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Padé approximant related to inequalities involving the constant *e* and a generalized Carleman-type inequality

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Abstract

Based on the Padé approximation method, in this paper we determine the coefficients a_j and b_j ($1 \le j \le k$) such that

$$\frac{1}{e}\left(1+\frac{1}{x}\right)^x = \frac{x^k + a_1x^{k-1} + \cdots + a_k}{x^k + b_1x^{k-1} + \cdots + b_k} + O\left(\frac{1}{x^{2k+1}}\right), \quad x \to \infty,$$

where $k \ge 1$ is any given integer. Based on the obtained result, we establish new upper bounds for $(1 + 1/x)^x$. As an application, we give a generalized Carleman-type inequality.

MSC: 26D15; 41A60

Keywords: Carleman's inequality; weight coefficient; Padé approximant

1 Introduction

Let $a_n \ge 0$ for $n \in \mathbb{N} := \{1, 2, \ldots\}$ and $0 < \sum_{n=1}^{\infty} a_n < \infty$. Then

$$\sum_{n=1}^{\infty} (a_1a_2\cdots a_n)^{1/n} < e\sum_{n=1}^{\infty} a_n. \tag{1.1}$$

The constant e is the best possible. The inequality (1.1) was presented in 1922 in [1] by Carleman and it is called Carleman's inequality. Carleman discovered this inequality during his important work on quasi-analytical functions.

Carleman's inequality (1.1) was generalized by Hardy [2] (see also [3, p.256]) as follows: If $a_n \ge 0$, $\lambda_n > 0$, $\Lambda_n = \sum_{m=1}^{n} \lambda_m$ for $n \in \mathbb{N}$, and $0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty$, then

$$\sum_{n=1}^{\infty} \lambda_n \left(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n}\right)^{1/\Lambda_n} < e\sum_{n=1}^{\infty} \lambda_n a_n. \tag{1.2}$$

Note that inequality (1.2) is usually referred to as a Carleman-type inequality or weighted Carleman-type inequality. In [2], Hardy himself said that it was Pólya who pointed out this inequality to him.

In [4–20], some strengthened and generalized results of (1.1) and (1.2) have been given by estimating the weight coefficient $(1+1/n)^n$. For example, Yang [17] proved that, for $n \in \mathbb{N}$,

$$e\left(1-\frac{1}{2(n+\frac{5}{6})}\right) < \left(1+\frac{1}{n}\right)^n < e\left(1-\frac{1}{2(n+1)}\right), \tag{1.3}$$

and then used it to obtain the following strengthened Carleman inequality:

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} < e\sum_{n=1}^{\infty}\left(1-\frac{1}{2(n+1)}\right)a_n. \tag{1.4}$$

Xie and Zhong [15] proved that, for $x \geq 1$,

$$e\left(1-\frac{7}{14x+12}\right) < \left(1+\frac{1}{x}\right)^x < e\left(1-\frac{6}{12x+11}\right), \tag{1.5}$$

and then used it to improve the Carleman-type inequality (1.2) as follows. If $0 < \lambda_{n+1} \leq \lambda_n$, $\Lambda_n = \sum_{m=1}^{n}\lambda_m$, $a_n \geq 0$ for $n \in \mathbb{N}$, and $0 < \sum_{n=1}^{\infty}\lambda_n a_n < \infty$, then

$$\sum_{n=1}^{\infty}\lambda_{n+1}\left(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n}\right)^{1/\Lambda_n} < e\sum_{n=1}^{\infty}\left(1-\frac{6}{12(\frac{\Lambda_n}{\lambda_n})+11}\right)\lambda_n a_n. \tag{1.6}$$

Taking $\lambda_n \equiv 1$ in (1.6) yields

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} < e\sum_{n=1}^{\infty}\left(1-\frac{6}{12n+11}\right)a_n, \tag{1.7}$$

which improves (1.4).

Recently, Mortici and Hu [14] proved that, for $x \geq 1$,

$$\begin{aligned}&\frac{x+\frac{5}{12}}{x+\frac{11}{12}}-\frac{5}{288x^3}+\frac{343}{8{,}640x^4}-\frac{2{,}621}{41{,}472x^5}\\ &\quad<\frac{1}{e}\left(1+\frac{1}{x}\right)^x<\frac{x+\frac{5}{12}}{x+\frac{11}{12}}-\frac{5}{288x^3}+\frac{343}{8{,}640x^4}-\frac{2{,}621}{41{,}472x^5}+\frac{300{,}901}{3{,}483{,}648x^6},\end{aligned} \tag{1.8}$$

and then they used it to establish the following improvement of Carleman's inequality:

$$\begin{aligned}&\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n}\\ &\quad< e\sum_{n=1}^{\infty}\left(\frac{12n+5}{12n+11}-\frac{5}{288n^3}+\frac{343}{8{,}640n^4}-\frac{2{,}621}{41{,}472n^5}+\frac{300{,}901}{3{,}483{,}648n^6}\right)a_n,\end{aligned}$$

which can be written as

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} < e\sum_{n=1}^{\infty}(1-\varepsilon_n)a_n, \tag{1.9}$$

where

$$\varepsilon_n = \frac{104{,}509{,}440n^6 + 3{,}628{,}800n^4 - 4{,}971{,}456n^3 + 5{,}603{,}472n^2 - 5{,}945{,}040n - 16{,}549{,}555}{17{,}418{,}240n^6(12n+11)}. \qquad (1.10)$$

For information as regards the history of Carleman-type inequalities, please refer to [21–24].

