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Sequence spaces $M(\phi)$ and $N(\phi)$ with application in clustering

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Abstract

Distance measures play a central role in evolving the clustering technique. Due to the rich mathematical background and natural implementation of l_p distance measures, researchers were motivated to use them in almost every clustering process. Beside l_p distance measures, there exist several distance measures. Sargent introduced a special type of distance measures $m(\phi)$ and $n(\phi)$ which is closely related to l_p. In this paper, we generalized the Sargent sequence spaces through introduction of $M(\phi)$ and $N(\phi)$ sequence spaces. Moreover, it is shown that both spaces are *BK*-spaces, and one is a dual of another. Further, we have clustered the two-moon dataset by using an induced $M(\phi)$-distance measure (induced by the Sargent sequence space $M(\phi)$) in the k-means clustering algorithm. The clustering result established the efficacy of replacing the Euclidean distance measure by the $M(\phi)$-distance measure in the k-means algorithm.

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1 Introduction

Clustering is a well-known procedure to deal with an unsupervised learning problem appearing in pattern recognition. Clustering is a process of organizing data into groups called clusters so that objects in the same cluster are similar to one another, but are dissimilar to objects in other clusters [1]. The main contribution in the field of clustering analysis was the pioneering work of MacQueen [1] and Bezdek [2]. They had introduced highly significant clustering algorithms such as k-means [1] and fuzzy c-means [2]. Among all clustering algorithms, k-means is the simplest unsupervised clustering algorithm that makes use of a minimum distance from the center, and it has many applications in scientific and industrial research [3–6] (for more information about the k-means clustering algorithm, see Section 5). K-means algorithm is distance dependent, so its outputs vary with changing distance measures. Among all distance measures, a clustering process was usually carried out through the Euclidean distance measure [7], but many times it failed to offer good results. In this paper, we define $M(\phi)$- and $N(\phi)$-distance measure. Further, $M(\phi)$-distance is used to cluster two-moon dataset. The output result is compared with the result of Euclidean distance measure to show the efficacy of $M(\phi)$-distance over the Euclidean distance measure. $M(\phi)$ and $N(\phi)$-distance measures are the generalization of $m(\phi)$- and $n(\phi)$-distance measures introduced by Sargent [8] and further studied by Mursaleen [9, 10] (to know more about $m(\phi)$ and $n(\phi)$, refer to [8–10]). The $M(\phi)$ and $N(\phi)$ spaces are closely related to l_p distance measures. l_p measures and its variance are mostly used to solve the problems evolving in the fields of Market prediction [11], Machine Learning [12], Pattern Recognition [13], Clustering [20] etc.

Throughout the paper, by ω we denote the set of all real or complex sequences. Moreover, by l_∞, c and c_0 we denote the Banach spaces of bounded, convergent and null sequences, respectively; and let l_p be the Banach space of absolutely p-summable sequences with p-norm $\|\cdot\|_p$. For the following notions, we refer to [14, 15]. A double sequence $x = (x_{jk})$ of real or complex numbers is said to be bounded if $\|x\|_\infty < \infty$, the space of all bounded double sequences is denoted by $\mathcal{L}_\infty$. A double sequence $x = (x_{jk})$ is said to converge to the limit L in Pringsheim's sense (shortly, convergent to L) if for every $\varepsilon > 0$, there exists an integer N such that $|x_{jk} - L| < \varepsilon$ whenever $j, k > N$. In this case L is called the p-limit of x. If in addition $x \in \mathcal{L}_\infty$, then x is said to be boundedly convergent to L in Pringsheim's sense (shortly, bp-convergent to L). A double sequence $x = (x_{jk})$ is said to converge regularly to L (shortly, r-convergent to L) if x is p-convergent and the limits $x_j := \lim_k x_{jk}$ $(j \in \mathbb{N})$ and $x^k := \lim_j x_{jk}$ $(k \in \mathbb{N})$ exist. Note that in this case the limits $\lim_j \lim_k x_{jk}$ and $\lim_k \lim_j x_{jk}$ exist and are equal to the p-limit of x. In general, for any notion of convergence ν, the space of all ν-convergent double sequences will be denoted by $\mathcal{C}_\nu$ and the limit of a ν-convergent double sequence x by $\nu\text{-}\lim_{j,k} x_{jk}$, where $\nu \in \{p, bp, r\}$.

Let Ω denote a vector space of all double sequences with the vector space operations defined coordinate-wise. Vector subspaces of Ω are called double sequence spaces. Let us consider a double sequence $x = \{x_{mn}\}$ and define the sequence $s = \{s_{mn}\}$ via x by

$$s_{mn} := \sum_{i,j}^{m,n} x_{ij} \quad (m, n \in \mathbb{N}).$$

Then the pair (x, s) and the sequence $s = \{s_{mn}\}$ are called a double series and a sequence of partial sums of the double series, respectively. Let λ be the space of double sequences converging with respect to some linear convergence rule $\mu\text{-}\lim : \lambda \to \mathbb{R}$. The sum of a double series $\sum_{i,j=1}^{\infty,\infty} x_{ij}$ with respect to this rule is defined by $\mu\text{-}\sum_{i,j=1}^{\infty,\infty} x_{ij} := \mu\text{-}\lim s_{mn}$. Başar and Şever introduced the space L_p in [16]

