(p, 0^+) \leq -\frac{194}{81} \times \left(\frac{3}{2}\right)^3 + 2 = -\frac{73}{12} < 0, \tag{2.42}$$

$$g(p, 1^-) \geq -\frac{4}{9}(p+1)^3 + 12p - 2 \tag{2.43}$$

for $p \in [3/2, 2]$.

Note that

$$\left[-\frac{4}{9}(p+1)^3 + 12p - 2\right]' = \frac{4}{3}(p+4)(2-p). \tag{2.44}$$

Inequality (2.43) and equation (2.44) imply that

$$g(p, 1^-) \geq -\frac{4}{9}\left(\frac{3}{2} + 1\right)^3 + 12 \times \frac{3}{2} - 2 = \frac{163}{18} > 0 \tag{2.45}$$

for $p \in [3/2, 2]$.

Therefore, Lemma 2.10 follows easily from (2.30), (2.42), (2.45) and the monotonicity of the function $x \to g(p, x)$ on the interval $(0, 1)$. □

Lemma 2.11 *Let $p \in [8/5, 9/5]$ and $f_2(p, x)$ be defined by (2.19). Then there exist $\eta_1(p), \eta_2(p) \in (0, 1)$ with $\eta_1(p) < \eta_2(p)$ such that $f_2(p, x) > 0$ for $x \in (0, \eta_1(p)) \cup (\eta_2(p), 1)$ and $f_2(p, x) < 0$ for $x \in (\eta_1(p), \eta_2(p))$.*

Proof It follows from (2.19) that

$$f_2(p, 0^+) = \frac{\pi^2}{6p^2}\left(p - \frac{\sqrt{6(\pi^2 + 6)} + 6}{\pi^2}\right)\left(p + \frac{\sqrt{6(\pi^2 + 6)} - 6}{\pi^2}\right) > 0, \tag{2.46}$$

$$f_2(p, 1^-) = \frac{(\pi^2 - 6)p^2 + 2(\pi^2 - 12)p + \pi^2}{6(p+1)^2} > 0 \tag{2.47}$$

for $p \in [8/5, 9/5]$.

From Lemma 2.10 and $[8/5, 9/5] \subset [3/2, 2]$ we know that there exists $\eta(p) \in (0, 1)$ such that the function $x \to f_2(p, x)$ is strictly decreasing on $(0, \eta(p))$ and strictly increasing

on $(\eta(p), 1)$. Then Lemma 2.7 leads to the conclusion that

$$f_2\big(p, \eta(p)\big) \leq f_2(p, 1/3) < 0. \tag{2.48}$$

Therefore, there exist $\eta_1(p) \in (0, \eta(p))$ and $\eta_2(p) \in (\eta(p), 1)$ such that $f_2(p, x) > 0$ for $x \in (0, \eta_1(p)) \cup (\eta_2(p), 1)$ and $f_2(p, x) < 0$ for $x \in (\eta_1(p), \eta_2(p))$ follow from (2.46)-(2.48) and the piecewise monotonicity of the function $x \to f_2(p, x)$ on the interval $(0, 1)$. □

3 Main results

Theorem 3.1 *Let $p > 0$ and $p_0 = \gamma/(1 - \gamma) = 1.365\ldots$. Then the inequality*

$$\Gamma(x + 1) > \frac{x^2 + p}{x + p} \tag{3.1}$$

holds for all $x \in (0, 1)$ if and only if $p \leq p_0$, and the inequality

$$\Gamma(x + 1) \leq \frac{\mu(x^2 + p_0)}{x + p_0} \tag{3.2}$$

holds for all $x \in (0, 1)$ if and only if $\mu \geq \mu_0$, where

$$\mu_0 = \frac{(x_0 + p_0)\Gamma(x_0 + 1)}{(x_0^2 + p_0)} = 1.027\ldots \tag{3.3}$$

and $x_0 = 0.346\ldots$ is the unique solution of the equation

$$\psi(x + 1) - \frac{2x}{x^2 + p_0} + \frac{1}{x + p_0} = 0 \tag{3.4}$$

on the interval $(0, 1)$.

