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On the least signless Laplacian eigenvalue of a non-bipartite connected graph with fixed maximum degree

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Abstract

In this paper, we determine the unique graph whose least signless Laplacian eigenvalue attains the minimum among all non-bipartite unicyclic graphs of order n with maximum degree Δ and among all non-bipartite connected graphs of order n with maximum degree Δ, respectively.

MSC: 05C50

Keywords: non-bipartite graph; signless Laplacian; least eigenvalue; maximum degree

1 Introduction

All graphs considered in this paper are finite, simple and undirected. Let G be a graph with vertex set $V = V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E = E(G)$. Write $A(G)$ for the adjacency matrix of G and let $D(G)$ be the diagonal matrix of the degrees of G. The matrix $Q(G) = D(G) + A(G)$ is called the signless Laplacian matrix of G. As usual, let $q_1(G) \geq q_2(G) \geq \cdots \geq q_n(G) \geq 0$ denote the eigenvalues of $Q(G)$ and call them the signless Laplacian eigenvalues of G. Denote by $\kappa(G)$ the least eigenvalue of G.

For a connected graph G, $\kappa(G) = 0$ if and only if G is bipartite. Desai and Rao [1] suggest the use of $\kappa(G)$ as a measure of non-bipartiteness of G. Fallat and Fan [2] introduce two parameters reflecting the graph bipartiteness, and establish a relationship between $\kappa(G)$ and the two parameters. de Lima, Nikiforov and Oliveira [3] point out that $\kappa(G)$ depends more on the distribution of the edges of a graph than on their number, so it may become a useful tool in extremal graph theory. *For a connected non-bipartite graph G with given order, how small can $\kappa(G)$ be?* Cardoso *et al.* [4] propose this problem and show that the minimum value of $\kappa(G)$ is attained uniquely in the unicyclic graph obtained from the cycle C_3 by attaching a path at one of its end vertices. Motivated by this problem, a good deal of attention has been devoted to finding all graphs with the minimal least signless Laplacian eigenvalue among a given class of graphs. For related results, one may refer to [5–14].

A unicyclic graph is a connected graph with a unique cycle. Let $\Delta = \Delta(G)$ be the maximum degree of a graph G. In this paper, we determine the unique graph whose least signless Laplacian eigenvalue attains the minimum among all non-bipartite unicyclic graphs of

order n with maximum degree Δ and among all non-bipartite connected graphs of order n with maximum degree Δ, respectively.

The rest of the paper is organized as follows. In Section 2, we recall some notions and lemmas used further, and prove three new lemmas. In Section 3, we prove two theorems which is our main result. In Section 4, we propose two problems for further research.

2 Preliminaries

Denote by C_n the cycle on n vertices. Let $G - uv$ denote the graph which arises from G by deleting the edge $uv \in E(G)$. Similarly, $G + uv$ is the graph that arises from G by adding an edge $uv \notin E(G)$, where $u, v \in V(G)$. For $v \in V(G)$, $N(v)$ denotes the neighborhood of v in G and $d(v) = |N(v)|$ denotes the degree of vertex v. A pendant vertex of G is a vertex of degree 1. $|x|$ denotes the absolute value of a real number x. The terminology not defined here can be found in [15].

Lemma 2.1 ([16]) *Let G be a graph on n vertices, e be an edge of G. Then*

$$q_1(G) \geq q_1(G - e) \geq q_2(G) \geq q_2(G - e) \geq \cdots \geq q_n(G) \geq q_n(G - e) \geq 0.$$

Given $x = (x_1, x_2, \ldots, x_n)^T \in R^n$, we can define a function on $V(G)$, that is, each vertex v_i is mapped to $x_i = x(v_i)$. If x is an eigenvector of $Q(G)$, then it is defined on G naturally, *i.e.* $x(v)$ is the entry of x corresponding to v. Clearly, for $x \in R^n$,

$$x^T Q(G) x = \sum_{uv \in E(G)} \big(x(u) + x(v) \big)^2.$$

Let $x \in R^n$ be an arbitrary unit vector. One can find in [10, 15] that

$$\kappa(G) \leq x^T Q(G) x, \tag{1}$$

with equality if and only if x is an eigenvector corresponding to $\kappa(G)$.