$$L_p := \left\{ \{x_{mn}\} \in \Omega : \sum_{m,n} |x_{mn}|^p < \infty \right\} \quad (1 \le p < \infty)$$

corresponding to the space l_p for $p \ge 1$ and examined some of its properties. Altay and Başar [17] have generalized the spaces of double sequences L_∞, C_p and C_{bp} to

$$L_\infty(t) = \left\{ \{x_{mn}\} \in \Omega : \sup_{m,n \in \mathbb{N}} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$C_p(t) = \left\{ \{x_{mn}\} \in \Omega : p\text{-}\lim_{m,n \to \infty} |x_{mn} - \ell|^{t_{mn}} = 0 \right\},$$

and

$$C_{bp}(t) = C_p(t) \cap L_\infty(t),$$

respectively, where $t = \{t_{mn}\}$ is the sequence of strictly positive reals t_{mn}. In the case $t_{mn} = 1$, for all $m, n \in \mathbb{N}$, $L_\infty(t)$, $C_p(t)$ and $C_{bp}(t)$ reduce to the sets L_∞, C_p and C_{bp}, respectively. Further, let C be the space whose elements are finite sets of distinct positive integers. Given any element σ of C, we denote by $c(\sigma)$ the sequence $\{c_n(\sigma)\}$ which is such that $c_n(\sigma) = 1$ if $n \in \sigma$, $c_n(\sigma) = 0$ otherwise. Further, let

$$C_s = \left\{\sigma \in C : \sum_{n=1}^{\infty} c_n(\sigma) \leq s\right\}$$

be the set of those σ whose support has cardinality at most s, and

$$\Phi = \left\{\phi = \{\phi_n\} \in \omega : \phi_1 > 0, \Delta\phi_n \geq 0 \text{ and } \Delta\left(\frac{\phi_n}{n}\right) \leq 0 \ (n = 1, 2, \ldots)\right\},$$

where $\Delta\phi_n = \phi_n - \phi_{n-1}$ and $\phi_0 = 0$.

For $\phi \in \Phi$, the following sequence spaces were introduced and studied in [8] by Sargent and further studied by Mursaleen in [9, 10]:

$$m(\phi) = \left\{x = \{x_n\} \in \omega : \sup_{s\geq 1} \sup_{\sigma \in C_s} \left(\frac{1}{\phi_s} \sum_{n\in\sigma} |x_n|\right) < \infty\right\},$$

and

$$n(\phi) = \left\{x = \{x_n\} \in \omega : \sup_{u\in S(x)} \left(\sum_{m,n=1,1}^{\infty,\infty} |u_n| \Delta\phi_n\right) < \infty\right\}.$$

Remark 1.1

(i) The spaces $m(\phi)$ and $n(\phi)$ are BK-spaces with their usual norms.
(ii) If $\phi_n = 1$ $(n = 1, 2, 3, \ldots)$, then $m(\phi) = l_1$ $[n(\phi) = l_\infty]$, and if $\phi_n = n$ $(n = 1, 2, 3, \ldots)$, then $m(\phi) = l_\infty$ $[n(\phi) = l_1]$.
(iii) $l_1 \subseteq m(\phi) \subseteq l_\infty$ $[l_1 \subseteq n(\phi) \subseteq l_\infty]$ for all $\phi \in \Phi$.
(iv) For any $\phi \in \Phi$, $m(\phi) \neq l_p$ $[n(\phi) \neq l_q]$, $1 < p < \infty$.

In this paper, we define Sargent's spaces for double sequences $x = \{x_{mn}\}$. For this we first suppose U to be the set whose elements are finite sets of distinct elements of $\mathbb{N} \times \mathbb{N}$ obtained by $\sigma \times \varsigma$, where $\sigma \in C_s$ and $\varsigma \in C_t$ for each $s, t \geq 1$. Therefore, any element ζ of U means (m, n); $m \in \sigma$ and $n \in \varsigma$ having cardinality at most st, where s is the cardinality with respect to m and t is the cardinality with respect to n. Given any element ζ of U, we denote by $c(\zeta)$ the sequence $\{c_{mn}(\zeta)\}$ such that

$$c_{mn}(\zeta) = \begin{cases} 1; & \text{if } (m, n) \in \zeta, \\ 0; & \text{otherwise.} \end{cases}$$

