Huang *Journal of Inequalities and Applications* (2017) 2017:99
DOI 10.1186/s13660-017-1376-1

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Higher order Riesz transforms for Hermite expansions

Jizheng Huang*

*Correspondence:
hjzheng@163.com
College of Sciences, North China University of Technology, Beijing, 100144, China

Abstract

In this paper, we consider the Riesz transform of higher order associated with the harmonic oscillator $L = -\Delta + |x|^2$, where Δ is the Laplacian on $\mathbb{R}^d$. Moreover, the boundedness of Riesz transforms of higher order associated with Hermite functions on the Hardy space is proved.

MSC: 42C10; 42B25

Keywords: Hermite expansions; Littlewood-Paley g-function; Riesz transform; Hardy space

1 Introduction

Let $H_k(x)$ denote the Hermite polynomials on $\mathbb{R}$, which can be defined as

$$H_k(x) = (-1)^k \frac{d^k}{dx^k}\bigl(e^{-x^2}\bigr)e^{x^2}, \quad k = 0, 1, 2, \ldots.$$

The normalized Hermite functions are defined by

$$h_k(x) = \bigl(\pi^{1/2} 2^k k!\bigr)^{-1/2} H_k(x) \exp\bigl(-x^2/2\bigr), \quad k = 0, 1, \ldots.$$

The high dimensional Hermite functions on $\mathbb{R}^d$ can be defined in the following way. For $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\alpha_i \in \{0, 1, \ldots\}$, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$,

$$h_\alpha(x) = \prod_{j=1}^{d} h_{\alpha_j}(x_j).$$

$\{h_\alpha\}$ forms a complete orthonormal basis of $L^2(\mathbb{R}^d)$. Let $|\alpha| = \alpha_1 + \cdots + \alpha_d$, then we have

$$Lh_\alpha = \bigl(2|\alpha| + d\bigr)h_\alpha.$$

A very famous reference for Hermite functions is [1].

The operator L is positive and symmetric on $L^2(\mathbb{R}^d)$. Let $\{T_t^L\}_{t\geq 0}$ be the heat kernel defined by

$$T_t^L f = e^{-tL} f = \sum_{n=0}^{\infty} e^{-t(2n+d)} \mathcal{P}_n f$$

for $f \in L^2(\mathbb{R}^d)$ and

$$\mathcal{P}_n f = \sum_{|\alpha|=n} \langle f, h_\alpha \rangle h_\alpha.$$

Then the Poisson semigroup is defined as

$$P_t^L f = e^{-tL^{1/2}} f = \sum_{n=0}^{\infty} e^{-t(2n+d)^{1/2}} \mathcal{P}_n f, \quad f \in L^2(\mathbb{R}^d).$$

The relation between the heat kernel and the Poisson kernel is

$$P_t^L f(x) = \frac{t}{\sqrt{4\pi}} \int_0^\infty s^{-3/2} \exp(-t^2/4s) T_s^L f(x)\, ds. \tag{1}$$

Let $A_j = \frac{\partial}{\partial x_j} + x_j$ and $A_{-j} = A_j^* = -\frac{\partial}{\partial x_j} + x_j$, $j = 1, 2, \ldots, d$. Then we can denote L as

$$L = -\frac{1}{2}\big[(\nabla + x) \cdot (\nabla - x) + (\nabla - x) \cdot (\nabla + x)\big] = \frac{1}{2} \sum_{j=1}^{d} A_j A_{-j} + A_{-j} A_j.$$

We define operators $R_{\pm j}^L$, $j = 1, 2, \ldots, d$

$$R_j^L = A_j L^{-1/2}, \qquad R_{-j}^L = A_{-j} L^{-1/2}.$$

R_j and R_{-j} are called the Riesz transforms associated with L. The definition was first suggested by Thangavelu in [2].

Let e_j be the coordinate vectors in $\mathbb{R}^d$, then

$$A_j h_\alpha = (2\alpha_j + 2)^{1/2} h_{\alpha+e_j}, \qquad A_{-j} h_\alpha = (2\alpha_j)^{1/2} h_{\alpha-e_j}.$$

Therefore, for $f \in L^2(\mathbb{R}^d)$,

$$\begin{aligned} R_j^L f &= \sum_\alpha \left(\frac{2\alpha_j}{2|\alpha| + d} \right)^{1/2} \langle f, h_\alpha \rangle h_{\alpha-e_j} \\ &= \sum_{n=0}^{\infty} \sum_{|\alpha|=n} \left(\frac{2\alpha_j}{2n + d} \right)^{1/2} \langle f, h_\alpha \rangle h_{\alpha-e_j}, \end{aligned} \tag{2}$$

and

$$\begin{aligned} R_{-j}^L f &= \sum_\alpha \left(\frac{2(\alpha_j + 1)}{2|\alpha| + d} \right)^{1/2} \langle f, h_\alpha \rangle h_{\alpha+e_j} \\ &= \sum_{n=0}^{\infty} \sum_{|\alpha|=n} \left(\frac{2(\alpha_j + 1)}{2n + d} \right)^{1/2} \langle f, h_\alpha \rangle h_{\alpha+e_j}. \end{aligned} \tag{3}$$

In [1], the author proved that R_j^L were bounded on the local Hardy spaces $h^1(\mathbb{R}^d)$ which were defined by Goldberg in [3]. Thangavelu asked one question: whether it was possible to characterize $h^1(\mathbb{R}^d)$ by R_j^L, i.e., whether the equality

$$h^1(\mathbb{R}^d) = \{f \in L^1(\mathbb{R}^d) : R_j^L f \in L^1(\mathbb{R}^d), j = 1, 2, \dots, d\}$$

is true. In [4], the author proved the boundedness of R_j^L on Hardy spaces $H_L^1(\mathbb{R}^d)$, $d \geq 3$, where $H_L^1(\mathbb{R}^d)$ are the Hardy spaces for L (cf. [5]).

Proposition 1 *Let $j = 1, 2, \dots, d$. Then the operators R_j^L are bounded on $H_L^1(\mathbb{R}^d)$, that is, there exists $C > 0$ satisfying*

$$\|R_j^L f\|_{H_L^1} \leq C\|f\|_{H_L^1}.$$

Moreover, he characterized $H_L^1(\mathbb{R}^d)$ by R_j^L, $j = 1, 2, \dots, d$. Therefore, we cannot characterize $h^1(\mathbb{R}^d)$ by R_j^L.