Let G_1 and G_2 be two vertex-disjoint connected graphs, and let $v_i \in V(G_i)$ for $i = 1, 2$. Identifying the v_1 with v_2 and forming a new vertex u (see [10] for details), the resulting graph is called coalescence of G_1 and G_2, and denoted by $G_1(v_1) \diamond G_2(v_2)$ or $G_1(u) \diamond G_2(u)$. If a connected graph G can be expressed in the form $G_1(u) \diamond G_2(u)$, where G_1 and G_2 are both nontrivial and connected, then G_1 is called a branch of G with root u. Clearly G_2 is also a branch of G with root u. Let $x \in R^n$ be a vector defined on $V(G)$. A branch G_i of G is called a zero branch with respect to x if $x(v) = 0$ for all $v \in V(G_i)$; otherwise it is called a nonzero branch with respect to x.

Lemma 2.2 ([10]) *Let G be a connected graph which contains a bipartite branch B with root u, and x be an eigenvector corresponding to $\kappa(G)$.*
  (i) *If $x(u) = 0$, then B is a zero branch of G with respect to x.*
  (ii) *If $x(u) \neq 0$, then $x(v) \neq 0$ for every vertex $v \in V(B)$.*

Lemma 2.3 ([10]) *Let G be a non-bipartite connected graph, and let x be an eigenvector corresponding to $\kappa(G)$. Let T be a tree, which is a nonzero branch of G with respect to x*

Figure 1 $C(v_0) \diamond B(v_0)$.

Figure 2 $U_n^k(g)$ and $U_n^*(3, \Delta)$.
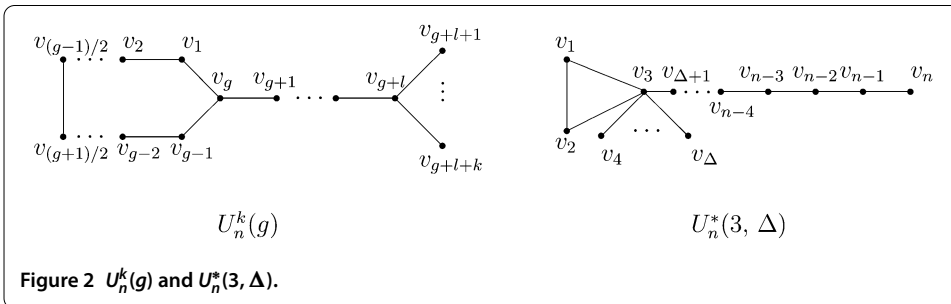
and with root u. Then $|x(q)| < |x(p)|$ whenever p, q are vertices of T such that q lies on the unique path from u to p.

Lemma 2.4 ([12]) *Let $G = C(v_0) \diamond B(v_0)$ (see Figure 1), where $C = v_0 v_1 v_2 \cdots v_k u_k u_{k-1} \cdots u_1 v_0$ is a cycle of length $2k + 1$ and B is a nontrivial connected bipartite graph. Let $x = (x(v_0), x(v_1), x(v_2), \ldots, x(v_k), x(u_1), x(u_2), \ldots, x(u_k), \ldots)^T$ be an eigenvector corresponding to $\kappa(G)$. Then*

(i) *$|x(v_0)| = \max\{|x(w)| \mid w \in V(C)\} > 0$;*

(ii) *$x(v_i) = x(u_i)$ for $i = 1, 2, \ldots, k$.*

Lemma 2.5 ([12]) *Let $G = G_1(v_2) \diamond T(u)$ and $G^* = G_1(v_1) \diamond T(u)$, where G_1 is a non-bipartite connected graph containing two distinct vertices v_1, v_2, and T is a nontrivial tree. If there exists an eigenvector $x = (x(v_1), x(v_2), \ldots, x(v_k), \ldots)^T$ corresponding to $\kappa(G)$ such that $|x(v_1)| > |x(v_2)|$ or $|x(v_1)| = |x(v_2)| > 0$, then $\kappa(G^*) < \kappa(G)$.*

For $k \geq 1$, let G' denote the graph obtained from G by deleting the edge uv, inserting k new vertices $v_1, v_2, \ldots, v_k$ and adding edges $uv_1, v_1 v_2, \ldots, v_{k-1} v_k, v_k v$. Then G' is called a k-subdivision graph of G by k-subdividing the edge uv.