It follows from (1.8) that

$$\frac{1}{e}\left(1+\frac{1}{x}\right)^x = \frac{x+\frac{5}{12}}{x+\frac{11}{12}} + O\left(\frac{1}{x^3}\right), \quad x \to \infty. \qquad (1.11)$$

Using the Padé approximation method, in Section 3 we derive (1.11) and the following approximation formula:

$$\frac{1}{e}\left(1+\frac{1}{x}\right)^x = \frac{x^2+\frac{87}{100}x+\frac{37}{240}}{x^2+\frac{137}{100}x+\frac{457}{1{,}200}} + O\left(\frac{1}{x^5}\right), \quad x \to \infty. \qquad (1.12)$$

Equation (1.12) motivates us to present the following inequality:

$$\left(1+\frac{1}{n}\right)^n < e\left(\frac{n^2+\frac{87}{100}n+\frac{37}{240}}{n^2+\frac{137}{100}n+\frac{457}{1{,}200}}\right) = e\left(1-\frac{8(75n+34)}{1{,}200n^2+1{,}644n+457}\right), \quad n \in \mathbb{N}. \qquad (1.13)$$

Following the same method used in the proof of Theorem 3.2, we can prove the inequality (1.13). We here omit it.

According to Pólya's proof of (1.1) in [25],

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} \le \sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n a_n, \qquad (1.14)$$

and then the following strengthened Carleman's inequality is derived directly from (1.13):

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} < e\sum_{n=1}^{\infty}\left(1-\frac{8(75n+34)}{1{,}200n^2+1{,}644n+457}\right)a_n, \qquad (1.15)$$

which improves (1.7).

Based on the Padé approximation method, we determine the coefficients a_j and b_j ($1 \le j \le k$) such that

$$\frac{1}{e}\left(1+\frac{1}{x}\right)^x = \frac{x^k+a_1x^{k-1}+\cdots+a_k}{x^k+b_1x^{k-1}+\cdots+b_k} + O\left(\frac{1}{x^{2k+1}}\right), \quad x \to \infty, \qquad (1.16)$$

where $k \ge 1$ is any given integer. Based on the obtained result, we establish new upper bounds for $(1+1/x)^x$. As an application, we give a generalization to the Carleman-type inequality.

The numerical values given have been calculated using the computer program MAPLE 13.

2 A useful lemma

For later use, we introduce the following set of partitions of an integer $n \in \mathbb{N} = \mathbb{N}_0 \setminus \{0\} := \{1, 2, 3, \ldots\}$:

$$\mathcal{A}_n := \{(k_1, k_2, \ldots, k_n) \in \mathbb{N}_0^n : k_1 + 2k_2 + \cdots + nk_n = n\}. \tag{2.1}$$

In number theory, the partition function $p(n)$ represents the number of possible partitions of $n \in \mathbb{N}$ (*e.g.*, the number of distinct ways of representing n as a sum of natural numbers regardless of order). By convention, $p(0) = 1$ and $p(n) = 0$ if n is a negative integer. For more information on the partition function $p(n)$, please refer to [26] and the references therein. The first values of the partition function $p(n)$ are (starting with $p(0) = 1$) (see [27]):

$$1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, \ldots.$$

It is easy to see that the cardinality of the set $\mathcal{A}_n$ is equal to the partition function $p(n)$. Now we are ready to present a formula which determines the coefficients a_j in (2.2) with the help of the partition function given by the following lemma.

Lemma 2.1 ([28]) *The following approximation formula holds true*:

$$\left(1 + \frac{1}{x}\right)^x = e \sum_{j=0}^{\infty} \frac{c_j}{x^j} \quad \textit{as } x \to \infty, \tag{2.2}$$

where the coefficients c_j $(j \in \mathbb{N})$ are given by

$$c_0 = 1 \quad \textit{and} \quad c_j = (-1)^j \sum_{(k_1, k_2, \ldots, k_j) \in \mathcal{A}_j} \frac{1}{k_1! k_2! \cdots k_j!} \left(\frac{1}{2}\right)^{k_1} \left(\frac{1}{3}\right)^{k_2} \cdots \left(\frac{1}{j+1}\right)^{k_j}, \tag{2.3}$$

where the $\mathcal{A}_j$ (for $j \in \mathbb{N}$) are given in (2.1).