Remark 1 When we consider the boundedness of Riesz transforms for L on Hardy spaces, the main tool is Littlewood-Paley characterizations of Hardy spaces. In fact, we have the following equality (cf. [4]):

$$t\partial_t e^{-t(L\pm 2)^{1/2}}(R_{\pm j}^L f) = -t\left(\pm\frac{\partial}{\partial x_j} + x_j\right)e^{-tL^{1/2}} f$$

for all $j = 1, 2, \dots, d$ and $f \in L^2(\mathbb{R}^d)$. If we prove the boundedness of Riesz transforms R_{-j}^L on Hardy spaces, we need to consider the operator $L - 2$. Since the Hardy spaces $H_L^1(\mathbb{R}^d)$, $d \geq 3$, associated with L defined in [5] are for nonnegative potentials, it is maybe natural to just consider R_j^L. In [6], the authors proved the boundedness of $R_{\pm j}^L$ on $L^p(\mathbb{R}^d)$, where they considered the semigroup generated by $L + b$ for $b < 0$ on $L^p(\mathbb{R}^d)$.

In this paper, we prove that the higher ordered Riesz transforms are bounded on the Hardy spaces associated with Hermite functions. More precisely, let

$$L^{-m/2} h_\alpha = (2|\alpha| + d)^{-m/2} h_\alpha,$$

and define the m-ordered Riesz transforms as

$$R_{i_1 i_2 \cdots i_m} = A_{i_1} A_{i_2} \cdots A_{i_m} L^{-m/2},$$

where $1 \leq i_j \leq d$ and $1 \leq j \leq m$.

We define Hardy space $H_L^1(\mathbb{R}^d)$ for $d \geq 3$ as follows (cf. [5]):

$$H_L^1(\mathbb{R}^d) = \{f \in L^1(\mathbb{R}^d) : \mathcal{M}_L f \in L^1(\mathbb{R}^d)\},$$

where $\mathcal{M}_L f(x) = \sup_{t>0} |T_t^L f(x)|$.

Define

$$\rho(x) = \frac{1}{1 + |x|}, \tag{4}$$

we say $a(x)$ is an atom for the space $H_L^1(\mathbb{R}^d)$ if there exists a ball $B(x_0, r)$ such that

(1) $\operatorname{supp} a \subset B(x_0, r)$,

(2) $\|a\|_{L^\infty} \leq |B(x_0, r)|^{-1}$,
(3) if $r < \rho(x_0)$, then $\int a(x)\,dx = 0$.

The atomic quasi-norm in $H^1_L(\mathbb{R}^d)$ can be defined as

$$\|f\|_{L\text{-atom}} = \inf\Big\{\sum |c_j|\Big\}.$$

In [5], the authors proved the following result.

Proposition 2 *There exists $C > 0$ satisfying*

$$C^{-1}\|f\|_{H^1_L} \leq \|f\|_{L\text{-}atom} \leq C\|f\|_{H^1_L}.$$

Let $b \in \mathbb{R}^d_+$. We define

$$G^b_t(x,y) = e^{-bt}G^L_t(x,y).$$

Then

$$G^b_t(f)(x) = \int_{\mathbb{R}^d} G^b_t(x,y)f(y)\,dy$$

is a semigroup for the spaces $L^p(\mathbb{R}^d)$, $1 \leq p < \infty$, and $\|G^b_t(f)\|_{L^p(\mathbb{R}^d)} \leq e^{-bt}\|f\|_{L^p(\mathbb{R}^d)}$. This semigroup is generated by the operator $-(L + b)$.

The subordination formula is

$$P^b_t(x,y) = \frac{t}{\sqrt{4\pi}}\int_0^\infty G^b_s(x,y)s^{-3/2}e^{-t^2/4s}\,ds. \tag{5}$$

The Poisson integral of $f(x)$ can be defined as

$$\begin{aligned} u_b(x,t) = P^b_t(f)(x) &= \int_{\mathbb{R}^d} P^b_t(x,y)f(y)\,dy \\ &= \frac{t}{\sqrt{4\pi}}\int_{\mathbb{R}^d}\int_0^\infty G^b_s(x,y)f(y)s^{-3/2}e^{-t^2/4s}\,ds\,dy. \end{aligned}$$

Let

$$\mathcal{G}_b(f)(x) = \left(\int_0^\infty \sum_{j=0}^d \big|tA_j u_b(x,t)\big|^2\frac{dt}{t}\right)^{1/2}$$

and

$$\mathcal{G}^1_b(f)(x) = \left(\int_0^\infty \big|t\partial_t u_b(x,t)\big|^2\frac{dt}{t}\right)^{1/2},$$

where $A_0 = \partial_t$.

The main results of this paper are as follows.

Theorem 1 *$f \in H^1_L(\mathbb{R}^d)$ is equivalent to $\mathcal{G}_b(f) \in L^1(\mathbb{R}^d)$ and $f \in L^1(\mathbb{R}^d)$. Moreover,*

$$\|f\|_{H^1_L} \sim \|\mathcal{G}_b(f)\|_{L^1} + \|f\|_{L^1}.$$

Theorem 2 *The operators $R_{i_1 i_2 \cdots i_m} = A_{i_1} A_{i_2} \cdots A_{i_m} L^{-m/2}$ are bounded on $H^1_L(\mathbb{R}^d)$ for all $1 \le i_j \le d$ for every $1 \le j \le m$, that is, there exists $C > 0$ satisfying*

$$\|R_{i_1 i_2 \cdots i_m} f\|_{H^1_L} \le C\|f\|_{H^1_L}.$$

The organization of this paper is as follows. In Section 2, we give some estimations of the heat kernel and the Poisson kernel associated with $L + b$. In Section 3, Theorem 1 is proved. In Section 4, we prove Theorem 2.

Throughout the article, we use A and C to denote the positive constants, which are independent of the main parameters and may be different at each occurrence. By $B_1 \sim B_2$, we mean that there exists a constant $C > 1$ such that $\frac{1}{C} \le \frac{B_1}{B_2} \le C$.

2 Estimations of the kernels

Let $G^b_t(x,y)$ be the heat kernel of $\{T^{L+b}_t\}$. Then the following inequality can be proved by the Feynman-Kac formula:

$$G^b_t(x,y) \le W_t(x-y),$$

where

$$W_t(x) = (4\pi t)^{-d/2} \exp\bigl(-|x|^2/(4t)\bigr)$$

is the heat kernel on $\mathbb{R}^d$.

Since $G^b_t(x,y) \le G^L_t(x,y)$, we have (cf. [7]) the following lemma.

Lemma 1

(a) *For $N \in \mathbb{N}$, there exists $C_N > 0$*

$$0 \le G^b_t(x,y) \le C_N t^{-\frac{d}{2}} e^{-(5t)^{-1}|x-y|^2} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}. \tag{6}$$

(b) *There are constants $0 < \delta < 1$ and $C > 0$, for $N > 0$, there is $C_N > 0$ which satisfies for all $|h| \le \frac{|x-y|}{2}$,*

$$\bigl|G^b_t(x+h,y) - G^b_t(x,y)\bigr| \le C_N \left(\frac{|h|}{\sqrt{t}}\right)^{\delta} t^{-\frac{d}{2}} e^{-At^{-1}|x-y|^2} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}. \tag{7}$$

By the subordination formula, we get the following.