Lemma 2.6 ([17]) *Let G' be a k-subdivision graph of a graph G. If k is even, then $\kappa(G') \leq \kappa(G)$.*

$U_n^k(g)$, showed in Figure 2, denotes the unicyclic graph on n vertices with odd girth g and k pendant vertices, where $g + l + k = n$. $U_n^*(3, \Delta)$, showed in Figure 2, denotes the unicyclic graph on n vertices obtained from the cycle $C_3 = v_1 v_2 v_3 v_1$ by attaching $\Delta - 3$ pendant edges and one pendant path at the vertex v_3.

Lemma 2.7 ([5, 9]) *Among all non-bipartite connected graphs on n vertices with k pendant vertices, $U_n^k(3)$ is the unique graph whose signless Laplacian eigenvalue attains the minimum.*

Lemma 2.8 ([5]) *Let $k \geq 2$, and $g \geq 3$ be an odd integer. Then $\kappa(U_n^{k-1}(g)) < \kappa(U_n^k(g))$.*

Lemma 2.9 *Let $G = G_1(v) \diamond B(v)$ be a connected graph, where G_1 is a graph of order n, and B is a bipartite graph of order s. Then $\kappa(G) \leq \kappa(G_1)$. Moreover, if $s > 1$, G_1 is non-bipartite and there exists an eigenvector x corresponding to $\kappa(G_1)$ such that $x(v) \neq 0$, then $\kappa(G) < \kappa(G_1)$.*

Proof Let $V(G_1) = \{v_1, v_2, \ldots, v_n\}$, and $x = (x(v_1), x(v_2), \ldots, x(v_n))^T$ be a unit eigenvector corresponding to $\kappa(G_1)$. Then

$$\kappa(G_1) = \sum_{v_i v_j \in E(G_1)} \left(x(v_i) + x(v_j) \right)^2.$$

Without loss generality, we may assume $v = v_n$. Let $V(B) = \{v_n, v_{n+1}, \ldots, v_{n+s-1}\}$, and let (U, W) be the two parts of the bipartite graph B, where $v \in U$. Let $y = (y(v_1), y(v_2), \ldots, y(v_n),$ $y(v_{n+1}), \ldots, y(v_{n+s-1}))^T \in R^{n+s-1}$ defined on $V(G)$ satisfy that $y(v_i) = x(v_i)$ for $i = 1, 2, \ldots, n$, $y(u) = x(v)$ if $u \in U$, and $y(u) = -x(v)$ if $u \in W$. Then

$$\|y\|^2 = \sum_{i=1}^{n+s-1} y(v_i)^2 = \sum_{i=1}^{n} x(v_i)^2 + (s-1)x(v)^2 \geq \sum_{i=1}^{n} x(v_i)^2 = \|x\|^2 = 1,$$

$$\kappa(G^*) \leq \frac{1}{\|y\|^2} \sum_{v_i v_j \in E(G^*)} \left(y(v_i) + y(v_j) \right)^2 \leq \frac{1}{\|x\|^2} \sum_{v_i v_j \in E(G)} \left(x(v_i) + x(v_j) \right)^2 = \kappa(G).$$

Clearly, if $s > 1$, G_1 is non-bipartite and $x(v) \neq 0$, we have $\|y\|^2 > \|x\|^2$. This implies that $\kappa(G) < \kappa(G_1)$. □

Lemma 2.10 *Let $n \geq 9$ and $s \geq 0$ be integer. G_1 and G_2, shown in Figure 3, are two unicyclic graphs of order n. Then $\kappa(G_2) < \kappa(G_1)$.*