3 Padé approximant related to asymptotics for the constant *e*

For later use, we introduce the Padé approximant (see [29–34]). Let f be a formal power series

$$f(t) = c_0 + c_1 t + c_2 t^2 + \cdots. \tag{3.1}$$

The Padé approximation of order (p, q) of the function f is the rational function, denoted by

$$[p/q]_f(t) = \frac{\sum_{j=0}^{p} a_j t^j}{1 + \sum_{j=1}^{q} b_j t^j}, \tag{3.2}$$

where $p \geq 0$ and $q \geq 1$ are two given integers, the coefficients a_j and b_j are given by (see [29–31, 33, 34])

$$\begin{cases} a_0 = c_0, \\ a_1 = c_0 b_1 + c_1, \\ a_2 = c_0 b_2 + c_1 b_1 + c_2, \\ \vdots \\ a_p = c_0 b_p + \cdots + c_{p-1} b_1 + c_p, \\ 0 = c_{p+1} + c_p b_1 + \cdots + c_{p-q+1} b_q, \\ \vdots \\ 0 = c_{p+q} + c_{p+q-1} b_1 + \cdots + c_p b_q, \end{cases} \tag{3.3}$$

and the following holds:

$$[p/q]_f(t) - f(t) = O\bigl(t^{p+q+1}\bigr). \tag{3.4}$$

Thus, the first $p + q + 1$ coefficients of the series expansion of $[p/q]_f$ are identical to those of f. Moreover, we have (see [32])

$$[p/q]_f(t) = \frac{\begin{vmatrix} t^q f_{p-q}(t) & t^{q-1} f_{p-q+1}(t) & \cdots & f_p(t) \\ c_{p-q+1} & c_{p-q+2} & \cdots & c_{p+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_p & c_{p+1} & \cdots & c_{p+q} \end{vmatrix}}{\begin{vmatrix} t^q & t^{q-1} & \cdots & 1 \\ c_{p-q+1} & c_{p-q+2} & \cdots & c_{p+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_p & c_{p+1} & \cdots & c_{p+q} \end{vmatrix}}, \tag{3.5}$$

with $f_n(x) = c_0 + c_1 x + \cdots + c_n x^n$, the nth partial sum of the series f (f_n is identically zero for $n < 0$).

Let

$$f(x) = \frac{1}{e}\left(1 + \frac{1}{x}\right)^x. \tag{3.6}$$

It follows from (2.2) that, as $x \to \infty$,

$$f(x) = \sum_{j=0}^{\infty} \frac{c_j}{x^j} = 1 - \frac{1}{2x} + \frac{11}{24x^2} - \frac{7}{16x^3} + \frac{2{,}447}{5{,}760x^4} - \frac{959}{2{,}304x^5} + \frac{238{,}043}{580{,}608x^6} - \cdots, \tag{3.7}$$

with the coefficients c_j given by (2.3). In what follows, the function f is given in (3.6).

We now give a derivation of equation (1.11). To this end, we consider

$$[1/1]_f(x) = \frac{\sum_{j=0}^{1} a_j x^{-j}}{1 + \sum_{j=1}^{1} b_j x^{-j}}.$$

Noting that

$$c_0 = 1, \qquad c_1 = -\frac{1}{2}, \qquad c_2 = \frac{11}{24}, \qquad c_3 = -\frac{7}{16}, \qquad c_4 = \frac{2{,}447}{5{,}760} \tag{3.8}$$

holds, we have, by (3.3),

$$\begin{cases} a_0 = 1, \\ a_1 = b_1 - \frac{1}{2}, \\ 0 = \frac{11}{24} - \frac{1}{2}b_1, \end{cases}$$

that is,

$$a_0 = 1, \qquad a_1 = \frac{5}{12}, \qquad b_1 = \frac{11}{12}.$$

We thus obtain

$$[1/1]_f(x) = \frac{1 + \frac{5}{12x}}{1 + \frac{11}{12x}} = \frac{x + \frac{5}{12}}{x + \frac{11}{12}}, \tag{3.9}$$

and we have, by (3.4),

$$\frac{1}{e}\left(1 + \frac{1}{x}\right)^x - \frac{x + \frac{5}{12}}{x + \frac{11}{12}} = O\left(\frac{1}{x^3}\right), \quad x \to \infty. \tag{3.10}$$

We now give a derivation of equation (1.12). To this end, we consider

$$[2/2]_f(x) = \frac{\sum_{j=0}^{2} a_j x^{-j}}{1 + \sum_{j=1}^{2} b_j x^{-j}}.$$