Lemma 2

(a) *For $N \in \mathbb{N}$, there is $C_N > 0$ satisfying*

$$0 \le P^b_t(x,y) \le C_N \frac{t}{(t^2 + A|x-y|^2)^{(d+1)/2}} \left(1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)}\right)^{-N}. \tag{8}$$

(b) *Let* $0 < \delta < 1$ *and* $|h| < \frac{|x-y|}{2}$. *Then, for* $N \in \mathbb{N}$, *there are* $C > 0$, $C_N > 0$ *satisfying*

$$\begin{aligned} &\left|P_t^b(x+h,y) - P_t^b(x,y)\right| \\ &\quad \le C_N\left(\frac{|h|}{t}\right)^{\delta} \frac{t}{(t^2 + A|x-y|^2)^{(d+1)/2}}\left(1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)}\right)^{-N}. \end{aligned} \tag{9}$$

Proof (a) By subordination formula and Lemma 1, we have

$$\begin{aligned} 0 \le P_t^b(x,y) &= \frac{1}{\sqrt{\pi}}\int_0^\infty G^b_{t^2/4\mu}(x,y)e^{-\mu}\mu^{-1/2}\,d\mu \\ &\le C_N\int_0^\infty \left(\frac{t^2}{4\mu}\right)^{-\frac{d}{2}} e^{-C_1t^{-2}(4\mu|x-y|^2)}\left(\frac{t}{\sqrt{4\mu}\rho(x)} + \frac{t}{\sqrt{4\mu}\rho(y)}\right)^{-N} e^{-\mu}\mu^{-1/2}\,d\mu \\ &= C_N\int_0^\infty \left(\frac{t^2}{4\mu}\right)^{-\frac{d}{2}} e^{-C_1t^{-2}(4\mu|x-y|^2)}\left(\frac{t}{\rho(x)} + \frac{t}{\rho(y)}\right)^{-N} e^{-\mu}\mu^{N/2-1/2}\,d\mu \\ &\le C_N\left(\frac{t}{\rho(x)} + \frac{t}{\rho(y)}\right)^{-N}\int_0^\infty \left(\frac{t^2}{4\mu}\right)^{-\frac{d}{2}} e^{-C_1t^{-2}(4\mu|x-y|^2)}e^{-\mu}\mu^{-1/2}\,d\mu \\ &= C_N\left(\frac{t}{\rho(x)} + \frac{t}{\rho(y)}\right)^{-N}\frac{t}{(t^2 + A|x-y|^2)^{(d+1)/2}}. \end{aligned} \tag{10}$$

By (10) and

$$P_t^b(x,y) \le \frac{t}{(t^2 + A|x-y|^2)^{(d+1)/2}},$$

we get

$$0 \le P_t^b(x,y) \le C_N\frac{t}{(t^2 + A|x-y|^2)^{(d+1)/2}}\left(1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)}\right)^{-N}.$$

(b) By subordination formula again, we know

$$\begin{aligned} &\left|P_t^b(x+h,y) - P_t^b(x,y)\right| \\ &\quad \le \frac{1}{\sqrt{\pi}}\int_0^\infty \left|G^b_{t^2/4\mu}(x+h,y) - G^b_{t^2/4\mu}(x,y)\right|e^{-\mu}\mu^{-1/2}\,d\mu \\ &\quad \le C_N\int_0^\infty \left(\frac{t^2}{4\mu}\right)^{-\frac{d}{2}} e^{-C_1t^{-2}(4\mu|x-y|^2)}\left(\frac{t}{\sqrt{4\mu}\rho(x)} + \frac{t}{\sqrt{4\mu}\rho(y)}\right)^{-N} \\ &\qquad \times \left(\frac{\sqrt{4\mu}|h|}{t}\right)^{\delta'} e^{-\mu}\mu^{-1/2}\,d\mu \\ &\quad = C_N\left(\frac{|h|}{t}\right)^{\delta'}\left(\frac{t}{\rho(x)} + \frac{t}{\rho(y)}\right)^{-N}\int_0^\infty \left(\frac{t^2}{4\mu}\right)^{-\frac{d}{2}} e^{-C_1t^{-2}(4\mu|x-y|^2)}e^{-\mu}\mu^{(N+\delta')/2-1/2}\,d\mu \\ &\quad \le C_N\left(\frac{|h|}{t}\right)^{\delta'}\left(\frac{t}{\rho(x)} + \frac{t}{\rho(y)}\right)^{-N}\int_0^\infty \left(\frac{t^2}{4\mu}\right)^{-\frac{d}{2}} e^{-C_1't^{-2}(4\mu|x-y|^2)}e^{-\mu}\mu^{-1/2}\,d\mu \\ &\quad = C_N\left(\frac{|h|}{t}\right)^{\delta'}\left(\frac{t}{\rho(x)} + \frac{t}{\rho(y)}\right)^{-N}\frac{t}{(t^2 + A|x-y|^2)^{(d+1)/2}}. \end{aligned} \tag{11}$$

We also have

$$\begin{aligned}
&\left|P_t^b(x+h,y)-P_t^b(x,y)\right| \\
&\le C_N\int_0^\infty\left(\frac{t^2}{4\mu}\right)^{-\frac{d}{2}}e^{-C_1t^{-2}(4\mu|x-y|^2)}\left(\frac{\sqrt{4\mu}|h|}{t}\right)^{\delta'}e^{-\mu}\mu^{-1/2}\,d\mu \\
&= C_N\left(\frac{|h|}{t}\right)^{\delta'}\int_0^\infty\left(\frac{t^2}{4\mu}\right)^{-\frac{d}{2}}e^{-C_1t^{-2}(4\mu|x-y|^2)}e^{-\mu}\mu^{\delta'/2-1/2}\,d\mu \\
&\le C_N\left(\frac{|h|}{t}\right)^{\delta'}\int_0^\infty\left(\frac{t^2}{4\mu}\right)^{-\frac{d}{2}}e^{-C_1't^{-2}(4\mu|x-y|^2)}e^{-\mu}\mu^{-1/2}\,d\mu \\
&= C_N\left(\frac{|h|}{t}\right)^{\delta'}\frac{t}{(t^2+A|x-y|^2)^{(d+1)/2}}.
\end{aligned}\tag{12}$$

Then (b) follows from (11) and (12). □

Let $D_t^{b,k}(x,y)=t^k\partial_t^kP_t^b(x,y)$. Then, by Lemma 2, we can prove (cf. [8] or [9]) the following.

