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Application of inequalities technique to dynamics analysis of a stochastic eco-epidemiology model

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Abstract

This paper formulates an infected predator-prey model with Beddington-DeAngelis functional response from a classical deterministic framework to a stochastic differential equation (SDE). First, we provide a global analysis including the global positive solution, stochastically ultimate boundedness, the persistence in mean, and extinction of the SDE system by using the technique of a series of inequalities. Second, by using Itô's formula and Lyapunov methods, we investigate the asymptotic behaviors around the equilibrium points of its deterministic system. The solution of the stochastic model has a unique stationary distribution, it also has the characteristics of ergodicity. Finally, we present a series of numerical simulations of these cases with respect to different noise disturbance coefficients to illustrate the performance of the theoretical results. The results show that if the intensity of the disturbance is sufficiently large, the persistence of the SDE model can be destroyed.

Keywords: stochastic eco-epidemiology model; Hölder inequality and Chebyshev inequality; asymptotic behavior; persistence in mean; stationary distribution

1 Introduction

Mathematical inequalities play a large role in mathematics analysis and its application. Recently, the inequality technique was applied to impulsive differential systems [1, 2] and stochastic differential systems [3–5], thus some new results were obtained.

Predation can have far-reaching effects on biological communities. Thus many scientists have studied the interaction between predator and prey [6–10]. Interaction between predator and prey is hard to avoid being influenced by some factors. One of the most common factors is the disease. Therefore, there are many scholars who have studied the infected predator-prey systems [11–17]. For instance, Hadeler and Freedman [16] considered a predator-prey system with parasitic infection. They proved the epidemic threshold theorem for where there is coexistence of the predator with the uninfected prey. Han and Ma [15] analyzed four modifications of a predator-prey model to include an SIS or SIR parasitic infection. They obtained the thresholds and global stability results of the four systems.

Species may be subject to uncertain environmental disturbances, such as fluctuations of birth rate and death rate, food, habitat and water, etc. These phenomena can be described by stochastic processes. Recently, the stochastic predator-prey systems have received much attention from scholars [18–21]. Zhang and Jiang [18] studied a stochastic three species eco-epidemiological system. They analyzed the stochastic stability and asymptotic behaviors around the equilibrium points of its deterministic model. Liu and Wang [19] considered a two-species non-autonomous predator-prey model with white noise. They obtained the sufficient criteria for extinction, non-persistence in the mean, and weak persistence in the mean.

The functional response of predator is a very important factor of predator-prey system, which reflects the average consumption rate of predator to prey. Therefore, many scholars prefer to study the predator-prey system with functional response [22–25]. For instance, Wang and Wei [22] explored a predator-prey system with strong Allee effect and an Ivlev-type functional response. Liu and Beretta [23] studied a predator-prey model with a Beddington-DeAngelis functional response. Some biologists have argued that in many instances, especially when predators have to hunt for food and, therefore, have to share or compete for food, the functional response in a prey-predator model should be predator-dependent. This view has been supported by some practical observations [26, 27]. Skalski and Gilliam [26] collected observation data from 19 predator-prey communities, they found that three predator-dependent functional responses (Crowley-Martin [28], Hassell-Varley [29] and Beddington-DeAngelis [30, 31]) were in agreement with the observation data, and in many instances, the Beddington-DeAngelis type looked better than the other two.

The Beddington-DeAngelis type functional response of per capita feeding rate can be expressed as follows:

$$F(x,y) = \frac{axy}{1 + px + qy},$$

where a (units: time^{-1}) represents the effects of capture rate on the feeding rate, p (units: prey^{-1}) denotes the effects of handling time on the feeding rate, q (units: predator^{-1}) represents the magnitude of interference among predators. Compared with the Holling II functional response, the Beddington-DeAngelis type functional response has an additional term qy in the denominator. In other words, this type of functional response is affected by both predator and prey. Therefore, the effect of mutual interference on the dynamics of population is worth studying.