Proof Let $\kappa = \kappa(G_1)$, and $x = (x_1, x_2, \ldots, x_n)^T$ be a unit eigenvector corresponding to κ. Then $\kappa = \sum_{v_i v_j \in E(G_1)} (x_i + x_j)^2$ and $0 < \kappa < 1$. By Lemmas 2.2 and 2.4, we have $x_n \neq 0$. From
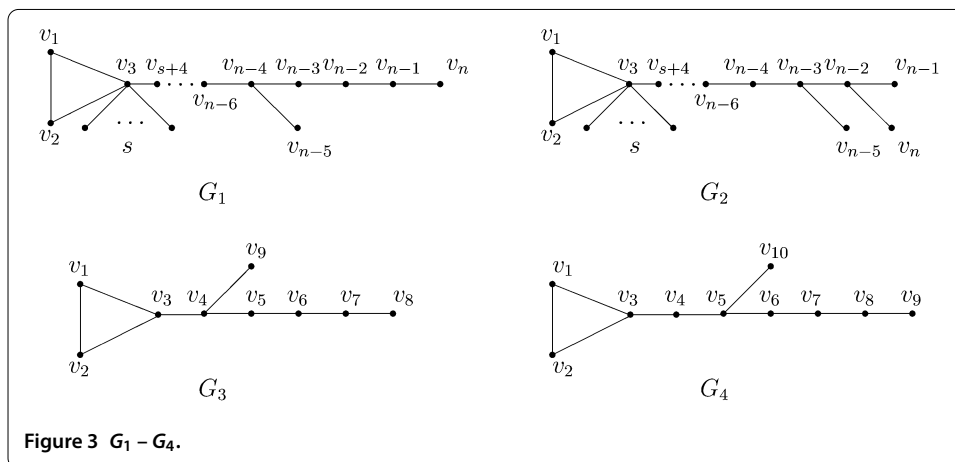
Figure 3 $G_1 - G_4$.

the eigenvalue equation $Q(G_1)x = \kappa x$, we have

$$x_{n-1} = (\kappa - 1)x_n,$$

$$x_{n-2} = (\kappa^2 - 3\kappa + 1)x_n,$$

$$x_{n-3} = (\kappa^3 - 5\kappa^2 + 6\kappa - 1)x_n,$$

$$x_{n-4} = (\kappa^4 - 7\kappa^3 + 15\kappa^2 - 10\kappa + 1)x_n,$$

$$x_{n-5} = (\kappa^3 - 6\kappa^2 + 9\kappa - 1)x_n.$$

Let $y = (y_1, y_2, \ldots, y_n)^T \in R^n$ defined on $V(G_2)$ satisfy that

$$y_{n-5} = -(x_{n-3} + x_{n-4} + x_{n-5}),$$

$$y_n = -(x_n + x_{n-1} + x_{n-2}),$$

and $y_i = x_i$ for $i = 1, 2, \ldots, n - 6, n - 4, n - 3, n - 2, n - 1$. Then

$$\sum_{v_i v_j \in E(G_2)} (y_i + y_j)^2 = \sum_{v_i v_j \in E(G_1)} (x_i + x_j)^2 = \kappa,$$

and

$$\|y\|^2 - \|x\|^2 = \sum_{i=1}^{n} y_i^2 - \sum_{i=1}^{n} x_i^2$$

$$= \kappa (\kappa^7 - 10\kappa^6 + 32\kappa^5 - 18\kappa^4 - 89\kappa^3 + 156\kappa^2 - 70\kappa + 4)x_n^2.$$

Let $f(t) = t^7 - 10t^6 + 32t^5 - 18t^4 - 89t^3 + 156t^2 - 70t + 4$. By a computation, $f(t) = 0$ has five real roots which are approximately equal to –1.7787, 0.0667, 0.6606, 2, 2.0890, respectively. By Lemma 2.9, we have

$$\kappa = \kappa(G_1) \leq \kappa(G_1 - v_4 - \cdots - v_{s+3}).$$

Note that $G_1 - v_4 - \cdots - v_{s+3}$ is a $2t$-subdividing graph of G_3 or G_4 (shown in Figure 3). By Lemma 2.6, we have

$$\kappa = \kappa(G_1) \leq \kappa(G_1 - v_4 - \cdots - v_{s+3}) \leq \min\{\kappa(G_3), \kappa(G_4)\}.$$

By a computation, we have $\kappa(G_3) \approx 0.0588025$ and $\kappa(G_5) \approx 0.0426304$. It follows that $\kappa < 0.0667$. Noting that $f(0) = 4$, we have $f(\kappa) > 0$. It follows that $\|y\|^2 > \|x\|^2$.