Proposition 3 *There are $C>0$, $0<\delta'<\delta$, for $N\in\mathbb{N}$, there is C_N such that*

(a) $$\left|D_t^{b,k}(x,y)\right|\le C_N\frac{t}{(t^2+C|x-y|^2)^{(d+1)/2}}\left(1+\frac{t}{\rho(x)}+\frac{t}{\rho(y)}\right)^{-N};$$

(b) $$\begin{aligned}&\left|D_t^{b,k}(x+h,y)-D_t^{b,k}(x,y)\right| \\ &\le C_N\left(\frac{|h|}{t}\right)^{\delta'}\frac{t}{(t^2+C|x-y|^2)^{(d+1)/2}}\left(1+\frac{t}{\rho(x)}+\frac{t}{\rho(y)}\right)^{-N}\end{aligned}$$

for all $|h|\le\frac{|x-y|}{2}$.

Let $t=\frac{1}{2}\ln\frac{1+s}{1-s}$, $s\in(0,1)$. Then

$$G_t(x,y)=\left(\frac{1-s^2}{4\pi s}\right)^{d/2}\exp\left(-\frac{1}{4}\left(s|x+y|^2+\frac{1}{s}|x-y|^2\right)\right)\doteq K_s(x,y).\tag{13}$$

The proof of the following proposition is motivated by [10].

Proposition 4 *There is $A>0$, for $N\in\mathbb{N}$ and $|x-x'|\le\frac{|x-y|}{2}$, we can find $C_N>0$ such that*

(a) $$\left|tA_jG_t^b(x,y)\right|\le C_Nt^{-\frac{d}{2}}\exp\left(-\frac{|x-y|^2}{At}\right)\left(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\right)^{-N};$$

(b) $$\begin{aligned}&\left|tA_jG_t^b(x,y)-tA_jG_t^b(x',y)\right| \\ &\le C_N\frac{|x-x'|}{t}t^{-\frac{d}{2}}\exp\left(-\frac{|x-y|^2}{At}\right)\left(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}.\end{aligned}$$

Proof By

$$\begin{aligned}\left|A_jG_t(x,y)\right| &= \left|\frac{\partial}{\partial x_j}G_t(x,y)+x_jG_t(x,y)\right| \\ &\le \left|\frac{\partial}{\partial x_j}G_t(x,y)\right| + \left|x_jG_t(x,y)\right| \doteq I_1+I_2,\end{aligned}$$

and $t=\frac{1}{2}\ln\frac{1+s}{1-s}\sim s$, $s\to 0^+$, for $s\in(0,\frac{1}{2}]$, we have

$$\begin{aligned}I_2 &\le C|x_j|s^{-\frac{d}{2}}\exp\left(-\frac{1}{4}s|x+y|^2\right)\exp\left(-\frac{1}{4}\frac{|x-y|^2}{s}\right) \\ &\le C|x|s^{-\frac{d}{2}}\exp\left(-\frac{1}{4}s|x+y|^2\right)\exp\left(-\frac{1}{4}\frac{|x-y|^2}{s}\right).\end{aligned}$$

If $x\cdot y\le 0$, then $|x|\le|x-y|$. So

$$\begin{aligned}I_2 &\le Cs^{-\frac{d}{2}}|x-y|\exp\left(-\frac{1}{4}\frac{|x-y|^2}{s}\right)\le Cs^{-\frac{d-1}{2}}\exp\left(-\frac{|x-y|^2}{8s}\right) \\ &\le Ct^{-\frac{d-1}{2}}\exp\left(-\frac{|x-y|^2}{8t}\right).\end{aligned}$$

If $x\cdot y\ge 0$, then $|x|\le|x+y|$. So

$$\begin{aligned}I_2 &\le Cs^{-\frac{d}{2}}|x+y|\exp\left(-\frac{1}{4}s|x+y|^2\right)\exp\left(-\frac{1}{4}\frac{|x-y|^2}{s}\right) \\ &\le Cs^{-\frac{d+1}{2}}\exp\left(-\frac{|x-y|^2}{4s}\right)\le Ct^{-\frac{d+1}{2}}\exp\left(-\frac{|x-y|^2}{4t}\right).\end{aligned}$$

Therefore,

$$|tI_2|\le C\left(t^{\frac{3}{2}}+t^{\frac{1}{2}}\right)e^{-bt}t^{-\frac{d}{2}}\exp\left(-\frac{|x-y|^2}{8t}\right)\le Ct^{-\frac{d}{2}}\exp\left(-\frac{|x-y|^2}{8t}\right). \tag{14}$$

When $s\in[\frac{1}{2},1)$,

$$\begin{aligned}I_2 &\le C|x_j|\exp\left(-\frac{1}{4}\left(s|x+y|^2+\frac{|x-y|^2}{s}\right)\right) \\ &\le C|x|s^{-\frac{d}{2}}\exp\left(-\frac{1}{4}\left(s|x+y|^2+\frac{|x-y|^2}{s}\right)\right) \\ &\le C(|x+y|+|x-y|)s^{-\frac{d}{2}}\exp\left(-\frac{1}{4}\left(s|x+y|^2+\frac{|x-y|^2}{s}\right)\right) \\ &\le C\exp\left(-\frac{|x-y|^2}{8s}\right).\end{aligned}$$

Since $t=\frac{1}{2}\ln\frac{1+s}{1-s}>s$ for $s\in[\frac{1}{2},1)$, we get

$$I_2\le C\exp\left(-\frac{|x-y|^2}{8t}\right).$$

Therefore,

$$|tI_2| \le Cte^{-bt}\exp\left(-\frac{|x-y|^2}{8t}\right) \le Ct^{-\frac{d}{2}}\exp\left(-\frac{|x-y|^2}{8t}\right). \tag{15}$$

By (13), we get

$$\frac{\partial}{\partial x_j}K_s(x,y) = -\frac{1}{2}\left(s(x_j+y_j)+\frac{1}{s}(x_j-y_j)\right)K_s(x,y),$$

and

$$I_1 \le C\left(s|x_j+y_j|+\frac{1}{s}|x_j-y_j|\right)K_s(x,y) \le C\left(s|x+y|+\frac{1}{s}|x-y|\right)K_s(x,y).$$

Therefore, when $s \in (0, \frac{1}{2}]$, we have

$$I_1 \le Cs^{-\frac{d}{2}}\exp\left(-\frac{|x-y|^2}{8s}\right) \le Ct^{-\frac{d}{2}}\exp\left(-\frac{|x-y|^2}{8t}\right).$$