Proof If inequality (3.1) holds for all $x \in (0, 1)$, then $p \leq p_0$ follows easily from

$$\lim_{x \to 1^-} \frac{\log \Gamma(x + 1) - \log(\frac{x^2 + p}{x + p})}{1 - x} = -\psi(2) + \frac{1}{1 + p} = \gamma - \frac{p}{1 + p} \geq 0.$$

Next, we prove that inequality (3.1) holds for all $x \in (0, 1)$ and $p = p_0$ and (3.2) holds for all $x \in (0, 1)$ if and only if $\mu \geq \mu_0$.

Let $f(p, x), f_1(p, x), f_2(p, x)$ be defined by (2.17)-(2.19) and

$$g(x) = \frac{(x^2 + p_0)^2}{x^2} f_2(p_0, x) = \frac{(x^2 + p_0)^2}{x^2} \psi'(x + 1) - \frac{2p_0}{x^2} - \frac{(x^2 + p_0)^2}{x^2(x + p_0)^2} + 2. \tag{3.5}$$

Then elaborated computations lead to

$$f\big(p_0, 0^+\big) = f\big(p_0, 1^-\big) = 0, \tag{3.6}$$

$$f_1\big(p_0, 0^+\big) = \frac{1 - \gamma - \gamma^2}{\gamma} > 0, \qquad f_1\big(p_0, 1^-\big) = 0, \tag{3.7}$$

$$g\big(0^+\big) = -\infty, \qquad g\big(1^-\big) = \frac{(\pi^2 - 6)p_0^2 + 2(\pi^2 - 12)p_0 + \pi^2}{6} > 0, \tag{3.8}$$

$$g'(x) = \frac{(x^2 + p_0)^2 \psi''(x + 1)}{x^2} - \frac{2(p_0^2 - x^4)\psi'(x + 1)}{x^3}$$
$$- \frac{2p_0[x^4 - 4x^3 - 6p_0 x^2 - 2p_0(3p_0 + 1)x - (2p_0 + 1)p_0^2]}{(x + p_0)^3 x^3}. \tag{3.9}$$

It follows from the second inequality in (2.1) and the first inequality in (2.2) together with (3.9) that

$$g'(x) \geq -\frac{(x^2 + p_0)^2}{x^2(x + 1/2)^2} - \frac{2(p_0^2 - x^4)}{(x + 1/2)x^3}$$
$$- \frac{2p_0[x^4 - 4x^3 - 6p_0 x^2 - 2p_0(3p_0 + 1)x - (2p_0 + 1)p_0^2]}{(x + p_0)^3 x^3}$$
$$= \frac{2}{x^3(2x + 1)^2(x + p_0)^3} g_1(x),$$

where

$$g_1(x) = 2x^8 + 2(3p_0 + 1)x^7 + 2p_0(3p_0 - 1)x^6 + 2p_0(p_0^2 - 3p_0 + 6)x^5$$
$$- p_0(10p_0^2 - 18p_0 - 15)x^4 - 2p_0(p_0 + 2)(2p_0^2 - 7p_0 - 1)x^3$$
$$- 2p_0^2(5p_0^2 - 11p_0 - 7)x^2 - 2p_0^2(p_0 + 1)(3p_0^2 - 4p_0 - 1)x$$
$$- p_0^3(2p_0^2 - 2p_0 - 1).$$

It is easy to verify that all the coefficients of the polynomial $g_1(x)$ are positive, which implies that $g(x)$ is strictly increasing on $(0, 1)$, then from (3.5) and (3.8) we know that there exists $\eta \in (0, 1)$ such that the function $f_1(p_0, x)$ is strictly decreasing on $(0, \eta)$ and strictly increasing on $(\eta, 1)$.

It follows from (2.18) and (3.7) together with the piecewise monotonicity of the function $f_1(p_0, x)$ on the interval $(0, 1)$ that there exists $x_0 \in (0, 1)$ such that $f(p_0, x)$ is strictly increasing on $(0, x_0)$ and strictly decreasing on $(x_0, 1)$ and x_0 is the unique solution of equation (3.4) on the interval $(0, 1)$.