To the best of our knowledge, the research on global asymptotic behaviors of a stochastic infected predator-prey system with Beddington-DeAngelis has not gone very far yet. Therefore, according to a deterministic predator-prey model, this paper investigates the stationary distribution and ergodic property of a stochastic infected predator-prey with Beddington-DeAngelis and explores the influence of white noise on the persistence in mean and extinction of the predator-prey-disease system.

First of all, a deterministic predator-prey system is described in [32] by

$$\begin{cases} \dot{X}(t) = X(t)[b - a_{11}X(t) - \frac{a_{12}S(t)}{1+pX(t)+qS(t)}], \\ \dot{S}(t) = S(t)[-c - a_{22}S(t) + \frac{a_{21}X(t)}{1+pX(t)+qS(t)} - \beta I(t)], \\ \dot{I}(t) = I(t)[-d - a_{33}I(t) + \beta S(t)], \end{cases} \tag{1}$$

where $X(t)$ is the population density of prey at time t, $S(t)$ and $I(t)$, respectively, stand for the densities of susceptible predator and infected predator at time t, b is the intrinsic growth rate of $X(t)$, c is the natural mortality rate of $S(t)$, d is the diseased death rate of $I(t)$. a_{11}, a_{22}, a_{33}, respectively, stand for the density coefficients of $X(t), S(t)$ and $I(t)$. a_{12} is the captured rate of $X(t)$, $\frac{a_{21}}{a_{12}}$ is the conversion rate from $X(t)$ to $S(t)$, β represents the infection rate from $S(t)$ to $I(t)$, $p, q > 0$ are constant coefficients.

Second, the world is full of uncertainty and random phenomena, so species in the ecosystem may be subject to different forms of random interference. In this paper, we assume that the disturbance in the environment affects not only the rate of predation but also the infection rate of the disease, so that

$$a_{12} \to a_{12} + \sigma_{12}\dot{B}_1, \qquad a_{21} \to a_{21} + \sigma_{21}\dot{B}_1, \qquad \beta \to \beta + \sigma\dot{B}_2,$$

where $B_1(t)$ and $B_2(t)$ are standard Brownian motions, $\sigma_{12}^2, \sigma_{21}^2$, and σ^2 are the intensities of the Brownian motions.

Taking into account the effects of random interference gives

$$\begin{cases} dX(t) = X(t)[b - a_{11}X(t) - \frac{a_{12}S(t)}{1+pX(t)+qS(t)}]\,dt - \frac{\sigma_{12}S(t)X(t)}{1+pX(t)+qS(t)}\,dB_1(t), \\ dS(t) = S(t)[-c - a_{22}S(t) + \frac{a_{21}X(t)}{1+pX(t)+qS(t)} - \beta I(t)]\,dt \\ \qquad + \frac{\sigma_{21}S(t)X(t)}{1+pX(t)+qS(t)}\,dB_1(t) - \sigma S(t)I(t)\,dB_2(t), \\ dI(t) = I(t)[-d - a_{33}I(t) + \beta S(t)]\,dt + \sigma S(t)I(t)\,dB_2(t). \end{cases} \tag{2}$$

The rest of this paper is organized as follows. In the next section, we consider the existence of a global positive solution and the stochastically ultimate boundedness of model (2). In Section 3, we study the global asymptotic behaviors of model (2) around the equilibrium points of its deterministic system. In addition, we explore the stationary distribution and ergodic property of model (2). In Section 4, we obtain the conditions for the persistence in mean and extinction of model (2). In the last section, we summarize our main results and give some numerical simulations.

Throughout this paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathcal{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (*i.e.* it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathcal{P}$-null sets). The function $B_i(t)$ ($i = 1, 2$) is a Brownian motion defined on the complete probability space Ω. For an integrable function $X(t)$ on $[0, +\infty)$, we define $\langle X(t)\rangle = \frac{1}{t}\int_0^t X(s)\,ds$, $\langle X(t)\rangle_* = \liminf_{t\to+\infty}\langle X(t)\rangle$, $\langle X(t)\rangle^* = \limsup_{t\to+\infty}\langle X(t)\rangle$.