When $s \in [\frac{1}{2}, 1)$, we have

$$I_1 \le C\exp\left(-\frac{|x-y|^2}{8s}\right) \le C\exp\left(-\frac{|x-y|^2}{8t}\right).$$

Then

$$\left|t\frac{\partial}{\partial x_j}G_t^b(x,y)\right| \le Ct\bigl(1+t^{-\frac{d}{2}}\bigr)e^{-bt}\exp\left(-\frac{|x-y|^2}{8t}\right) \le Ct^{-\frac{d}{2}}\exp\left(-\frac{|x-y|^2}{8t}\right). \tag{16}$$

By (14)-(16), we get

$$\bigl|tA_jG_t^b(x,y)\bigr| \le Ct^{-\frac{d}{2}}\exp\left(-\frac{|x-y|^2}{8t}\right). \tag{17}$$

Similar to the proof of (17), for any $N > 0$, we can prove

$$\bigl(\sqrt{t}|x|\bigr)^N\bigl|tA_jG_t^b(x,y)\bigr| \le C_Nt^{-\frac{d}{2}}\exp\left(-\frac{|x-y|^2}{8t}\right)$$

and

$$t^N\bigl|tA_jG_t^b(x,y)\bigr| \le C_Nt^{-\frac{d}{2}}\exp\left(-\frac{|x-y|^2}{8t}\right).$$

Since $\rho(x) = \frac{1}{1+|x|}$, we get $\frac{\sqrt{t}}{\rho(x)} = \sqrt{t}(1+|x|)$. Then, for $N > 0$,

$$\left(\frac{\sqrt{t}}{\rho(x)}\right)^N\bigl|tA_jG_t^b(x,y)\bigr| \le C_Nt^{-\frac{d}{2}}\exp\left(-\frac{|x-y|^2}{8t}\right). \tag{18}$$

Since x and y are symmetric, we also have

$$\left(\frac{\sqrt{t}}{\rho(y)}\right)^N \left|tA_j G_t^b(x,y)\right| \le C_N t^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{8t}\right). \tag{19}$$

Then (a) follows from (17)-(19).

(b) Note that

$$\begin{aligned}
&\left|tA_j G_t^b(x',y) - tA_j G_t^b(x,y)\right| \\
&\quad \le \left|t\frac{\partial}{\partial x_j} G_t^b(x',y) - t\frac{\partial}{\partial x_j} G_t^b(x,y)\right| + \left|tx_j' G_t^b(x',y) - tx_j G_t^b(x,y)\right| \\
&\quad \doteq J_1 + J_2.
\end{aligned}$$

For J_2, let

$$\varphi(z) = \varphi_{y,s}(z) = z_j \exp\left(-\frac{1}{4}\alpha(s,z,y)\right),$$

where $\alpha(s,z,y) = s|z+y|^2 + \frac{1}{s}|z-y|^2$.

Then

$$\frac{\partial\varphi}{\partial z_k}(z) = \left(\delta_{jk} - \frac{s}{2}z_j(z_k+y_k) - \frac{1}{2s}z_j(z_k-y_k)\right)\exp\left(-\frac{1}{4}\alpha(s,z,y)\right).$$

Therefore

$$\begin{aligned}
\left|\frac{\partial\varphi}{\partial z_k}(z)\right| &\le C\left(1 + s|z||z+y| + \frac{1}{s}|z||z-y|\right)\exp\left(-\frac{1}{4}\alpha(s,z,y)\right) \\
&\le C\left(1 + s^{1/2}|z| + \frac{1}{s^{1/2}}|z|\right)\exp\left(-\frac{1}{8}\alpha(s,z,y)\right) \\
&\le C\left(1 + s^{1/2}\left(|z-y| + |z+y|\right) + \frac{1}{s^{1/2}}\left(|z-y| + |z+y|\right)\right)\exp\left(-\frac{1}{8}\alpha(s,z,y)\right) \\
&\le C\left(1 + s + \frac{1}{s}\right)\exp\left(-\frac{1}{16s}|z-y|^2\right) \\
&\le Cs^{-1}\exp\left(-\frac{1}{16s}|z-y|^2\right).
\end{aligned} \tag{20}$$

Let $\theta = \lambda x + (1-\lambda)x'$, $0 < \lambda < 1$. Then

$$\begin{aligned}
J_2 &= te^{-bt}\left|x_j' K_s(x',y) - x_j K_s(x,y)\right| \\
&\le Ct^{-d/2}\left|x-x'\right| \sup_\theta \left|\nabla\varphi(\theta)\right| \\
&\le Ct^{-d/2}\frac{|x-x'|}{s}\sup_\theta \exp\left(-\frac{|\theta-y|^2}{16s}\right) \\
&\le Ct^{-d/2}\frac{|x-x'|}{t}\sup_\theta \exp\left(-\frac{|\theta-y|^2}{16t}\right).
\end{aligned}$$

When $|x-x'| \leq \frac{|x-y|}{2}$, we can get $|\theta - y| \sim |x-y|$. Therefore, there exists $A > 0$ such that

$$J_2 \leq Ct^{-d/2} \frac{|x-x'|}{t} \exp\left(-\frac{|x-y|^2}{At}\right). \tag{21}$$

For J_1,

$$\begin{aligned} J_1 &= \left| t\frac{\partial}{\partial x_j} G_t^b(x', y) - t\frac{\partial}{\partial x_j} G_t^b(x,y) \right| \\ &= te^{-bt} \left| \frac{\partial}{\partial x_j} K_s(x', y) - \frac{\partial}{\partial x_j} K_s(x,y) \right| \\ &= te^{-bt} \left| \left(s(x_j + y_j) + \frac{1}{s}(x_j - y_j) \right) \exp\left(-\frac{1}{4}\alpha(s,x,y) \right) \right. \\ &\quad \left. - \left(s(x'_j + y_j) + \frac{1}{s}(x'_j - y_j) \right) \exp\left(-\frac{1}{4}\alpha(s,x',y) \right) \right|. \end{aligned}$$

Let

$$\psi(z) = \psi_{y,s}(z) = \left(s(z_j + y_j) + \frac{1}{s}(z_j - y_j) \right) \exp\left(-\frac{1}{4}\alpha(s,z,y) \right).$$

Then

$$\begin{aligned} \frac{\partial \psi}{\partial z_k}(z) &= \left[\left(s + \frac{1}{s} \right) \delta_{jk} - \frac{1}{2} \left(s(z_j + y_j) + \frac{1}{s}(z_j - y_j) \right) \right. \\ &\quad \left. \times \left(s(z_k + y_k) + \frac{1}{s}(z_k - y_k) \right) \right] \exp\left(-\frac{1}{4}\alpha(s,z,y) \right). \end{aligned}$$