Further, we write

$$U_{st} = \left\{\zeta \in U : \sum_{m,n=1}^{\infty,\infty} c_{mn}(\zeta) \leq st\right\}$$

for the set of those ζ whose support has cardinality at most st; and

$$\Theta = \left\{ \phi = \{\phi_{mn}\} \in \Omega : \phi_{11} > 0, \Delta_{11}\phi_{mn} \geq 0 \text{ and } \Delta_{11}\left(\frac{\phi_{mn}}{mn}\right) \leq 0 \ (m,n = 1,2,\ldots) \right\},$$

where $\Delta_{11}\phi_{mn} = \phi_{mn} - \phi_{m-1,n} - \phi_{m,n-1} + \phi_{m-1,n-1}$ and $\phi_{00}, \phi_{0m}, \phi_{n0} = 0, \forall m,n \in \mathbb{I}^+$. Throughout the paper, we write $\sum_{m,n\in\zeta}$ for $\sum_{m\in\sigma}\sum_{n\in\varsigma}$, and $S(x)$ is used to denote the set of all double sequences that are rearrangements of $x = \{x_{mn}\} \in \Omega$. For $\phi \in \Theta$, we define the following sequence spaces:

$$M(\phi) = \left\{ x = \{x_{mn}\} \in \Omega : \|x\|_{M(\phi)} = \sup_{s,t\geq 1} \sup_{\zeta\in U_{st}} \left(\frac{1}{\phi_{st}} \sum_{m,n\in\zeta} |x_{mn}| \right) < \infty \right\}$$

and

$$N(\phi) = \left\{ x = \{x_{mn}\} \in \Omega : \|x\|_{N(\phi)} = \sup_{u\in S(x)} \left(\sum_{m,n=1}^{\infty,\infty} |u_{mn}| \Delta_{11}\phi_{mn} \right) < \infty \right\}.$$

Then the distances between $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ induced by $M(\phi)$ and $N(\phi)$ can be expressed as

$$d_{M(\phi)} = \sup_{s,t\geq 1} \sup_{\zeta\in U_{st}} \left(\frac{1}{\phi_{st}} \sum_{m,n\in\zeta} |x_{mn} - y_{mn}| \right)$$

and

$$d_{N(\phi)} = \sup_{u,v\in S(x)} \left(\sum_{m,n=1}^{\infty,\infty} |u_{mn} - v_{mn}| \Delta_{11}\phi_{mn} \right).$$

Remark 1.2 If $\phi_{st} = 1$ $(s,t = 1,2,3,\ldots)$, then $M(\phi) = L_1$ $[N(\phi) = L_\infty]$, and if $\phi_{st} = st$ $(s,t = 1,2,3,\ldots)$, then $M(\phi) = L_\infty$ $[N(\phi) = L_1]$.

We now state the following known results of [18] for single sequences (series) which can also be proved easily for double sequences (series).

Lemma 1.1 *If the series $\sum u_n x_n$ is convergent for every x of a BK-space E, then the functional $\sum_{n=1}^{\infty} u_n x_n$ is linear and continuous in E.*

Lemma 1.2 *If E and F are BK-spaces, and if $E \subseteq F$, then there is a real number K such that, for all x of E,*

$$\|x\|_F \leq K\|x\|_E.$$

2 Properties of the spaces $M(\phi)$ and $N(\phi)$

Theorem 2.1 *The space $M(\phi)$ is a BK-space with the norm.*

$$\|x\|_{M(\phi)} = \sup_{s,t\geq 1} \sup_{\zeta\in U_{st}} \frac{1}{\phi_{st}} \left(\sum_{m,n\in\zeta} |x_{mn}| \right). \tag{2.1}$$

Proof It is a routine verification to show that $M(\phi)$ is a normed space with the given norm (2.1), and so we omit it. Now, we proceed to showing that $M(\phi)$ is complete. Let $\{x^l\}$ be a Cauchy sequence in $M(\phi)$, where $x^l = \{x^l_{mn}\}_{m,n=1,1}^{\infty,\infty}$ for every fixed $l \in \mathbb{N}$. Then, for a given $\varepsilon > 0$, there exists a positive integer $n_0(\varepsilon) > 0$ such that

$$\|x^l - x^r\|_{M(\phi)} = \sup_{s,t\geq 1} \sup_{\zeta \in U_{st}} \frac{1}{\phi_{st}} \left(\sum_{m,n\in\zeta} |x^l_{mn} - x^r_{mn}| \right) < \varepsilon$$

for all $l, r > n_0(\varepsilon)$, which yields, for each fixed $s, t \geq 1$ and $\zeta \in U_{st}$,

$$\sum_{m,n\in\zeta} |x^l_{mn} - x^r_{mn}| \leq \varepsilon\phi_{11} \quad \text{for all } l, r > n_0(\varepsilon). \tag{2.2}$$

Therefore

$$\left| \sum_{m,n\in\zeta} |x^l_{mn}| - \sum_{m,n\in\zeta} |x^r_{mn}| \right| < \varepsilon\phi_{11} \quad \text{for all } l, r > n_0(\varepsilon). \tag{2.3}$$

This means that $\{\sum_{m,n\in\zeta} |x^l_{mn}|\}_{l\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ for every fixed $s, t \geq 1$ and $\zeta \in U_{st}$. Since $\mathbb{R}$ is complete, it converges, say

$$\sum_{m,n\in\zeta} |x^l_{mn}| \to \sum_{m,n\in\zeta} |x_{mn}| \quad \text{as } l \to \infty.$$

Since absolute convergence implies convergence in $\mathbb{R}$, hence

$$\sum_{m,n\in\zeta} x^l_{mn} \to \sum_{m,n\in\zeta} x_{mn} \quad \text{as } l \to \infty. \tag{2.4}$$