Noting that (3.8) holds, we have, by (3.3),

$$\begin{cases} a_0 = 1, \\ a_1 = b_1 - \frac{1}{2}, \\ a_2 = b_2 - \frac{1}{2}b_1 + \frac{11}{24}, \\ 0 = -\frac{7}{16} + \frac{11}{24}b_1 - \frac{1}{2}b_2, \\ 0 = \frac{2{,}447}{5{,}760} - \frac{7}{16}b_1 + \frac{11}{24}b_2, \end{cases}$$

that is,

$$a_0 = 1, \qquad a_1 = \frac{87}{100}, \qquad a_2 = \frac{37}{240}, \qquad b_1 = \frac{137}{100}, \qquad b_2 = \frac{457}{1{,}200}.$$

We thus obtain

$$[2/2]_f(x) = \frac{1 + \frac{87}{100x} + \frac{37}{240x^2}}{1 + \frac{137}{100x} + \frac{457}{1{,}200x^2}} = \frac{x^2 + \frac{87}{100}x + \frac{37}{240}}{x^2 + \frac{137}{100}x + \frac{457}{1{,}200}} \tag{3.11}$$

and we have, by (3.4),

$$\frac{1}{e}\left(1+\frac{1}{x}\right)^x - \frac{x^2+\frac{87}{100}x+\frac{37}{240}}{x^2+\frac{137}{100}x+\frac{457}{1{,}200}} = O\left(\frac{1}{x^5}\right), \quad x\to\infty. \tag{3.12}$$

Using the Padé approximation method and the expansion (3.7), we now present a general result given by Theorem 3.1. As a consequence, we obtain (1.16).

Theorem 3.1 *The Padé approximation of order (p,q) of the asymptotic formula of the function $f(x)=\frac{1}{e}(1+\frac{1}{x})^x$ (at the point $x=\infty$) is the following rational function*:

$$[p/q]_f(x) = \frac{1+\sum_{j=1}^{p} a_j x^{-j}}{1+\sum_{j=1}^{q} b_j x^{-j}} = x^{q-p}\left(\frac{x^p+a_1x^{p-1}+\cdots+a_p}{x^q+b_1x^{q-1}+\cdots+b_q}\right), \tag{3.13}$$

where $p\ge 1$ and $q\ge 1$ are two given integers, the coefficients a_j and b_j are given by

$$\begin{cases} a_1 = b_1 + c_1, \\ a_2 = b_2 + c_1 b_1 + c_2, \\ \vdots \\ a_p = b_p + \cdots + c_{p-1}b_1 + c_p, \\ 0 = c_{p+1} + c_p b_1 + \cdots + c_{p-q+1} b_q, \\ \vdots \\ 0 = c_{p+q} + c_{p+q-1} b_1 + \cdots + c_p b_q, \end{cases} \tag{3.14}$$

c_j is given in (2.3), *and the following holds*:

$$f(x) - [p/q]_f(x) = O\left(\frac{1}{x^{p+q+1}}\right), \quad x\to\infty. \tag{3.15}$$

Moreover, we have

$$[p/q]_f(x) = \frac{\begin{vmatrix} \frac{1}{x^q}f_{p-q}(x) & \frac{1}{x^{q-1}}f_{p-q+1}(x) & \cdots & f_p(x) \\ c_{p-q+1} & c_{p-q+2} & \cdots & c_{p+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_p & c_{p+1} & \cdots & c_{p+q} \end{vmatrix}}{\begin{vmatrix} \frac{1}{x^q} & \frac{1}{x^{q-1}} & \cdots & 1 \\ c_{p-q+1} & c_{p-q+2} & \cdots & c_{p+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_p & c_{p+1} & \cdots & c_{p+q} \end{vmatrix}}, \tag{3.16}$$

with $f_n(x)=\sum_{j=0}^{n}\frac{c_j}{x^j}$, the nth partial sum of the asymptotic series (3.7).

Remark 3.1 Using (3.16), we can also derive (3.9) and (3.11). Indeed, we have

$$
\begin{aligned}
[1/1]_f(x) &= \frac{\begin{vmatrix} \frac{1}{x}f_0(x) & f_1(x) \\ c_1 & c_2 \end{vmatrix}}{\begin{vmatrix} \frac{1}{x} & 1 \\ c_1 & c_2 \end{vmatrix}} = \frac{\begin{vmatrix} \frac{1}{x} & 1-\frac{1}{2x} \\ -\frac{1}{2} & \frac{11}{24} \end{vmatrix}}{\begin{vmatrix} \frac{1}{x} & 1 \\ -\frac{1}{2} & \frac{11}{24} \end{vmatrix}} \\
&= \frac{x+\frac{5}{12}}{x+\frac{11}{12}}
\end{aligned}
$$