2 Global positive solution and stochastically ultimate boundedness

2.1 Global positive solution

Due to the physical meaning, variables $S(t)$, $I(t)$, and $Y(t)$ in model (2) should remain nonnegative for $t \geq 0$. We next prove that this is actually the case and, furthermore, the positive solution is unique.

Lemma 2.1 *For any initial value* $(X(0), S(0), I(0)) \in R_+^3$, *model* (2) *has a local unique positive solution* $(X(t), S(t), I(t))$ *on* $t \in [0, \tau_e)$, *where* τ_e *is the explosion time*.

Theorem 2.1 *For any initial value* $(X(0), S(0), I(0)) \in R_+^3$, *model* (2) *has a unique positive solution* $(X(t), S(t), I(t)) \in R_+^3$ *on* $t \geq 0$ *with probability* 1.

Proof By Lemma 2.1, we only need to prove that $\tau_e = \infty$ a.s. To this end, let $k_0 > 0$ be a sufficiently large constant such that $X(0), S(0)$ and $I(0)$ all lie in $[\frac{1}{k_0}, k_0]$. For each $k \geq k_0$ ($k \in N_+$), define the stopping time

$$\tau_k = \inf\left\{t \in [0, \tau_e] : X(t) \notin \left(\frac{1}{k_0}, k_0\right), S(t) \notin \left(\frac{1}{k_0}, k_0\right) \text{ or } I(t) \notin \left(\frac{1}{k_0}, k_0\right)\right\}.$$

As is easy to see, τ_k is a monotonically increasing function when $k \to \infty$. Let $\tau_\infty = \lim_{k\to\infty}\tau_k$, thus $\tau_\infty \leq \tau_e$ a.s. Now we need to prove $\tau_\infty = \infty$ a.s., otherwise, there exist two constants $T > 0$ and $\epsilon \in (0,1)$ such that $P\{\tau_\infty \leq T\} > \epsilon$. Thus, there is an integer $k_1 \geq k_0$ such that

$$P\{\tau_\infty \leq T\} > \epsilon, \quad k \geq k_1. \tag{3}$$

Define a C^3-function $V : R_+^3 \to R_+$,

$$V(X,S,I) = X - 1 - \ln X + S - 1 - \ln S + I - 1 - \ln I.$$

The non-negativity of the function $V(X,S,I)$ can be seen by $u - 1 - \ln u \geq 0, u > 0$.

Applying Itô's formula to the stochastic differential system (2) yields

$$\begin{aligned} dV &= LV\,dt - \frac{\sigma_{12}(X-1)S}{1+pX+qS}\,dB_1(t) + \frac{\sigma_{21}(S-1)X}{1+pX+qS}\,dB_1(t) \\ &\quad - \sigma(S-1)I\,dB_2(t) + \sigma(I-1)S\,dB_2(t), \end{aligned}$$

where

$$\begin{aligned} LV &= (X-1)\left(b - a_{11}X - \frac{a_{12}S}{1+pX+qS}\right) + \frac{\sigma_{12}^2 S^2}{2(1+pX+qS)^2} \\ &\quad + (S-1)\left(-c - a_{22}S + \frac{a_{21}X}{1+pX+qS} - \beta I\right) + \frac{\sigma_{21}^2 X^2}{2(1+pX+qS)^2} \\ &\quad + (I-1)(-d - a_{33}I + \beta S) + \frac{1}{2}\sigma^2 S^2 + \frac{1}{2}\sigma^2 I^2 \\ &= bX - a_{11}X^2 - \frac{a_{12}SX}{1+pX+qS} - b + a_{11}X + \frac{a_{12}S}{1+pX+qS} + \frac{\sigma_{12}^2 S^2}{2(1+pX+qS)^2} \\ &\quad - cS - a_{22}S^2 + \frac{a_{21}XS}{1+pX+qS} + c + a_{22}S - \frac{a_{21}X}{1+pX+qS} + \beta I \\ &\quad + \frac{\sigma_{21}^2 X^2}{2(1+pX+qS)^2} + \frac{1}{2}\sigma^2 I^2 - dI - a_{33}I^2 + d + a_{33}I - \beta S + \frac{1}{2}\sigma^2 S^2 \\ &\leq -\left[a_{11}X^2 - (b + a_{11})X - \frac{a_{12}}{q} - \frac{\sigma_{12}^2}{2q^2}\right] - \left[a_{22}S^2 - \left(a_{22} + \frac{a_{21}}{p}\right)S - c - \frac{\sigma_{21}^2}{2p^2}\right] \\ &\quad - \left[a_{33}I^2 - (\beta + a_{33})I - d\right] + \frac{1}{2}\sigma^2\left(S^2 + I^2\right). \end{aligned}$$