Therefore, the desired results follow easily from (2.17), (3.3), (3.6) and the piecewise monotonicity of the function $f(p_0, x)$ on the interval $(0, 1)$ together with the fact that the function $p \to (x^2 + p)/(x + p)$ is strictly increasing.

Numerical computations show that $x_0 = 0.346\ldots$ and $\mu_0 = (x_0 + p_0)\Gamma(x_0 + 1)/(x_0^2 + p_0) = 1.027\ldots$. □

Theorem 3.2 *The inequality*

$$\Gamma(x + 1) > \frac{x^2 + \frac{1}{\gamma}}{x + \frac{1}{\gamma}}$$

holds for all $x \in (0, x^*)$, and its reverse inequality

$$\Gamma(x + 1) < \frac{x^2 + \frac{1}{\gamma}}{x + \frac{1}{\gamma}}$$

holds for all $x \in (x^*, 1)$, where $x^* = 0.385\ldots$ is the unique solution of the equation

$$\Gamma(x+1) - \frac{x^2 + \frac{1}{\gamma}}{x + \frac{1}{\gamma}} = 0$$

on the interval $(0,1)$.

Proof Let $f(p,x)$, $f_1(p,x)$ and $f_2(p,x)$ be, respectively, defined by (2.17), (2.18) and (2.19). Then simple computations lead to

$$f\left(\frac{1}{\gamma}, 0^+\right) = f\left(\frac{1}{\gamma}, 1^-\right) = 0, \tag{3.10}$$

$$f_1\left(\frac{1}{\gamma}, 0^+\right) = 0, \qquad f_1\left(\frac{1}{\gamma}, 1^-\right) = \frac{1 - \gamma - \gamma^2}{1 + \gamma} > 0. \tag{3.11}$$

From Lemma 2.11 and $1/\gamma = 1.732\ldots \in [8/5, 9/5]$ we know that there exist $\eta_1(1/\gamma), \eta_2(1/\gamma) \in (0,1)$ with $\eta_1(1/\gamma) < \eta_2(1/\gamma)$ such that $f_1(1/\gamma, x)$ is strictly increasing on $(0, \eta_1(1/\gamma)) \cup (\eta_2(1/\gamma), 1)$ and strictly decreasing on $(\eta_1(1/\gamma), \eta_2(1/\gamma))$. We claim that

$$f_1\left(1/\gamma, \eta_2(1/\gamma)\right) < 0. \tag{3.12}$$

Indeed, if $f_1(1/\gamma, \eta_2(1/\gamma)) \geq 0$, then the piecewise monotonicity of the function $f_1(1/\gamma, x)$ on the interval $(0,1)$ and (3.11) lead to the conclusion that $f(1/\gamma, x)$ is strictly increasing on $(0,1)$, which contradicts (3.10).

It follows from (3.11) and (3.12) together with the piecewise monotonicity of the function $f_1(1/\gamma, x)$ on the interval $(0,1)$ that there exist $\eta_1^*(1/\gamma) \in (\eta_1(1/\gamma), \eta_2(1/\gamma))$ and $\eta_2^*(1/\gamma) \in (\eta_2(1/\gamma), 1)$ such that $f(1/\gamma, x)$ is strictly increasing on $(0, \eta_1^*(1/\gamma)) \cup (\eta_2^*(1/\gamma), 1)$ and strictly decreasing on $(\eta_1^*(1/\gamma), \eta_2^*(1/\gamma))$.

Therefore, Theorem 3.2 follows easily from (2.17) and (3.10) together with the piecewise monotonicity of $f(1/\gamma, x)$ on $(0,1)$. Numerical computations show that $x^* = 0.385\ldots$. □

Theorem 3.3 *The double inequality*

$$\frac{\lambda(x^2 + \frac{9}{5})}{x + \frac{9}{5}} \leq \Gamma(x+1) < \frac{x^2 + \frac{9}{5}}{x + \frac{9}{5}} \tag{3.13}$$

holds for all $x \in (0,1)$ with the best possible constant

$$\lambda = \frac{(5\tau_0 + 9)\Gamma(\tau_0 + 1)}{5\tau_0^2 + 9} = 0.991\ldots, \tag{3.14}$$

where $\tau_0 = 0.719\ldots$ is the unique solution of the equation

$$\psi(x+1) - \frac{2x}{x^2 + \frac{9}{5}} + \frac{1}{x + \frac{9}{5}} = 0 \tag{3.15}$$

on the interval $(0,1)$.