and

$$
\begin{aligned}
[2/2]_f(x) &= \frac{\begin{vmatrix} \frac{1}{x^2}f_0(x) & \frac{1}{x}f_1(x) & f_2(x) \\ c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \end{vmatrix}}{\begin{vmatrix} \frac{1}{x^2} & \frac{1}{x} & 1 \\ c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \end{vmatrix}} = \frac{\begin{vmatrix} \frac{1}{x^2} & \frac{1}{x}(1-\frac{1}{2x}) & 1-\frac{1}{2x}+\frac{11}{24x^2} \\ -\frac{1}{2} & \frac{11}{24} & -\frac{7}{16} \\ \frac{11}{24} & -\frac{7}{16} & \frac{2,447}{5,760} \end{vmatrix}}{\begin{vmatrix} \frac{1}{x^2} & \frac{1}{x} & 1 \\ -\frac{1}{2} & \frac{11}{24} & -\frac{7}{16} \\ \frac{11}{24} & -\frac{7}{16} & \frac{2,447}{5,760} \end{vmatrix}} \\
&= \frac{x^2+\frac{87}{100}x+\frac{37}{240}}{x^2+\frac{137}{100}x+\frac{457}{1,200}}.
\end{aligned}
$$

Remark 3.2 Setting $(p,q) = (k,k)$ in (3.15), we obtain (1.16).

Setting

$$(p,q) = (3,3) \quad \text{and} \quad (p,q) = (4,4),$$

respectively, we obtain by Theorem 3.1, as $x \to \infty$,

$$\frac{1}{e}\left(1+\frac{1}{x}\right)^x = \frac{x^3+\frac{162,713}{121,212}x^2+\frac{13,927}{26,936}x+\frac{41,501}{786,240}}{x^3+\frac{223,319}{121,212}x^2+\frac{237,551}{242,424}x+\frac{3,950,767}{29,090,880}} + O\left(\frac{1}{x^7}\right) \tag{3.17}$$

and

$$
\begin{aligned}
\frac{1}{e}\left(1+\frac{1}{x}\right)^x &= \frac{x^4+\frac{1,157,406,727}{634,301,284}x^3+\frac{8,452,872,239}{7,611,615,408}x^2+\frac{81,587,251,465}{319,687,847,136}x+\frac{15,842,677}{924,376,320}}{x^4+\frac{1,474,557,369}{634,301,284}x^3+\frac{13,811,559,391}{7,611,615,408}x^2+\frac{170,870,679,559}{319,687,847,136}x+\frac{1,724,393,461,793}{38,362,541,656,320}} \\
&\quad + O\left(\frac{1}{x^9}\right).
\end{aligned}
\tag{3.18}
$$

Equations (3.17) and (3.18) motivate us to establish the following theorem.

Theorem 3.2 *For $x > 0$,*

$$\left(1+\frac{1}{x}\right)^x < e\left(\frac{x^3+\frac{162,713}{121,212}x^2+\frac{13,927}{26,936}x+\frac{41,501}{786,240}}{x^3+\frac{223,319}{121,212}x^2+\frac{237,551}{242,424}x+\frac{3,950,767}{29,090,880}}\right) \tag{3.19}$$

and

$$\left(1+\frac{1}{x}\right)^{x} < e\left(\frac{x^4+\frac{1{,}157{,}406{,}727}{634{,}301{,}284}x^3+\frac{8{,}452{,}872{,}239}{7{,}611{,}615{,}408}x^2+\frac{81{,}587{,}251{,}465}{319{,}687{,}847{,}136}x+\frac{15{,}842{,}677}{924{,}376{,}320}}{x^4+\frac{1{,}474{,}557{,}369}{634{,}301{,}284}x^3+\frac{13{,}811{,}559{,}391}{7{,}611{,}615{,}408}x^2+\frac{170{,}870{,}679{,}559}{319{,}687{,}847{,}136}x+\frac{1{,}724{,}393{,}461{,}793}{38{,}362{,}541{,}656{,}320}}\right). \tag{3.20}$$

Proof We only prove the inequality (3.20). The proof of (3.19) is analogous. In order to prove (3.20), it suffices to show that

$$F(x) < 0 \quad \text{for } x > 0,$$

where

$$F(x) = x\ln\left(1+\frac{1}{x}\right) - 1 - \ln\left(\frac{x^4+\frac{1{,}157{,}406{,}727}{634{,}301{,}284}x^3+\frac{8{,}452{,}872{,}239}{7{,}611{,}615{,}408}x^2+\frac{81{,}587{,}251{,}465}{319{,}687{,}847{,}136}x+\frac{15{,}842{,}677}{924{,}376{,}320}}{x^4+\frac{1{,}474{,}557{,}369}{634{,}301{,}284}x^3+\frac{13{,}811{,}559{,}391}{7{,}611{,}615{,}408}x^2+\frac{170{,}870{,}679{,}559}{319{,}687{,}847{,}136}x+\frac{1{,}724{,}393{,}461{,}793}{38{,}362{,}541{,}656{,}320}}\right).$$