Therefore, similar to the proofs of (20) and (21), we can prove

$$\left| \frac{\partial \psi}{\partial z_k}(z) \right| \leq Cs^{-1} \exp\left(-\frac{1}{4}\alpha(s,z,y) \right)$$

and

$$\begin{aligned} J_1 &\leq Ce^{-bt} \sup_\theta \left| \nabla \psi(\theta) \right| \left| x - x' \right| \\ &\leq Ct^{-d/2} \frac{|x-x'|}{t} \exp\left(-\frac{|x-y|^2}{At} \right). \end{aligned} \tag{22}$$

Inequalities (21) and (22) show

$$\left| tA_j G_t^b(x,y) - tA_j G_t^b(x',y) \right| \leq C_N \frac{|x-x'|}{t} t^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{At} \right).$$

Then, similar to the proof of (a), we have

$$\left| tA_j G_t^b(x,y) - tA_j G_t^b(x',y) \right| \leq C_N \frac{|x-x'|}{t} t^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{At} \right) \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)} \right)^{-N}.$$

This completes the proof of Proposition 4. □

The subordination formula gives the following lemma.

Lemma 3

(a) *For $N \in \mathbb{N}$, there is $C_N > 0$ satisfying*

$$\left|tA_j P_t^b(x,y)\right| \le C_N \frac{t}{(t^2 + A|x-y|^2)^{(d+1)/2}} \left(1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)}\right)^{-N}. \tag{23}$$

(b) *For any $N > 0$ and $|x - x'| \le \frac{|x-y|}{2}$, there are $C > 0$, $C_N > 0$, so that*

$$\begin{aligned}&\left|tA_j P_t^b(x,y) - tA_j P_t^b(x',y)\right| \\ &\quad \le C_N \left(\frac{|x-x'|}{t}\right) \frac{t}{(t^2 + A|x-y|^2)^{(d+1)/2}} \left(1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)}\right)^{-N}.\end{aligned} \tag{24}$$

3 Square function characterizations of $H_L^1(\mathbb{R}^d)$

We define square functions

$$\mathcal{G}_L^{b,k} f(x) = \left(\int_0^\infty \left|D_t^{b,k} f(x)\right|^2 \frac{dt}{t}\right)^{1/2}$$

and

$$S_L^{b,k} f(x) = \left(\int_0^\infty \int_{|x-y|<t} \left|D_t^{b,k} f(y)\right|^2 \frac{dy\,dt}{t^{d+1}}\right)^{1/2},$$

where $D_t^{b,k} f(x) = t^k (\partial_t^k P_t^b f)(x)$ for $k = 1, 2, \ldots$.

The proof of the following lemma can be found in [4].

Lemma 4 *If $f \in L^1(\mathbb{R}^d)$, we have $f \in H_L^1(\mathbb{R}^d)$ is equivalent to $f \in H_{L+b}^1(\mathbb{R}^d)$ for $b > 0$.*

Then, by Lemma 4, we can prove (cf. Section 8 in [11] or [12]) the following.

Proposition 5 *$f \in H_L^1(\mathbb{R}^d)$ is equivalent to its area integral $S_L^{b,k} f \in L^1(\mathbb{R}^d)$ and $f \in L^1(\mathbb{R}^d)$. Moreover,*

$$\|f\|_{H_{L+b}^1} \sim \|f\|_{H_L^1} \sim \left\|S_L^{b,k} f\right\|_{L^1} + \|f\|_{L^1}.$$

Motivated by [13], we can prove the following.

Lemma 5 *There is $C > 0$ satisfying*

$$\left\|S_L^{b,k+1} f\right\|_{L^1} \le C \left\|\mathcal{G}_L^{b,k} f\right\|_{L^1}.$$

Proof Let

$$F(x)(t) = \left(\partial_t^k e^{-t\sqrt{L+b}} f\right)(x), \qquad V(x,s) = e^{-s\sqrt{L+b}} F(x).$$

Then

$$V(x,s)(t) = e^{-s\sqrt{L+b}}\big(\partial_t^k e^{-t\sqrt{L+b}}f\big)(x) = \big(\partial_t^k e^{-(s+t)\sqrt{L+b}}f\big)(x).$$

Therefore

$$\begin{aligned}\int_0^{+\infty}\big|V(x,s)(t)\big|^2 t^{2k-1}\,dt &= \int_0^{+\infty}\big|\big(\partial_t^k e^{-(s+t)\sqrt{L+b}}f\big)(x)\big|^2 t^{2k-1}\,dt\\ &= \int_s^{+\infty}\big|\big(\partial_t^k e^{-t\sqrt{L+b}}f\big)(x)\big|^2 (t-s)^{2k-1}\,dt.\end{aligned}$$

Hence

$$\sup_{s>0}\int_0^{+\infty}\big|V(x,s)(t)\big|^2 t^{2k-1}\,dt \le \int_0^{+\infty}\big|\big(t^k\partial_t^k e^{-t\sqrt{L+b}}f\big)(x)\big|^2\frac{dt}{t} = \big(\mathcal{G}_L^{b,k}f(x)\big)^2.$$

Let $\mathbf{X} = L^2((0,\infty), t^{2k-1}\,dt)$. Then

$$\sup_{s>0}\big\|e^{-s\sqrt{L+b}}F(x)\big\|_{\mathbf{X}} = \mathcal{G}_L^{b,k}f(x) \in L^1\big(\mathbb{R}^d\big).$$

Therefore $F \in H^1_{\mathbf{X}}(\mathbb{R}^d)$, here $H^1_{\mathbf{X}}(\mathbb{R}^d)$ is a vector-valued Hardy space. Therefore $\widetilde{S_L^{b,1}}F(x) \in L^1(\mathbb{R}^d)$, where

$$\widetilde{S_L^{b,1}}F(x) = \left(\int_0^{+\infty}\int_{|z-y|<2t}\big\|D_t^{b,1}F(y)\big\|_{\mathbf{X}}^2\frac{dy\,dt}{t^{d+1}}\right)^{1/2}.$$