Proof Let $f(p,x)$, $f_1(p,x)$ and $f_2(p,x)$ be, respectively, defined by (2.17), (2.18) and (2.19). Then simple computations lead to

$$f\left(\frac{9}{5},0^+\right) = f\left(\frac{9}{5},1^-\right) = 0, \tag{3.16}$$

$$f_1\left(\frac{9}{5},0^+\right) = \frac{5}{9} - \gamma < 0, \qquad f_1\left(\frac{9}{5},1^-\right) = \frac{9}{14} - \gamma > 0. \tag{3.17}$$

It follows from Lemma 2.11 that there exist $\eta_1(9/5), \eta_2(9/5) \in (0,1)$ with $\eta_1(9/5) < \eta_2(9/5)$ such that $f_2(9/5,x) > 0$ for $x \in (0, \eta_1(9/5)) \cup (\eta_2(9/5), 1)$ and $f_2(9/5,x) < 0$ for $x \in (\eta_1(9/5), \eta_2(9/5))$, and $f_1(9/5,x)$ is strictly increasing on $(0, \eta_1(9/5)) \cup (\eta_2(9/5), 1)$ and strictly decreasing on $(\eta_1(9/5), \eta_2(9/5))$. Then Lemmas 2.7-2.9 lead to the conclusion that $\eta_1(9/5) \in (7/50, 1/3)$ and

$$f_1\big(9/5, \eta_1(9/5)\big) < 0. \tag{3.18}$$

From (2.18), (3.17), (3.18) and the piecewise monotonicity of $f_1(9/5,x)$ on $(0,1)$ we clearly see that there exists τ_0 such that τ_0 is the unique solution of equation (3.15) on the interval $(0,1)$, and $f(9/5,x)$ is strictly decreasing on $(0, \tau_0)$ and strictly increasing on $(\tau_0, 1)$.

Equation (3.16) and the piecewise monotonicity of the function $f(9/5,x)$ on the interval $(0,1)$ lead to the conclusion that

$$f(9/5, \tau_0) \leq f(9/5, x) < 0 \tag{3.19}$$

for all $x \in (0,1)$.

Therefore, inequality (3.13) holds for all $x \in (0,1)$ follows from (2.17) and (3.19). We clearly see that the parameter λ given by (3.14) is the best possible constant such that the first inequality in (3.13) holds for all $x \in (0,1)$. Numerical computations show that $\tau_0 = 0.719\ldots$ and $\lambda = 0.991\ldots$. $\qquad\square$

Remark 3.4 From Theorems 3.1 and 3.3 we clearly see that the double inequality

$$\frac{x^2 + p_0}{x + p_0} < \Gamma(x+1) < \frac{x^2 + p_1}{x + p_1} \tag{3.20}$$

holds for all $x \in (0,1)$ with $p_0 = \gamma/(1-\gamma) = 1.365\ldots$ and $p_1 = 9/5$, the constant p_0 appears to be the best possible, but this is not true for p_1, and a slightly smaller value for p_1 is possible. Unfortunately, we cannot find the best possible constant p_1 in the article; we leave this as an open problem for the reader.

Remark 3.5 From the monotonicity of the function $p \mapsto (x^2 + p)/(x + p)$ we clearly see that both the upper and lower bounds for $\Gamma(x+1)$ given in (3.20) are better than that given in (1.3), and the first (second) inequality in Theorem 3.2 is the improvement of the first (second) inequality in (1.3) for $x \in (0, x^*)$ ($x \in (x^*, 1)$), where $x^* = 0.385\ldots$ is given by Theorem 3.2.