Differentiation yields

$$F'(x) = \ln\left(1+\frac{1}{x}\right) - \frac{P_8(x)}{P_9(x)},$$

where

$$\begin{aligned} P_8(x) = {} & 4{,}534{,}960{,}145{,}139{,}175{,}220{,}907{,}601 + 89{,}156{,}435{,}404{,}854{,}709{,}617{,}164{,}400x \\ & + 753{,}611{,}422{,}427{,}554{,}143{,}580{,}166{,}880x^2 \\ & + 3{,}400{,}732{,}641{,}706{,}885{,}239{,}015{,}784{,}320x^3 \\ & + 8{,}959{,}898{,}009{,}119{,}992{,}740{,}647{,}591{,}680x^4 \\ & + 14{,}212{,}846{,}466{,}921{,}911{,}377{,}490{,}790{,}400x^5 \\ & + 13{,}355{,}464{,}865{,}044{,}929{,}241{,}744{,}281{,}600x^6 \\ & + 6{,}842{,}437{,}276{,}900{,}714{,}847{,}214{,}796{,}800x^7 \\ & + 1{,}471{,}684{,}602{,}332{,}887{,}248{,}995{,}942{,}400x^8 \end{aligned}$$

and

$$\begin{aligned} P_9(x) = {} & \big(38{,}362{,}541{,}656{,}320x^4 + 69{,}999{,}958{,}848{,}960x^3 + 42{,}602{,}476{,}084{,}560x^2 \\ & + 9{,}790{,}470{,}175{,}800x + 657{,}486{,}938{,}177\big)\big(38{,}362{,}541{,}656{,}320x^4 \\ & + 89{,}181{,}229{,}677{,}120x^3 + 69{,}610{,}259{,}330{,}640x^2 + 20{,}504{,}481{,}547{,}080x \\ & + 1{,}724{,}393{,}461{,}793\big)(x+1). \end{aligned}$$

Differentiating $F'(x)$, we find

$$F''(x) = -\frac{Q_8(x)}{Q_{19}(x)},$$

where

$$\begin{aligned}Q_8(x) = {}& 1{,}285{,}425{,}745{,}031{,}439{,}744{,}924{,}351{,}944{,}181{,}267{,}498{,}830{,}297{,}392{,}321 \\ &+ 28{,}378{,}097{,}964{,}665{,}213{,}870{,}448{,}253{,}775{,}917{,}974{,}735{,}833{,}555{,}915{,}520x \\ &+ 247{,}639{,}239{,}538{,}550{,}650{,}618{,}428{,}925{,}475{,}351{,}177{,}418{,}903{,}828{,}519{,}360x^2 \\ &+ 1{,}131{,}116{,}309{,}072{,}948{,}249{,}686{,}419{,}776{,}599{,}013{,}563{,}965{,}352{,}036{,}853{,}760x^3 \\ &+ 2{,}998{,}129{,}273{,}934{,}033{,}621{,}834{,}452{,}343{,}529{,}577{,}599{,}070{,}175{,}646{,}117{,}120x^4 \\ &+ 4{,}775{,}194{,}702{,}079{,}256{,}668{,}486{,}950{,}292{,}217{,}012{,}539{,}098{,}845{,}384{,}867{,}840x^5 \\ &+ 4{,}503{,}188{,}365{,}939{,}207{,}771{,}317{,}966{,}173{,}833{,}346{,}921{,}724{,}385{,}791{,}590{,}400x^6 \\ &+ 2{,}315{,}562{,}242{,}935{,}704{,}170{,}341{,}114{,}308{,}201{,}588{,}127{,}064{,}283{,}807{,}744{,}000x^7 \\ &+ 500{,}009{,}489{,}498{,}922{,}911{,}594{,}629{,}442{,}997{,}057{,}334{,}195{,}586{,}408{,}448{,}000x^8\end{aligned}$$

and

$$\begin{aligned}Q_{19}(x) = {}& x\bigl(38{,}362{,}541{,}656{,}320x^4 + 69{,}999{,}958{,}848{,}960x^3 + 42{,}602{,}476{,}084{,}560x^2 \\ &+ 9{,}790{,}470{,}175{,}800x + 657{,}486{,}938{,}177\bigr)^2\bigl(38{,}362{,}541{,}656{,}320x^4 \\ &+ 89{,}181{,}229{,}677{,}120x^3 + 69{,}610{,}259{,}330{,}640x^2 + 20{,}504{,}481{,}547{,}080x \\ &+ 1{,}724{,}393{,}461{,}793\bigr)^2(x+1)^2.\end{aligned}$$

Hence, $F''(x) < 0$ for $x > 0$, and we have

$$F'(x) > \lim_{t\to\infty} F'(t) = 0 \quad \Longrightarrow \quad F(x) < \lim_{t\to\infty} F(t) = 0 \quad \text{for } x > 0.$$