Combining the above arguments, we have

$$\kappa(G_2) \leq \frac{1}{\|y\|^2} \sum_{v_i v_j \in E(G_2)} (y_i + y_j)^2 < \frac{1}{\|x\|^2} \sum_{v_i v_j \in E(G_1)} (x_i + x_j)^2 = \kappa(G_1).$$

This completes the proof. □

Lemma 2.11 *Let $n \geq 9$, and $U_n^{n-5}(3)$, $U_n^{n-4}(3)$, $U_n^*(3, n-4)$, $U_n^*(3, n-3)$ be shown in Figure 2. Then*

$$\kappa\left(U_n^{n-5}(3)\right) < \kappa\left(U_n^*(3, n-4)\right), \qquad \kappa\left(U_n^{n-4}(3)\right) < \kappa\left(U_n^*(3, n-3)\right).$$

Proof Let $\kappa = \kappa(U_n^*(3, n-4))$, and $x = (x_1, x_2, \ldots, x_n)^T$ be a unit eigenvector corresponding to κ. By Corollary 1.3 of [18], it is easy to see $\kappa(G) < 1/2$. From the eigenvalue equation $Q(U_n^*(3, n-4))x = \kappa x$, we have $x_1 = x_2$, $x_4 = \cdots = x_{n-4}$,

$$(\kappa - 2)x_1 = x_1 + x_3,$$

$$(\kappa - n + 4)x_3 = 2x_1 + (n - 7)x_4 + x_{n-3},$$

$$(\kappa - 1)x_4 = x_3,$$

$$(\kappa - 2)x_{n-3} = x_3 + x_{n-2},$$

$$(\kappa - 2)x_{n-2} = x_{n-3} + x_{n-1},$$

$$(\kappa - 2)x_{n-1} = x_{n-2} + x_n,$$

$$(\kappa - 1)x_n = x_{n-1}.$$

Since $x = (x_1, x_2, \ldots, x_n)^T$ is an eigenvector, $x \neq 0$. It follows that

$$\begin{vmatrix} \kappa - 3 & -1 & 0 & 0 & 0 & 0 & 0 \\ -2 & \kappa - n + 4 & 7 - n & -1 & 0 & 0 & 0 \\ 0 & -1 & \kappa - 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & \kappa - 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & \kappa - 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & \kappa - 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & \kappa - 1 \end{vmatrix} = 0.$$

This implies that κ is the least root of the following equation:

$$f(x) \triangleq x^7 - (n + 7)x^6 + (10n + 6)x^5 - (36n - 48)x^4 + (55n - 99)x^3$$
$$- (31n - 15)x^2 + (3n + 40)x - 4 = 0.$$

Similarly, we can see that $\kappa(U_n^{n-5}(3))$ is the least root of the following equation:

$$g(x) \triangleq x^5 - (n + 5)x^4 + (8n - 6)x^3 - (18n - 42)x^2 + (11n - 28)x - 4 = 0.$$

Noting that $g(0) = -4 < 0$ and

$$f(x) - (x - 1)^2 g(x) = x(x - 1)\left(x^3 - nx^2 - (n - 19)x + 8n - 60\right) < 0$$

for $0 < x < 1/2$, we have $g(\kappa) > 0$, and so

$$\kappa\left(U_n^{n-5}(3)\right) < \kappa = \kappa\left(U_n^*(3, n-4)\right).$$

By a similar reasoning to above, we can see that $\kappa(U_n^*(3, n-3))$ and $\kappa(U_n^{n-4}(3))$ are the least root of the following equations respectively:

$$h(x) \triangleq x^6 - (n+6)x^5 + (8n+5)x^4 - (21n-18)x^3 + (19n-10)x^2 - (3n+24)x + 4 = 0,$$