Since

$$\begin{aligned}
&\frac{d(\frac{\sigma_{21}}{\sigma_{12}}X+S+I)}{dt}+\left(\frac{\sigma_{21}}{\sigma_{12}}X+S+I\right)\\
&\quad=\frac{\sigma_{21}}{\sigma_{12}}X\left(1+b-a_{11}X-\frac{a_{12}S}{1+pX+qS}\right)+S\left(1-c-a_{22}S+\frac{a_{21}X}{1+pX+qS}-\beta I\right)\\
&\qquad+I[1-d-a_{33}I+\beta S]\\
&\quad\le-\frac{\sigma_{21}}{\sigma_{12}}\bigl[a_{11}X^2-(b+1)X\bigr]-\left[a_{22}S^2-\left(1+\frac{a_{21}}{p}\right)S\right]-\bigl(a_{33}I^2-I\bigr)\\
&\quad\le 3\cdot\max\left\{\frac{\sigma_{21}(b+1)^2}{4\sigma_{12}a_{11}},\frac{(1+\frac{a_{21}}{p})^2}{4a_{22}},\frac{1}{4a_{33}}\right\}\\
&\quad\le C_0,
\end{aligned}$$

where C_0 is a positive constant.

Then we have

$$\begin{aligned}
\frac{\sigma_{21}}{\sigma_{12}}X(t)+S(t)+I(t)&\le\left(\frac{\sigma_{21}}{\sigma_{12}}X(0)+S(0)+I(0)\right)e^{-t}+C_0\bigl(1-e^{-t}\bigr)\\
&\le e^{-t}\left(\frac{\sigma_{21}}{\sigma_{12}}(0)+S(0)+I(0)+C_0\bigl(e^t-1\bigr)\right)\\
&\le\max\left\{\frac{\sigma_{21}}{\sigma_{12}}X(0)+S(0)+I(0),C_0\right\}
\end{aligned}$$

and

$$\limsup_{t\to\infty}\left(\frac{\sigma_{21}}{\sigma_{12}}X(t)+S(t)+I(t)\right)\le C_0. \tag{4}$$

Therefore, we have

$$\begin{aligned}
LV&\le-\left[a_{11}X^2-(b+a_{11})X-\frac{a_{12}}{q}-\frac{\sigma_{12}^2}{2q^2}\right]-\left[a_{22}S^2-\left(a_{22}+\frac{a_{21}}{p}\right)S-c-\frac{\sigma_{21}^2}{2p^2}\right]\\
&\quad-\bigl[a_{33}I^2-(\beta+a_{33})I-d\bigr]+\sigma^2C_0^2\\
&\le K_0,
\end{aligned}$$

where K_0 is a positive constant.

So we have

$$\begin{aligned}
dV&\le K_0\,dt-\frac{\sigma_{12}(X-1)S}{1+pX+qS}\,dB_1(t)+\frac{\sigma_{21}(S-1)X}{1+pX+qS}\,dB_1(t)\\
&\quad-\sigma(S-1)I\,dB_2(t)+\sigma(I-1)S\,dB_2(t).
\end{aligned}\tag{5}$$

Integrating (5) from 0 to $\tau_k\wedge T$ and taking expectation on both sides, we have

$$EV\bigl(X(\tau_k\wedge T),S(\tau_k\wedge T),I(\tau_k\wedge T)\bigr)\le V\bigl(X(0),S(0),I(0)\bigr)+K_0T. \tag{6}$$