Hence we have

$$\lim_{l\to\infty} \left\| \sum_{m,n\in\zeta} x^l_{mn} - \sum_{m,n\in\zeta} x_{mn} \right\|_{M(\phi)} = 0. \tag{2.5}$$

Let $y^l = \sum_{m,n\in\zeta} |x^l_{mn}|$. Then $\{y^l\} \in l_\infty$. Therefore

$$\sup_{l\in\mathbb{N}} \sum_{m,n\in\zeta} |x^l_{mn}| \leq k.$$

Since $\sum_{m,n\in\zeta} |x_{mn}| \leq \sum_{m,n\in\zeta} |x_{mn} - x^l_{mn}| + \sum_{m,n\in\zeta} |x^l_{mn}| \leq \varepsilon\phi_{11} + k$, it follows that $x = \{x_{mn}\} \in M(\phi)$. Since $\{x^l\}_{l\in\mathbb{N}}$ was an arbitrary Cauchy sequence, the space $M(\phi)$ is complete. Now we prove that $M(\phi)$ has continuous coordinate projections p_{mn}, where $p_{mn} : \Omega \to K$ and $p_{mn}(x) = x_{mn}$. The coordinate projections p_{mn} are continuous since $|x_{mn}| \leq \sup_{s,t\geq 1} \sup_{\zeta\in U_{st}} \phi_{st} \|x\|_{M(\phi)}$ for each $m, n \in \mathbb{N}$. □

Remark 2.1 The space $N(\phi)$ is a BK-space with the norm

$$\|x\|_{N(\phi)} = \sup_{u\in S(x)} \left(\sum_{m,n=1}^{\infty,\infty} |u_{mn}| \Delta_{11}\phi_{mn} \right).$$

Lemma 2.1

(i) *If* $x \in M(\phi)$ $[x \in N(\phi)]$ *and* $u \in S(x)$, *then* $u \in M(\phi)$ $[u \in N(\phi)]$ *and* $\|u\| = \|x\|$.

(ii) *If* $x \in M(\phi)$ $[x \in N(\phi)]$ *and* $|u_{mn}| \leq |x_{mn}|$ *for every positive integer* m, n, *then* $u \in M(\phi)$ $[u \in N(\phi)]$ *and* $\|u\| \leq \|x\|$.

Proof (i) Let $x \in M(\phi)$, then $\sup_{s,t\geq 1} \sup_{\zeta \in U_{st}} \frac{1}{\phi_{st}} \sum_{m,n\in\zeta} |x_{mn}| < \infty$. So, we have

$$\frac{1}{\phi_{st}} \sum_{m,n\in\zeta} |x_{mn}| < \infty \quad \text{for each } \zeta \in U_{st} \text{ and } s,t \geq 1.$$

Since the sum of a finite number of terms remains the same for all the rearrangements,

$$\frac{1}{\phi_{st}} \sum_{m,n\in\zeta} |u_{mn}| = \frac{1}{\phi_{st}} \sum_{m,n\in\zeta} |x_{mn}| \quad \text{for each } u \in S(x) \text{ and } \zeta \in U_{st}, s,t \geq 1.$$

Hence

$$\sup_{s,t\geq 1} \sup_{\zeta \in U_{st}} \frac{1}{\phi_{st}} \sum_{m,n\in\zeta} |u_{mn}| = \sup_{s,t\geq 1} \sup_{\zeta \in U_{st}} \frac{1}{\phi_{st}} \sum_{m,n\in\zeta} |x_{mn}| < \infty,$$

thus $u \in M(\phi)$ and $\|u\| = \|x\|$.

(ii) By using the definition, easy to prove. □

Theorem 2.2 *For arbitrary* $\phi \in \Theta$, *we have* $\Delta_{11}\phi \in M(\phi)$ *and* $\|\Delta_{11}\phi\|_{M(\phi)} \leq 2$.

Proof Let s and t be arbitrary positive integers, let $\sigma, \varsigma \in U_{st}$, and let τ_1, τ_2 constitute the element of σ and ς exceed by s and t respectively, also from the definition we have $\Delta_{11}\phi \geq 0$ and $\Delta_{11}(\frac{\phi_{mn}}{mn}) \leq 0$. Then

$$\begin{aligned}
\sum_{n\in\sigma, m\in\varsigma} |\Delta_{11}\phi_{mn}| &\leq \sum_{n=1,m=1}^{s,t} \Delta_{11}\phi_{mn} + \sum_{n\in\tau_1, m\in\tau_2} \Delta_{11}\phi_{mn} \\
&\leq \phi_{st} + \sum_{n\in\tau_1, m\in\tau_2} \left(\frac{\phi_{m-1,n-1}}{(m-1)(n-1)} \right) \\
&\leq \phi_{st} + \left.\begin{matrix}
\frac{\phi_{st}}{st} & + \frac{\phi_{s,t+1}}{s(t+1)} & + \frac{\phi_{s,t+2}}{s(t+2)} & +. & . \\
\frac{\phi_{s+1,t}}{(s+1)t} & + \frac{\phi_{s+1,t+1}}{(s+1)(t+1)} & + \frac{\phi_{s+1,t+2}}{(s+1)(t+2)} & +. & . \\
+ & + & + & . & \\
. & . & . & . &
\end{matrix}\right\} \max(st)\text{-terms} \\
&\leq \phi_{st} + st\frac{\phi_{st}}{st} = 2\phi_{st}.
\end{aligned}$$