By

$$\begin{aligned}\big(\widetilde{S_L^{b,1}}F(x)\big)^2 &= \int_0^{+\infty}\int_{|x-y|<2t}\big\|D_t^{b,1}(x)\big\|_{\mathbf{X}}^2\frac{dy\,dt}{t^{d+1}}\\ &= \int_0^{+\infty}\int_{|x-y|<2t}\int_0^{+\infty}\big|(-t\sqrt{L+b})e^{-t\sqrt{L+b}}F(y)(s)\big|^2 s^{2k-1}\,ds\frac{dy\,dt}{t^{d+1}}\\ &= \int_0^{+\infty}\int_0^{+\infty}\int_{|x-y|<2t}\big|(-\sqrt{L+b})^{k+1}e^{-(s+t)\sqrt{L+b}}f(y)\big|^2\\ &\quad\times t^{1-d}s^{2k-1}\,dy\,dt\,ds\\ &= \int_0^{+\infty}\int_s^{+\infty}\int_{|x-y|<2(t-s)}\big|(-\sqrt{L+b})^{k+1}e^{-t\sqrt{L+b}}f(y)\big|^2\\ &\quad\times (t-s)^{1-d}s^{2k-1}\,dy\,dt\,ds\\ &= \int_0^{+\infty}\int_0^{t}\int_{|x-y|<2(t-s)}\big|(-\sqrt{L+b})^{k+1}e^{-t\sqrt{L+b}}f(y)\big|^2\\ &\quad\times (t-s)^{1-d}s^{2k-1}\,dy\,ds\,dt\\ &\ge \int_0^{+\infty}\int_0^{t/2}\int_{|x-y|<2(t-s)}\big|(-\sqrt{L+b})^{k+1}e^{-t\sqrt{L+b}}f(y)\big|^2\\ &\quad\times (t-s)^{1-d}s^{2k-1}\,dy\,ds\,dt\\ &\ge \int_0^{+\infty}\int_0^{t/2}\int_{|x-y|<t}\big|(-\sqrt{L+b})^{k+1}e^{-t\sqrt{L+b}}f(y)\big|^2 t^{1-d}s^{2k-1}\,dy\,ds\,dt\end{aligned}$$

$$= \frac{1}{2k2^{2k}} \int_0^{+\infty} \int_{|x-y|<t} \left|(-t\sqrt{L+b})^{k+1} e^{-t\sqrt{L+b}} f(y)\right|^2 t^{-1-2n}\, dy\, dt$$
$$= \frac{1}{2k2^{2k}} \int_0^{+\infty} \int_{|x-y|<t} \left|D_t^{b,k+1} f(y)\right|^2 \frac{dy\, dt}{t^{d+1}} = \frac{1}{2k2^{2k}} \left(S_L^{b,k+1} f(x)\right)^2,$$

we get $\|S_L^{b,k+1} f\|_{L^1} \leq C\|\mathcal{G}_L^{b,k}(f)\|_{L^1}$. □

By Lemma 5, we can prove the following.

Proposition 6 *$f \in H_L^1(\mathbb{R}^d)$ is equivalent to $\mathcal{G}_L^{b,k} f \in L^1(\mathbb{R}^d)$ and $f \in L^1(\mathbb{R}^d)$. Moreover,*

$$\|f\|_{H^1_{L+b}} \sim \left\|\mathcal{G}_L^{b,k} f\right\|_{L^1} + \|f\|_{L^1}.$$

Similar to the proof of Lemma 14 in [9], we have the following.

Lemma 6 *Let a be an $H_L^{1,\infty}$-atom. Then we can find a constant $C > 0$ satisfying*

$$\left\|\mathcal{G}_b(a)\right\|_{L^1} \leq C.$$

As pointed out in [14], we cannot get that an operator is bounded on $H_L^p(\mathbb{R}^d)$ by just proving that it is uniformly bounded on atoms. But we have the following lemma (cf. p.316, Theorem 7.3 in [15]).

Lemma 7 *Let T be an integral operator with the kernel in the Campanato space $\Lambda_{d(1/p-1)}$ and satisfy $\|Ta\|_{L^p} \leq C$ for all the $H_L^{p,q}$-atom $a(x)$, then T is a bounded operator from $H_L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$.*

In the following, we prove $D_t^b(x,y) = tA_j P_t^b(x,y)$ belongs to BMO_L, which is defined in [8].

Lemma 8 *For every $t > 0$ and $x \in \mathbb{R}^d$, we have $D_t^b(x,y) \in BMO_L$.*

Proof For any ball $B(y_0, r)$, if $r < \rho(y_0)$ and $r < t$, then by Lemma 3(b) we have

$$\begin{aligned} &\frac{1}{|B|^{1/2}} \left(\int_B \left|D_t^b(x,y) - D_t^b(x,y_0)\right|^2 dy \right)^{1/2} \\ &\leq C r^{-d/2} \left(\int_B \left(\frac{|y-y_0|}{t}\right)^2 \frac{t^{-2d}}{(1+t^{-2}|x-y_0|^2)^{d+1}}\, dy \right)^{1/2} \\ &\leq C t^{-d} \left(\frac{r}{t}\right) \leq C t^{-d}. \end{aligned} \tag{25}$$

If $t \leq r < \rho(y_0)$, then by Lemma 3(a)

$$\frac{1}{|B|^{1/2}} \left(\int_B \left|D_t^b(x,y) - D_t^b(x,y_0)\right|^2 dy \right)^{1/2} \leq C t^{-d}. \tag{26}$$

If $r \geq \rho(y_0)$, then by Lemma 3(a) we have

$$\begin{aligned} &\frac{1}{|B|^{1/2}}\left(\int_B \left|D_t^b(x,y)\right|^2 dy\right)^{1/2} \\ &\quad \leq Cr^{-d/2}\left(\int_B \frac{t^{-2d}}{(1+t^{-2}|x-y_0|^2)^{d+1}}\,dy\right)^{1/2} \\ &\quad \leq Ct^{-d}r^{-d/2}|B|^{1/2} \leq Ct^{-d}. \end{aligned} \tag{27}$$

Then Lemma 8 follows from (25)-(27). □

Now, let us prove Theorem 1.

Proof of Theorem 1 When $f \in L^1(\mathbb{R}^d)$ and $\mathcal{G}_b(f) \in L^1(\mathbb{R}^d)$, by Proposition 5 and Lemma 5, we have

$$\begin{aligned} \|f\|_{H_L^1} &\leq C\|f\|_{H_{L+b}^1} \leq C\left\{\left\|S_L^{b,2}(f)\right\|_{L^1} + \|f\|_{L^1}\right\} \leq C\left\{\left\|\mathcal{G}_L^{b,1}(f)\right\|_{L^1} + \|f\|_{L^1}\right\} \\ &\leq C\left\{\left\|\mathcal{G}_b(f)\right\|_{L^1} + \|f\|_{L^1}\right\}. \end{aligned}$$

Therefore, $f \in H_L^1(\mathbb{R}^d)$.

The reverse can be proved by Lemmas 6, 7 and 8.