$$r(x) \triangleq x^4 - (n+4)x^3 + (6n-5)x^2 - (7n-12)x + 4 = 0.$$

Noting that $r(0) = 4 > 0$ and

$$h(x) - (x-1)^2 r(x) = x\left(x^3 - nx^2 + (n-15)x + 4n - 28\right) > 0$$

for $0 < x < 1/2$, we have $r(\kappa(U_n^*(3, n-3))) < 0$, and so

$$\kappa\left(U_n^{n-4}(3)\right) < \kappa\left(U_n^*(3, n-3)\right).$$

This completes the proof.   □

3 Main results

Let $\mathcal{U}(n, \Delta)$ be the set of non-bipartite unicyclic graphs of order n with maximum degree Δ, and $\mathcal{G}(n, \Delta)$ be the set of non-bipartite connected graphs of order n with maximum degree Δ. In this section, we firstly determine the unicyclic graph whose signless Laplacian eigenvalue attains the minimum among all graphs in $\mathcal{U}(n, \Delta)$.

Theorem 3.1 *Let $4 \leq \Delta \leq n-3$. Among all graphs in $\mathcal{U}(n, \Delta)$, $U_n^{\Delta-1}(3)$ is the unique graph whose signless Laplacian eigenvalue attains the minimum.*

Proof Let $G \in \mathcal{U}(n, \Delta)$, and $C_g = v_1 v_2 \ldots v_g v_1$ be the unique cycle of G. Then g is odd, and G can be obtained by attaching trees $T_1, T_2, \ldots, T_g$ to the vertices $v_1, v_2, \ldots, v_g$ of C_g, respectively, where T_i contains the root vertex v_i for $i = 1, 2, \ldots, g$. $|V(T_i)| = 1$ means $V(T_i) = \{v_i\}$. Suppose that G has k pendant vertices. It is easy to see $\Delta \leq k + 2$. Let $x = (x_1, x_2, \ldots, x_n)^T$ be a unit eigenvector corresponding to $\kappa(G)$.

Case 1. $\Delta \leq k + 1$. By Lemma 2.7, we have $\kappa(U_n^k(3)) \leq \kappa(G)$ with equality if and only if $G = U_n^k(3)$. By Lemma 2.8, we have $\kappa(U_n^{\Delta-1}(3)) \leq \kappa(U_n^k(3))$ with equality if and only if $\Delta = k + 1$. It follows that $\kappa(U_n^{\Delta-1}(3)) \leq \kappa(G)$ with equality if and only if $G = U_n^{\Delta-1}(3)$.

Case 2. $\Delta = k + 2$. Then G must be the graph obtained from the cycle C_g with k pendant paths $P_{i_1}, \ldots, P_{i_k}$ attached at the same vertex v_1 of C_g, and $k \geq 2$.

If $g \geq 5$, by Lemma 2.4, we have $x_{(g-3)/2} = x_{(g+3)/2}$ and $|x_2| \leq |x_1|$. Let

$$G' = G - v_{(g-1)/2}v_{(g-3)/2} + v_{(g-1)/2}v_{(g+3)/2}.$$

Then $\Delta(G') = \Delta$, G' has $k + 1$ pendant vertices, and from (1) we have

$$\kappa\left(G'\right) \leq x^T Q\left(G'\right)x = x^T Q(G)x = \kappa(G).$$

If $\kappa(G') = \kappa(G)$, then $x = (x_1, x_2, \ldots, x_n)^T$ is also an eigenvector corresponding to $\kappa(G')$. By Lemmas 2.4 and 2.3, we have $|x_2| > |x_1| > 0$, a contradiction. Therefore $\kappa(G') < \kappa(G)$. By Lemma 2.7, we have $\kappa(U_n^{k+1}(3)) \leq \kappa(G')$. It follows that

$$\kappa\left(U_n^{\Delta-1}(3)\right) = \kappa\left(U_n^{k+1}(3)\right) < \kappa(G).$$