The proof is complete. □

The inequality (3.20) can be written as

$$\left(1 + \frac{1}{x}\right)^x < e\bigl(1 - \mathcal{E}(x)\bigr), \quad x > 0, \tag{3.21}$$

where

$$\begin{aligned}\mathcal{E}(x) = {}& 48\bigl(399{,}609{,}808{,}920x^3 + 562{,}662{,}150{,}960x^2 \\ &+ 223{,}208{,}570{,}235x + 22{,}227{,}219{,}242\bigr)/\bigl(38{,}362{,}541{,}656{,}320x^4 \\ &+ 89{,}181{,}229{,}677{,}120x^3 + 69{,}610{,}259{,}330{,}640x^2 + 20{,}504{,}481{,}547{,}080x \\ &+ 1{,}724{,}393{,}461{,}793\bigr).\end{aligned} \tag{3.22}$$

4 A generalized Carleman-type inequality

Theorem 4.1 *Let* $0 < \lambda_{n+1} \le \lambda_n$, $\Lambda_n = \sum_{m=1}^{n} \lambda_m$ $(\Lambda_n \ge 1)$, $a_n \ge 0$ $(n \in \mathbb{N})$ *and* $0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty$. *Then, for* $0 < p \le 1$,

$$\sum_{n=1}^{\infty} \lambda_{n+1}\bigl(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n}\bigr)^{1/\Lambda_n} < \frac{e^p}{p} \sum_{n=1}^{\infty} \left(1 - \mathcal{E}\left(\frac{\Lambda_n}{\lambda_n}\right)\right)^p \lambda_n a_n^p \Lambda_n^{p-1} \left(\sum_{k=1}^{n} \lambda_k (c_k a_k)^p\right)^{(1-p)/p}, \tag{4.1}$$

where $\mathcal{E}(x)$ *is given in* (3.22) *and*

$$c_n^{\lambda_n} = \frac{(\Lambda_{n+1})^{\Lambda_n}}{(\Lambda_n)^{\Lambda_{n-1}}}.$$

Proof The inequality

$$\sum_{n=1}^{\infty} \lambda_{n+1}\bigl(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n}\bigr)^{1/\Lambda_n} \le \frac{1}{p} \sum_{m=1}^{\infty} \left(1 + \frac{1}{\Lambda_m/\lambda_m}\right)^{p\Lambda_m/\lambda_m} \lambda_m a_m^p \Lambda_m^{p-1} \left(\sum_{k=1}^{m} \lambda_k (c_k a_k)^p\right)^{(1-p)/p} \tag{4.2}$$

has been proved in Theorem 2.2 of [9] (see also [11, p.96]). From the above inequality and (3.20), we obtain (4.1). The proof is complete. □

Remark 4.1 In Theorem 2.2 of [9], $c_k^{\lambda_n} = \frac{(\Lambda_{n+1})^{\Lambda_n}}{(\Lambda_n)^{\Lambda_{n-1}}}$ should be $c_n^{\lambda_n} = \frac{(\Lambda_{n+1})^{\Lambda_n}}{(\Lambda_n)^{\Lambda_{n-1}}}$; see [9, p.44, line 3]. Likewise, $c_s^{\lambda_n} = \frac{(\Lambda_{n+1})^{\Lambda_n}}{(\Lambda_n)^{\Lambda_{n-1}}}$ in Theorem 3.1 of [11] should be $c_n^{\lambda_n} = \frac{(\Lambda_{n+1})^{\Lambda_n}}{(\Lambda_n)^{\Lambda_{n-1}}}$; see [11, p.96, equation (9)].

Remark 4.2 Taking $p = 1$ in (4.1) yields

$$\sum_{n=1}^{\infty} \lambda_{n+1}\bigl(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n}\bigr)^{1/\Lambda_n} < e \sum_{n=1}^{\infty} \left(1 - \mathcal{E}\left(\frac{\Lambda_n}{\lambda_n}\right)\right) \lambda_n a_n, \tag{4.3}$$

which improves (1.6). Taking $\lambda_n \equiv 1$ in (4.3) yields

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \bigl(1 - \mathcal{E}(n)\bigr) a_n, \tag{4.4}$$

which improves (1.9).