Theorem 1 is proved. □

4 Riesz transform associated with *L*

We introduce the following version of Riesz transform:

$$R_j^{L,b} = A_j(L+b)^{-1/2}, \quad j = 1,2,\ldots,d, b > 0.$$

If $f \in L^2(\mathbb{R}^d)$, then

$$\begin{aligned} R_j^{L,b}f &= \sum_\alpha \left(\frac{2\alpha_j}{2|\alpha|+d+b}\right)^{1/2}\langle f,h_\alpha\rangle h_{\alpha-e_j} \\ &= \sum_{n=0}^{\infty}\sum_{|\alpha|=n}\left(\frac{2\alpha_j}{2n+d+b}\right)^{1/2}\langle f,h_\alpha\rangle h_{\alpha-e_j}. \end{aligned} \tag{28}$$

We can prove the following.

Theorem 3 *Let* $j = 1,2,\ldots,d$. *Then* $R_j^{L,b}$ *are bounded operators on* $H_L^1(\mathbb{R}^d)$, *that is, there is* $C > 0$ *satisfying*

$$\left\|R_j^{L,b}f\right\|_{H_L^1} \leq C\|f\|_{H_L^1}.$$

Proof When $f \in L^2(\mathbb{R}^d)$, following (28), it is not difficult to check

$$D_t^{b+2,1}R_j^{L,b}f = -tA_ju_b(x,t) \tag{29}$$

for $j = 1, 2, \ldots, d$. As $H_L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ is dense in $H_L^1(\mathbb{R}^d)$ (see [11]), we can assume $f \in H_L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Then, by Lemma 5, Theorem 1, Proposition 6 and (29), we get

$$\begin{aligned}\|R_j^{L,b} f\|_{H_L^1} &\le C\|R_j^{L,b} f\|_{H_{L+b+2}^1} \le C\|\mathcal{G}_L^{b+2,1}(R_j^{L,b} f)\|_{L^1} \\ &= C\left\|\left(\int_0^\infty |tA_j u_b(x,t)|^2 \frac{dt}{t}\right)^{1/2}\right\|_{L^1} \\ &\le C\|\mathcal{G}_b(f)\|_{L^1} \le C\|f\|_{H_L^1}.\end{aligned}$$

This proves Theorem 3. □

The proof of the following lemma can be found in [16].

Lemma 9 *If $\beta \in \mathbb{R}$ and $f \in L^2(\mathbb{R}^d)$, then*

$$A_j L^\beta f = (L+2)^\beta A_j f,$$

for $j = 1, 2, \ldots, d$.

Now, we can prove Theorem 2.

Proof of Theorem 2 Since $H_L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ is dense in $H_L^1(\mathbb{R}^d)$ (see [11]), we can assume $f \in H_L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. We prove Theorem 2 by an inductive argument.

When $m = 1$, Theorem 2 has been proved in [4]. We assume that Theorem 2 holds for $m - 1$, by Lemma 9 and Theorem 3,

$$\begin{aligned}\|R_{i_1 i_2 \cdots i_m} L^{-m/2} f\|_{H_L^1} &= \|A_{i_1}(L + 2(m-1))^{-1/2} A_{i_2} \cdots A_{i_m} L^{-(m-1)/2} f\|_{H_L^1} \\ &\le \|A_{i_2} \cdots A_{i_m} L^{-(m-1)/2} f\|_{H_L^1} \le \|f\|_{H_L^1}.\end{aligned}$$

Therefore Theorem 2 holds. □

5 Conclusions

In this paper, we consider the Riesz transforms of higher order associated with a harmonic oscillator and prove the boundedness of them on the Hardy space. It is well known that the Riesz transforms play an important role in the study of harmonic analysis and partial differential equations. These results are very good progress on the harmonic analysis of Hermite operators.

Competing interests
The author declares that they have no competing interests.

Acknowledgements
This paper is supported by the National Natural Science Foundation of China (11471018), the Beijing Natural Science Foundation (1142005).

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 25 December 2016 Accepted: 21 April 2017 Published online: 03 May 2017

References

1. Thangavelu, S: Lectures on Hermite and Laguerre Expansions. Mathematical Notes, vol. 42. Princeton University Press, Princeton (1993)
2. Thangavelu, S: Riesz transforms and the wave equation for the Hermite operator. Commun. Partial Differ. Equ. **15**, 1199-1215 (1990)
3. Goldberg, D: A local version of real Hardy spaces. Duke Math. J. **46**(1), 27-42 (1979)
4. Huang, JZ: The boundedness of Riesz transforms for Hermite expansions on the Hardy spaces. J. Math. Anal. Appl. **385**(1), 559-571 (2012)
5. Dziubański, J, Zienkiewicz, J: The Hardy space H^1 associated to Schrödinger operator with potential satisfying reverse Hölder inequality. Rev. Mat. Iberoam. **15**, 279-296 (1999)
6. Harboure, E, de Rosa, L, Segovia, C, Torrea, JL: L^p-Dimension free boundedness for Riesz transforms associated to Hermite functions. Math. Ann. **328**, 653-682 (2004)
7. Dziubański, J, Zienkiewicz, J: H^p spaces for Schrödinger operators. In: Fourier Analysis and Related Topics. Banach Center Publications, vol. 56, pp. 45-53. Polish Acad. Sci. Inst. Math., Warsaw (2002)
8. Dziubański, J, Garrigós, G, Martínez, T, Torrea, JL, Zienkiewicz, J: BMO spaces related to Schrödinger operator with potential satisfying reverse Hölder inequality. Math. Z. **249**, 329-356 (2005)
9. Lin, CC, Liu, HP: $BMO_L(\mathbb{H}^n)$ spaces and Carleson measures for Schrödinger operators. Adv. Math. **228**, 1631-1688 (2011)
10. Stempak, K, Torrea, JL: Poisson integrals and Riesz transforms for the Hermite function expansions with weights. J. Funct. Anal. **202**, 443-472 (2003)
11. Hofmann, S, Lu, GZ, Mitres, D, Yan, LX: Hardy spaces associated to non-negative self-adjoint operators satisfying Davies-Gaffney estimates. Mem. Am. Math. Soc. **214**, 1007 (2011)
12. Huang, JZ, Xing, Z: New real-variable characterizations of Hardy spaces associated with twisted convolution. J. Inequal. Appl. **2015**, 170 (2015)
13. Folland, GB, Stein, EM: Hardy Spaces on Homogeneous Groups. Princeton University Press, Princeton (1982)
14. Bownik, M: Boundedness of operators on Hardy spaces via atomic decomposition. Proc. Am. Math. Soc. **133**, 3535-3542 (2005)
15. Garcia-Cuerva, J, Rubio de Francia, JL: Weighted Norm Inequalities and Related Topics. North-Holland Mathematics Studies, vol. 116. North-Holland, Amsterdam (1985)
16. Bongioanni, B, Torrea, JL: Sobolev spaces associated to the harmonic oscillator. Proc. Indian Acad. Sci. Math. Sci. **116**, 337-360 (2006)