□

Lemma 2.2 *If* $x \in M(\phi)$ *and* $\{c_{11}, c_{12}, \dots, c_{1n}, c_{21}, c_{22}, \dots, c_{2n}, \dots, c_{m1}, c_{m2}, \dots, c_{mn}\}$ *is a rearrangement of* $\{b_{11}, b_{12}, \dots, b_{1n}, b_{21}, b_{22}, \dots, b_{2n}, \dots, b_{m1}, b_{m2}, \dots, b_{mn}\}$ *such that* $|c_{11}| \geq |c_{12}| \geq \cdots \geq |c_{1n}|$, $|c_{21}| \geq |c_{22}| \geq \cdots \geq |c_{2n}|, \dots, |c_{m1}| \geq |c_{m2}| \geq \cdots \geq |c_{mn}|$ *and* $|c_{11}| \geq |c_{21}| \geq \cdots \geq |c_{m1}|$, $|c_{12}| \geq |c_{22}| \geq \cdots \geq |c_{m2}|, |c_{n1}| \geq |c_{n2}| \geq \cdots \geq |c_{nm}|$, *then*

$$\sum_{i,j=1,1}^{m,n} |b_{ij}x_{ij}| \leq \|x\|_{M(\phi)} \sum_{i,j=1,1}^{m,n} |c_{ij}|\Delta_{11}\phi_{ij}.$$

Proof In view of Lemma 2.1(i), it is sufficient to consider the case when $b_{ij} = c_{ij}$ ($i = 1,2,\ldots,m$; $j = 1,2,\ldots,n$). Then writing $X_{mn} = \sum_{i,j=1}^{m,n} |x_{ij}|$, we get

$$\begin{aligned}\sum_{i,j=1,1}^{m,n} |b_{ij}x_{ij}| &= \sum_{i=1}^{m-1}\sum_{j=1}^{n-1}\bigl(|c_{ij}| - |c_{i,j+1}| - |c_{i+1,j}| + |c_{i+1,j+1}|\bigr)X_{ij} + |c_{mn}|X_{mn} \\ &\leq \|x\|_{M(\phi)}\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}\bigl(|c_{ij}| - |c_{i,j+1}| - |c_{i+1,j}| + |c_{i+1,j+1}|\bigr)\phi_{ij} + \|x\|_{M(\phi)}|c_{mn}|X_{mn} \\ &= \|x\|_{M(\phi)}\sum_{i,j=1,1}^{m,n} |c_{ij}|\Delta_{11}\phi_{ij}.\end{aligned}$$

Hence we have $\sum_{i,j=1,1}^{m,n} |b_{ij}x_{ij}| \leq \|x\|_{M(\phi)}\sum_{i,j=1,1}^{m,n} |c_{ij}|\Delta_{11}\phi_{ij}$. □

Theorem 2.3 *In order that $\sum u_{ij}x_{ij}$ be convergent [absolutely convergent] whenever $x \in M(\phi)$, it is necessary and sufficient that $u \in N(\phi)$. Further, if $x \in M(\phi)$ and $u \in N(\phi)$, then*

$$\sum_{i,j=1,1}^{\infty,\infty} |u_{ij}x_{ij}| \leq \|u\|_{N(\phi)}\|x\|_{M(\phi)}. \tag{2.6}$$

Proof Necessity. We now suppose that $\sum u_{ij}x_{ij}$ is convergent whenever $x \in M(\phi)$, then from Lemma 1.1 we have

$$\left|\sum_{i,j=1,1}^{\infty,\infty} u_{ij}x_{ij}\right| \leq K\|x\|_{M(\phi)}$$

for some real number K and all x of $M(\phi)$. In view of Lemma 2.1(ii), we may replace x_{ij} by $x_{ij}\operatorname{sgn}\{u_{ij}\}$, obtaining

$$\sum_{i,j=1,1}^{\infty,\infty} |u_{ij}x_{ij}| \leq K\|x\|_{M(\phi)}. \tag{2.7}$$

Let $v \in S(u)$. Then taking x to be a suitable rearrangement of $\Delta_{11}\phi$, it follows from Eq. (2.7) and Theorem 2.2 and Lemma 2.1(i) that

$$\sum_{i,j=1,1}^{\infty,\infty} |v_{ij}|\Delta_{11}\phi_{ij} \leq 4K,$$

and thus $u \in N(\phi)$.