Now we assume that $g = 3$. If $G \neq U_n^*(3, \Delta)$, then there are two paths attached at the vertex v_1 with length more than 1. Without loss of generality, we may assume that $i_1 \geq 3$ and $i_2 \geq 3$. Let $P_{i_1} = v_1 \ldots v_b v_a$ and $P_{i_2} = v_1 \ldots v_d v_c$. Without loss of generality, we may assume that $|x_b| \geq |x_d| > 0$. Let $G' = G - v_d v_c + v_b v_c$. Then $\Delta(G') = \Delta$, G' has $k + 1$ pendant vertices. By Lemma 2.5, we have $\kappa(G') < \kappa(G)$. It follows from Lemma 2.7 that

$$\kappa\left(U_n^{\Delta-1}(3)\right) = \kappa\left(U_n^{k+1}(3)\right) \leq \kappa\left(G'\right) < \kappa(G).$$

If $G = U_n^*(3, \Delta)$ and $\Delta \leq n - 5$, by Lemma 2.3, we have $|x_{n-4}| > |x_3|$. Let

$$G_1 = U_n^*(3, \Delta) - v_3 v_\Delta + v_{n-4} v_\Delta.$$

Let $s = \Delta - 4$. Then by Lemma 2.5, we have $\kappa(G_1) < \kappa(U_n^*(3, \Delta))$. Let

$$G_2 = G_1 - v_{n-4} v_\Delta - v_{n-1} v_n + v_{n-3} v_\Delta + v_{n-2} v_n.$$

By Lemma 2.10, we have $\kappa(G_2) < \kappa(G_1)$. Noting that G_2 has $\Delta - 1$ pendant vertices, by Lemma 2.7, we have

$$\kappa\left(U_n^{\Delta-1}(3)\right) \leq \kappa(G_2) < \kappa(G_1) < \kappa(G).$$

If $G = U_n^*(3, n - 4)$ or $U_n^*(3, n - 3)$, by Lemma 2.11, we have

$$\kappa\left(U_n^{n-5}(3)\right) < \kappa\left(U_n^*(3, n - 4)\right) = \kappa(G),$$
$$\kappa\left(U_n^{n-4}(3)\right) < \kappa\left(U_n^*(3, n - 3)\right) = \kappa(G).$$

This completes the proof. □

Secondly, we determine the graph whose least signless Laplacian eigenvalue attains the minimum among all graphs in $\mathcal{G}(n, \Delta)$.

Theorem 3.2 *Let $4 \leq \Delta \leq n - 3$. Among all graphs in $\mathcal{G}(n, \Delta)$, $U_n^{\Delta-1}(3)$ is the unique graph whose least signless Laplacian eigenvalue attains the minimum.*

Proof Let $G \in \mathcal{G}(n, \Delta)$ such that $\kappa(G)$ as small as possible, and let $v \in V(G)$ such that $d_G(v) = \Delta$. By deleting edges from G, we can get a non-bipartite unicyclic spanning subgraph of G, denoted by G', such that $d_{G'}(v) = \Delta$. By Lemma 2.1, we have $\kappa(G') \leq \kappa(G)$. By Theorem 3.1, we have $\kappa(U_n^{\Delta-1}(3)) \leq \kappa(G')$ with equality if and only if $G' = U_n^{\Delta-1}(3)$. Therefore

$$\kappa\left(U_n^{\Delta-1}(3)\right) \leq \kappa\left(G'\right) \leq \kappa(G).$$

Noting that G is the graph whose least signless Laplacian eigenvalue attains the minimum among all graphs in $\mathcal{G}(n, \Delta)$, we have $\kappa(G) \leq \kappa(U_n^{\Delta-1}(3))$. It follows that $\kappa(U_n^{\Delta-1}(3)) = \kappa(G)$.