Remark 3.6 From Lemma 2.6, $\gamma + \gamma^2 < 1$, $1/\gamma > 1$, $\gamma/(1-\gamma) > 1$ and $(x^2 + 1)/(x + 1) < 1$ for $x \in (0,1)$ one has

$$\frac{x^2 + \frac{\gamma}{1-\gamma}}{x + \frac{\gamma}{1-\gamma}} > \left(\frac{x^2 + 1}{x + 1}\right)^{2(1-\gamma)},$$

$$\left(\frac{x^2 + 1}{x + 1}\right)^{\gamma} > \frac{x^2 + \frac{1}{\gamma}}{x + \frac{1}{\gamma}} > \left(\frac{x^2 + 1}{x + 1}\right)^{2\gamma/(1+\gamma)} > \left(\frac{x^2 + 1}{x + 1}\right)^{2(1-\gamma)}.$$

Therefore, the lower bound for $\Gamma(x + 1)$ given in (3.20) is better than that given in (1.4), the first inequality in Theorem 3.2 is an improvement of the first inequality in (1.4) for $x \in (0, x^*)$ and the second inequality in Theorem 3.2 is an improvement of the second inequality in (1.4) for $x \in (x^*, 1)$, where $x^* = 0.385\ldots$ is given by Theorem 3.2.

Remark 3.7 It is not difficult to verify that

$$\min_{x \in (0,1)} \left(\frac{x^2 + 1}{x + 1}\right)^{\gamma} = \left[2(\sqrt{2} - 1)\right]^{\gamma} = 0.897\ldots,$$

$$\min_{x \in (0,1)} \left(2^{1-x} x^x\right) = 2e^{-2/e} = 0.958\ldots,$$

$$\frac{x^2 + \frac{9}{5}}{x + \frac{9}{5}} < 0.89$$

for $x \in (0.44, 0.45)$ and

$$\frac{x^2 + \frac{9}{5}}{x + \frac{9}{5}} < 0.95$$

for $x \in (\theta_0, \theta_1)$, where $\theta_0 = (0.95 - \sqrt{0.5425})/2 = 0.106\ldots$ and $\theta_1 = (0.95 + \sqrt{0.5425})/2 = 0.843\ldots$. Therefore, the upper bound $(x^2 + 9/5)/(x + 9/5)$ for $\Gamma(x + 1)$ given in (3.20) is better than that given in (1.4) for $x \in (0.44, 0.45)$, and it is also better than that given in (1.2) for $x \in (\theta_0, \theta_1)$.

Remark 3.8 Let

$$L_3(x) = \left(\frac{1}{2} + \sqrt{\frac{1}{4} + x}\right)^{1-x} x^x, \qquad L_4(x) = \frac{x^2 + \frac{\gamma}{1-\gamma}}{x + \frac{\gamma}{1-\gamma}}.$$

Then numerical computations show that

$$L_3(1/8) = 0.846\ldots < L_4(1/8) = 0.926\ldots,$$

$$L_3(1/4) = 0.814\ldots < L_4(1/4) = 0.883\ldots,$$

$$L_3(3/8) = 0.811\ldots < L_3(3/8) = 0.865\ldots,$$

$$L_3(1/2) = 0.826\ldots < L_4(1/2) = 0.865\ldots,$$

$$L_3(5/8) = 0.794\ldots < L_4(5/8) = 0.882\ldots,$$

$$L_3(3/4) = 0.891\ldots < L_4(3/4) = 0.911\ldots,$$

$$L_3(7/8) = 0.913\ldots < L_4(7/8) = 0.951\ldots.$$

   Therefore, there exists $\delta \in (0, 1/8)$ such that the lower bound for $\Gamma(x + 1)$ given in (3.20) is better than that given in (1.2) for $x \in (\delta, 1/8 + \delta) \cup (1/4 - \delta, 1/4 + \delta) \cup (3/8 - \delta, 3/8 + \delta) \cup (1/2 - \delta, 1/2 + \delta) \cup (5/8 - \delta, 5/8 + \delta) \cup (3/4 - \delta, 3/4 + \delta) \cup (7/8 - \delta, 7/8 + \delta)$.

4 Results and discussion

In this paper, we provide the accurate bounds for the classical gamma function in terms of very simple rational functions, which can be used to estimate the value of the gamma function in the area of engineering and technology.

5 Conclusion

In the article, we present several very simple and practical rational bounds for the gamma function, which can be regarded as a simple estimation of the value of the gamma function. The given results are improvements of some well-known results.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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