Sufficiency. If $x \in M(\phi)$ and $u \in N(\phi)$, it follows from Lemma 2.2 that for every positive integer m and n,

$$\sum_{i,j=1,1}^{\infty,\infty} |u_{ij}x_{ij}| \leq \|u\|_{N(\phi)}\|x\|_{M(\phi)}.$$

□

Theorem 2.4 *In order that $\sum u_{mn}x_{mn}$ be convergent [absolutely convergent] whenever $x \in N(\phi)$, it is necessary and sufficient that $u \in M(\phi)$.*

Proof Since sufficiency is included in Theorem 2.3, we only consider necessity. We therefore suppose that $\sum u_{mn}x_{mn}$ is convergent whenever $x \in N(\phi)$. By arguments similar to those used in Theorem 2.3, we may therefore have that

$$\sum_{m,n=1,1}^{\infty,\infty} |u_{mn}x_{mn}| \leq K\|x\|_{N(\phi)} \tag{2.8}$$

for some real number K and all x of $N(\phi)$. Let $x = c(\zeta)$, where $\zeta \in U_{st}$. Then $x \in N(\phi)$, and

$$\|x\|_{N(\phi)} = \sup_{\xi \in U_{st}} \sum_{m,n\in\xi} \Delta_{11}\phi_{mn} \leq 4\phi_{st},$$

from Theorem 2.2 and Eq. (2.8) we have

$$\sum_{m,n\in\zeta} |u_{mn}| \leq 4K\phi_{st} \quad (\zeta \in U_{st}; s,t = 1,2,3,\ldots),$$

and thus $u \in M(\phi)$. □

3 Inclusion relations for $M(\phi)$ and $N(\phi)$

Lemma 3.1 *In order that* $M(\phi) \subseteq M(\psi)$ $[N(\phi) \supseteq N(\psi)]$, *it is necessary and sufficient that*

$$\sup_{s,t\geq 1}\left(\frac{\phi_{st}}{\psi_{st}}\right) < \infty.$$

Proof Since each of the spaces $M(\phi)$ and $N(\phi)$ is the dual of the other, by Theorems 2.3 and 2.4, the second version is equivalent to the first. Moreover, sufficiency follows from the definition of an $M(\phi)$ space. We therefore suppose that $M(\phi) \subseteq M(\psi)$. Since $\Delta\phi \in M(\phi)$, it follows that $\Delta\psi \in M(\psi)$, and hence we find that, for every positive integer $s,t \geq 1$,

$$\phi_{st} = \sum_{i,j=1,1}^{s,t} \Delta_{11}\phi_{ij} \leq \psi_{st}\|\Delta\phi\|_{M(\psi)}, \quad \text{where } \Delta = \Delta_{11}.$$

□

Theorem 3.1

(i) $L_1 \subseteq M(\phi) \subseteq L_\infty$ $[L_1 \subseteq N(\phi) \subseteq L_\infty]$ *for all* ϕ *of* Θ.

(ii) $M(\phi) = L_1$ $[N(\phi) = L_\infty]$ *if and only if* $bp\text{-}\lim_{s,t}\phi_{st} < \infty$.

(iii) $M(\phi) = L_\infty$ $[N(\phi) = L_1]$ *if and only if* $bp\text{-}\lim_{s,t}(\phi_{st}/st) > 0$.

Proof We prove here the first version, while the second version follows by Theorems 2.3 and 2.4. Since $\phi_{11} \leq \phi_{mn} \leq mn\phi_{mn}$ for all ϕ of Θ, we have by Lemma 3.1 that (i) is satisfied. Further, from Lemma 3.1, it follows that $M(\phi) \subseteq L_1$ if and only if $\sup_{s,t\geq 1}\phi_{st} < \infty$, while $L_\infty \subseteq M(\phi)$ if and only if $\sup_{s,t\geq 1}(\phi_{st}/st) < \infty$; since the sequences $\{\phi_{st}\}$ and $\{st/\phi_{st}\}$ are monotonic, (ii) and (iii) are also satisfied. □

Theorem 3.2 *Suppose that* $1 < p < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

(i) *Given any* ϕ *of* Θ, $M(\phi) \neq L_p$ $[N(\phi) \neq L_q]$.

(ii) *In order that* $L_p \subset M(\phi)$ $[N(\phi) \subset L_q]$, *it is necessary and sufficient that* $\sup_{s,t\geq 1}(\frac{(st)^{1/q}}{\phi_{st}}) < \infty$.

(iii) *In order that* $M(\phi) \subset L_p$ $[N(\phi) \supset L_q]$, *it is necessary and sufficient that* $\Delta\phi \in L_p$.
(iv) $\bigcup_{\Delta\phi\in L_p} M(\phi) = L_p$ $[\bigcap_{\Delta\phi\in L_p} N(\phi) = L_q]$.

Proof (i) Let us suppose that $M(\phi) = L_p$.

Then, by Lemma 1.2, there exist real numbers r_1 and r_2 ($r_1 > 0$, $r_2 > 0$) such that, for all x of $M(\phi)$,

$$r_1\|x\|_{L_p} \leq \|x\|_{M(\phi)} \leq r_2\|x\|_{L_p}.$$

Taking $x = c(\zeta)$, where $\zeta \in \mathcal{U}_{st}$, we have that

$$r_1(st)^{\frac{1}{p}} \leq \frac{st}{\phi_{st}} \leq r_2(st)^{\frac{1}{p}} \quad (s,t = 1,2,3,\ldots),$$

and hence that

$$r_1 \leq \frac{(st)^{\frac{1}{q}}}{\phi_{st}} \leq r_2 \quad (s,t = 1,2,3,\ldots).$$

In view of Lemma 3.1, this implies that $M(\phi) = M(\psi)$, where $\psi = \{(mn)^{\frac{1}{q}}\}$. Since $\Delta\psi \in M(\psi)$ by Theorem 2.2, but $\Delta\psi \notin L_q$, this leads to a contradiction. Hence (i) follows.