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References

1. Carleman, T: Sur les fonctions quasi-analytiques. In: Comptes rendus du V^e Congres des Mathematiciens Scandinaves. Helsingfors, pp. 181-196 (1922)
2. Hardy, GH: Notes on some points in the integral calculus. Messenger Math. **54**, 150-156 (1925)
3. Hardy, GH, Littlewood, JE, Pólya, G: Inequalities, 2nd edn. Cambridge University Press, Cambridge (1952)
4. Čižmešija, A, Pečarić, J, Persson, LE: On strengthened Carleman's inequality. Bull. Aust. Math. Soc. **68**, 481-490 (2003)
5. Chen, HW: On an infinite series for $(1 + 1/x)^x$ and its application. Int. J. Math. Math. Sci. **29**, 675-680 (2002)
6. Chen, CP: Generalization of weighted Carleman-type inequality. East J. Approx. **12**, 63-69 (2006)
7. Chen, CP, Qi, F: Generalization of Hardy's inequality. Proc. Jangjeon Math. Soc. **7**, 57-61 (2004)
8. Chen, CP, Cheung, WS, Qi, F: Note on weighted Carleman type inequality. Int. J. Math. Math. Sci. **3**, 475-481 (2005)
9. Dragomir, SS, Kim, YH: The strengthened Hardy inequalities and their new generalizations. Filomat **20**, 39-49 (2006)
10. Hu, Y: A strengthened Carleman's inequality. Commun. Math. Anal. **1**, 115-119 (2006)
11. Lü, Z, Gao, Y, Wei, Y: Note on the Carleman's inequality and Hardy's inequality. Comput. Math. Appl. **59**, 94-97 (2010)
12. Li, JL: Notes on an inequality involving the constant *e*. J. Math. Anal. Appl. **250**, 722-725 (2000)
13. Liu, HP, Zhu, L: New strengthened Carleman's inequality and Hardy's inequality. J. Inequal. Appl. **2007**, Article ID 84104 (2007). http://link.springer.com/article/10.1155/2007/84104/fulltext.html
14. Mortici, C, Hu, Y: On some convergences to the constant *e* and improvements of Carleman's inequality. Carpath. J. Math. **31**, 249-254 (2015)
15. Xie, Z, Zhong, Y: A best approximation for constant *e* and an improvement to Hardy's inequality. J. Math. Anal. Appl. **252**, 994-998 (2000)
16. Yan, P, Sun, GZ: A strengthened Carleman's inequality. J. Math. Anal. Appl. **240**, 290-293 (1999)
17. Yang, BC, Debnath, L: Some inequalities involving the constant *e*, and an application to Carleman's inequality. J. Math. Anal. Appl. **223**, 347-353 (1998)
18. Yang, BC: On Hardy's inequality. J. Math. Anal. Appl. **234**, 717-722 (1999)
19. Yang, XJ: On Carleman's inequality. J. Math. Anal. Appl. **253**, 691-694 (2001)
20. Yang, XJ: Approximations for constant *e* and their applications. J. Math. Anal. Appl. **262**, 651-659 (2001)
21. Johansson, M, Persson, LE, Wedestig, A: Carleman's inequality - history, proofs and some new generalizations. J. Inequal. Pure Appl. Math. **4**(3), Article ID 53 (2003). http://jipam.vu.edu.au/article.php?sid=291
22. Kaijser, S, Persson, LE, Öberg, A: On Carleman and Knopp's inequalities. J. Approx. Theory **117**, 140-151 (2002)
23. Kufner, A, Persson, LE: Weighted Inequalities of Hardy Type. World Scientific, Singapore (2003)
24. Pečarić, J, Stolarsky, KB: Carleman's inequality: history and new generalizations. Aequ. Math. **61**, 49-62 (2001)
25. Pólya, G: Proof of an inequality. Proc. Lond. Math. Soc. **24**, 57 (1926)
26. Wikipedia contributors: Partition (number theory), Wikipedia, the free encyclopedia. http://en.wikipedia.org/wiki/Partition_function_(number_theory)#Partition_function
27. Sloane, NJA: $a(n)$ = number of partitions of n (the partition numbers). Maintained by The OEIS Foundation. http://oeis.org/A000041
28. Chen, CP, Choi, J: Asymptotic formula for $(1 + 1/x)^x$ based on the partition function. Am. Math. Mon. **121**, 338-343 (2014)
29. Bercu, G: Padé approximant related to remarkable inequalities involving trigonometric functions. J. Inequal. Appl. **2016** 99 (2016). http://www.doc88.com/p-0037658479714.html
30. Bercu, G: The natural approach of trigonometric inequalities-Padé approximant. J. Math. Inequal. **11**, 181-191 (2017)
31. Bercu, G, Wu, S: Refinements of certain hyperbolic inequalities via the Padé approximation method. J. Nonlinear Sci. Appl. **9**, 5011-5020 (2016)
32. Brezinski, C, Redivo-Zaglia, M: New representations of Padé, Padé-type, and partial Padé approximants. J. Comput. Appl. Math. **284**, 69-77 (2015)
33. Li, X, Chen, CP: Padé approximant related to asymptotics for the gamma function. J. Inequal. Appl. **2017**, 53 (2017). http://journalofinequalitiesandapplications.springeropen.com/articles/10.1186/s13660-017-1315-1
34. Liu, J, Chen, CP: Padé approximant related to inequalities for Gauss lemniscate functions. J. Inequal. Appl. **2016**, 320 (2016). http://journalofinequalitiesandapplications.springeropen.com/articles/10.1186/s13660-016-1262-2