This implies that G may be obtained from $U_n^{\Delta-1}(3)$ by adding edges. Let $x = (x_1, x_2, \ldots, x_n)^T$ be a unit eigenvector corresponding to $\kappa(G)$. Then

$$
\begin{aligned}
\kappa(G) &= \sum_{uv \in E(G)} \big(x(u) + x(v)\big)^2 \\
&= \sum_{uv \in E(U_n^{\Delta-1}(3))} \big(x(u) + x(v)\big)^2 + \sum_{uv \in E(G) \backslash E(U_n^{\Delta-1}(3))} \big(x(u) + x(v)\big)^2 \\
&\geq \sum_{uv \in E(U_n^{\Delta-1}(3))} \big(x(u) + x(v)\big)^2 \geq \kappa\big(U_n^{\Delta-1}(3)\big).
\end{aligned}
$$

Since $\kappa(G) = \kappa(U_n^{\Delta-1}(3))$, it follows that

$$
\sum_{uv \in E(G) \backslash E(U_n^{\Delta-1}(3))} \big(x(u) + x(v)\big)^2 = 0, \qquad \sum_{uv \in E(U_n^{\Delta-1}(3))} \big(x(u) + x(v)\big)^2 = \kappa\big(U_n^{\Delta-1}(3)\big).
$$

Therefore $x = (x_1, x_2, \ldots, x_n)^T$ is also an eigenvector corresponding to $\kappa(U_n^{\Delta-1}(3))$. By Lemmas 2.4 and 2.3, we have

$$
|x_1| = |x_2| \leq |x_3| < |x_4| < \cdots < |x_{n-\Delta+2}| = \cdots = |x_n|.
$$

From the eigenvalue equation $Q(U_n^{\Delta-1}(3))x = \kappa(U_n^{\Delta-1}(3))x$, we have $x_{n-\Delta+2} = \cdots = x_n$. If $E(G) \backslash E(U_n^{\Delta-1}(3)) \neq \emptyset$, then

$$
\sum_{uv \in E(G) \backslash E(U_n^{\Delta-1}(3))} \big(x(u) + x(v)\big)^2 \neq 0,
$$

which yields a contradiction. So $E(G) \backslash E(U_n^{\Delta-1}(3)) = \emptyset$. Therefore, $G = U_n^{\Delta-1}(3)$. □

Remark 3.3 For $\Delta = 2$, we know that $\mathcal{G}(n, 2) = \{C_n\}$ with n being odd. For $\Delta = 3$, from [4], we know that $U_n^1(3)$ is the unique graph whose least signless Laplacian eigenvalue attains the minimum among all graphs in $\mathcal{G}(n, 3)$. For $\Delta = n - 1$, $\mathcal{U}(n, n-1) = \{S_n^*\}$, where S_n^* is obtained by adding one edge to the star $K_{1,n-1}$. Let $G \in \mathcal{G}(n, n-1) \backslash \{S_n^*\}$, then G is obtained from S_n^* by adding at least one edge. By a similar reasoning to that of Theorem 3.2, we can show that $\kappa(S_n^*) < \kappa(G)$. For $\Delta = n - 2$, $\mathcal{U}(n, n-2) = \{S_{n-1}^{*+1}, S_{n-1}^{*+2}\}$, where S_{n-1}^{*+1} is obtained from S_{n-1}^* by adding one pendant edge to a vertex of degree 1, and S_{n-1}^{*+2} is obtained from S_{n-1}^* by adding one pendant edge to a vertex of degree 2. From Lemmas 2.5, 2.4 and 2.6, we may obtain $\kappa(S_{n-1}^{*+1}) < \kappa(S_{n-1}^{*+2})$. Let $G \in \mathcal{G}(n, n-2) \backslash \{S_{n-1}^{*+1}, S_{n-1}^{*+2}\}$, then G is obtained from S_{n-1}^{*+1} or S_{n-1}^{*+2} by adding at least one edge. By a similar reasoning to that of Theorem 3.2, we can show that $\kappa(S_{n-1}^{*+1}) < \kappa(G)$.

4 Discussion

Recalling that $\kappa(G)$ depends more on the distribution of the edges of a graph than on their number, we propose the following problems for further research.

(1) Characterize all extremal graphs whose least signless Laplacian eigenvalue attains the minimum among all non-bipartite unicyclic graphs with a given degree sequence.

(2) Characterize all extremal graphs whose least signless Laplacian eigenvalue attains the minimum among all non-bipartite connected graphs with a given degree sequence.

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