(ii) If $L_q \subset M(\phi)$, arguments similar to those used in the proof of (i) show that

$$(st)^{1/q} \leq K\phi_{st} \quad (s,t = 1,2,3,\ldots). \tag{3.1}$$

For sufficiency, we suppose that (3.1) is satisfied. Then, whenever $x \in L_p$ and $\zeta \in \mathcal{U}_{st}$,

$$\sum_{m,n\in\zeta} |x_{mn}| \leq \left(\sum_{m,n\in\zeta} |x_{mn}|^p\right)^{\frac{1}{p}} \left(\sum_{m,n\in\zeta} 1\right)^{\frac{1}{q}} \leq \|x\|_{L_q}(st)^{\frac{1}{q}} < K\phi_{st}\|x\|_{L_q},$$

and hence $x \in M(\phi)$. In view of (i), it follows that $L_q \subset M(\phi)$.

(iii) By Theorem 2.2, we have $\Delta\phi \in M(\phi)$. For sufficiency, we suppose that $\Delta\phi \in L_p$ and that $x \in M(\phi)$. Then $\{u_{mn}\Delta_{11}\phi_{mn}\} \in L_1$ whenever $u \in L_q$, and it therefore follows from Lemma 2.2 that $\{u_{mn}x_{mn}\} \in L_1$ whenever $u \in L_q$. Since L_p is the dual of L_q and since $M(\phi) \neq L_p$, it follows that $M(\phi) \subset L_q$.

(iv) By using (iii) we have $\bigcup_{\Delta\phi\in L_p} M_\phi \subseteq L_p$. Now, for obtaining the complementary relation $L_p \subseteq \bigcup_{\Delta\phi\in L_p} M_\phi$, let us suppose that $x \in L_p$. Then $\lim_{m,n\to\infty} x_{mn} = 0$, and hence there is an element u of $S(x)$ such that $\{|u_{mn}|\}$ is a non-increasing sequence. If we take $\psi = \{\sum_{i,j=1,1}^{m,n} |u_{ij}|\}$, then it is easy to verify that $\psi \in \Theta$ and that $x \in M(\phi)$. Since $\Delta\psi \in L_p$, the complementary relation is satisfied. □

4 Application of $M(\phi)$ and $N(\phi)$ in clustering

In this section, we implement a k-means clustering algorithm by using $M(\phi)$-distance measure. Further, we apply the k-means algorithm into clustering to cluster two-moon data. The clustering result obtained by the $M(\phi)$-distance measure is compared with the results derived by the existing Euclidean distance measures (l_2).

4.1 Algorithm to compute $M(\phi)$ distance

Let $x = [x_1, x_2, x_3, \ldots, x_n]_{1\times n}$ and $y = [y_1, y_2, y_3, \ldots, y_n]_{1\times n}$ be two matrices of size $1 \times n$, and let $\phi_{m,n} = \phi_{1,n} = n$.

(1) Calculate $a_i = \frac{1}{\phi_{1,i}}|x_i - y_i|$, $i = 1, 2, 3, \ldots, n$.
(2) The $M(\phi)$-distance between x and y is d, where

$$d = \max\{a_1, a_1 + a_2, \ldots, a_1 + a_2 + \cdots + a_n\}.$$

4.2 K-means clustering algorithm for $M(\phi)$-distance measure

Let $X = [x_1, x_2, x_3, \ldots, x_n]$ be the data set.

(1) Randomly/judiciously select k cluster centers (in this paper we choose first k data points as the cluster center $y = [x_1, x_2, \ldots, x_k]$).
(2) By using $M(\phi)$ or $N(\phi)$ distance measure (since both are dual of each other, in application point of view, we only consider $M(\phi)$), compute the distance between each data points and cluster centers.
(3) Put data points into the cluster whose $M(\phi)$-distance with its center is minimum.
(4) Define cluster centers for the new clusters evolved due to steps 1-3, the new cluster centers are computed as follows: $c_i = \frac{1}{k_i}\sum_{j=1}^{k_i} x_i$, where k_i denotes the number of points in the ith cluster.
(5) Repeat the above process until the difference between two consecutive cluster centers reaches less than a small number ε.

4.3 Two-moon dataset clustering by using $M(\phi)$-distance measure in k-means algorithm

Two-moon dataset is a well-known nonconvex data set. It is an artificially designed two dimensional dataset consisting of 373 data points [19]. Two-moon dataset is visualized as moon-shaped clusters (see Figure 1).

By using $M(\phi)$-distance measure in the k-means clustering algorithm, the obtained result is represented in Figure 2. In Figure 3, we represent the result obtained by using the Euclidean distance measure in the k-means algorithm (we measure the accuracy of the cluster

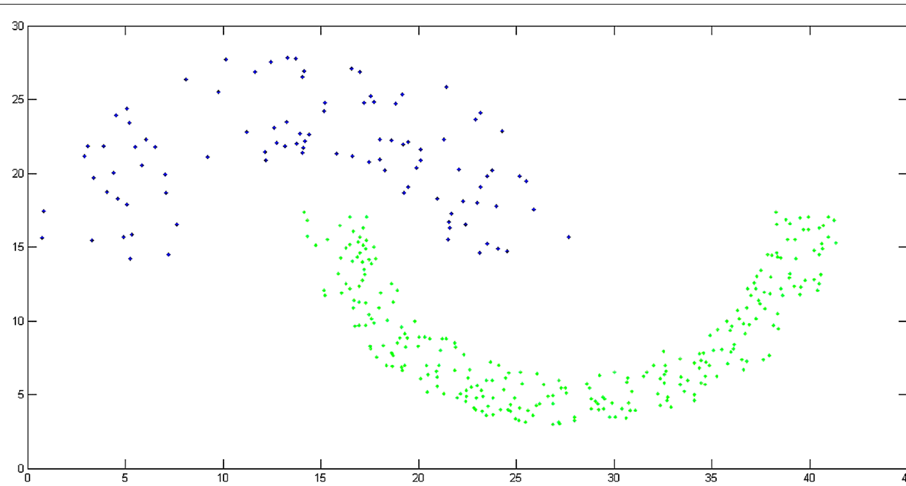

Figure 1 Original shape of two-moon dataset.

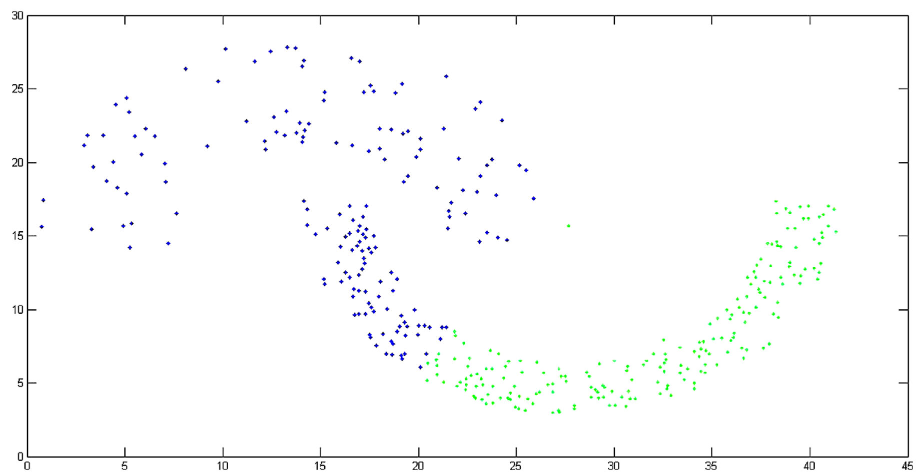

Figure 2 Clustering induced by $M(\phi)$-distance measure.

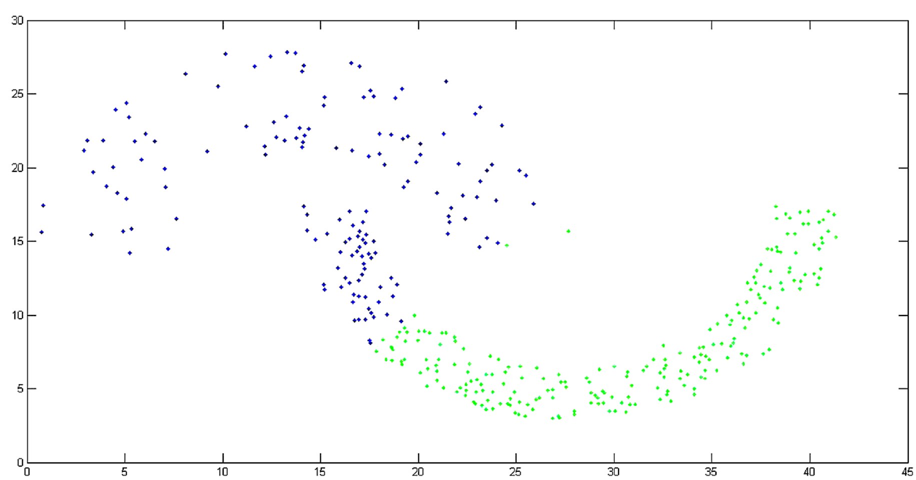

Figure 3 Clustering induced by Euclidean distance l_2.

by using the formula, accuracy = (number of data points in the right cluster/total number of data points)). The experimental result shows that cluster accuracy of $M(\phi)$-distance measure is 84.72% while l_2-distance measure's clustering accuracy is 78.55%. Thus, $M(\phi)$-distance measure substantially improves the clustering accuracy.

5 Conclusions

In this paper, we defined Banach spaces $M(\phi)$ and $N(\phi)$ with discussion of their mathematical properties. Further, we proved some of their inclusion relation. Furthermore, we applied the distance measure induced by the Banach space $M(\phi)$ into clustering to cluster the two-moon data by using the k-means clustering algorithm; the result of the experiment shows that the $M(\phi)$-distance measure extensively improves the clustering accuracy.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors of the manuscript have read and agreed to its content and are accountable for all aspects of the accuracy and integrity of the manuscript.

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