Let $\Omega_k = \{\tau_k \le T\}$, from inequality (3) we can see that $P(\Omega_k) \ge \epsilon$. Note that, for every $\omega \in \Omega_k$, there exists at least one of $X(\tau_k,\omega), S(\tau_k,\omega), I(\tau_k,\omega)$ that equals either k or $\frac{1}{k}$. As a result, we have

$$V\bigl(X(\tau_k \wedge T), S(\tau_k \wedge T), I(\tau_k \wedge T)\bigr) \ge (k - 1 - \ln k) \wedge \left(\frac{1}{k} - 1 - \ln\frac{1}{k}\right). \tag{7}$$

Applying equation (6) and equation (7), we get

$$\begin{aligned} &V\bigl(X(0), S(0), I(0)\bigr) + K_0 T \\ &\quad\ge E\bigl[1\Omega_k(\omega) V\bigl(X(\tau_k \wedge T), S(\tau_k \wedge T), I(\tau_k \wedge T)\bigr)\bigr] \\ &\quad\ge \epsilon(k - 1 - \ln k) \wedge \left(\frac{1}{k} - 1 - \ln\frac{1}{k}\right), \end{aligned}$$

where $1\Omega_k$ is the indicator function of Ω_k.

When $k \to \infty$, we have

$$\infty > V\bigl(X(0), S(0), I(0)\bigr) + K_0 T = \infty.$$

This is a contradiction. So $\tau_\infty = \infty$. □

2.2 Stochastically ultimate boundedness

Theorem 2.1 shows that R_+^3 is the positive invariant set of model (2). Now we prove the stochastically ultimate boundedness of model (2).

Definition 2.1 Let $(X(t), S(t), I(t))$ be the solution of model (2) with initial value $(X(0), S(0), I(0)) \in R_+^3$. If, for any $\varepsilon \in (0,1)$, there exists a $\chi(= \chi(\omega)) > 0$ such that the solution of model (2) satisfies

$$\limsup_{t\to\infty} P\bigl\{\bigl|\bigl(X(t), S(t), I(t)\bigr)\bigr| > \chi\bigr\} < \varepsilon,$$

then model (2) has stochastically ultimate boundedness.

Lemma 2.2 *The following elementary inequality will be used frequently in the sequel.*

(1) $x^r \le 1 + r(x-1), x \ge 0, 1 \ge r \ge 0$,

(2) $n^{(1-p/2)\wedge 0}|x|^p \le \sum_{i=1}^n x_i^p \le n^{(1-p/2)\vee 0}|x|^p$,

where $R_+^n := \{x \in R^n : x_i > 0, 1 \le i \le n\}, n \in R_+, p > 0$.

Theorem 2.2 *Let* $(X(t), S(t), I(t))$ *be the solution of model* (2) *with initial value* $(X(0), S(0), I(0)) \in R_+^3$, *then* $(X(t), S(t), I(t))$ *is stochastically ultimate boundedness.*

Proof Define

$$V(X, S, I) = X^{\frac{1}{2}} + S^{\frac{1}{2}} + I^{\frac{1}{2}}, \quad \bigl(X(t), S(t), I(t)\bigr) \in R_+^3.$$

Applying Itô's formula to stochastic differential system (2) yields

$$\begin{aligned} dV &= LV\,dt - \frac{\sigma_{12}X^{\frac{1}{2}}S}{2(1+pX+qS)}\,dB_1(t) + \frac{\sigma_{21}S^{\frac{1}{2}}X}{2(1+pX+qS)}\,dB_1(t) \\ &\quad - \frac{1}{2}\sigma S^{\frac{1}{2}}I\,dB_2(t) + \frac{1}{2}\sigma I^{\frac{1}{2}}S\,dB_2(t), \end{aligned}$$

where

$$\begin{aligned} LV &= \frac{1}{2}X^{\frac{1}{2}}\left(b - a_{11}X - \frac{a_{12}S}{1+pX+qS}\right) - \frac{\sigma_{12}^2X^{\frac{1}{2}}S}{8(1+pX+qS)^2} \\ &\quad + \frac{1}{2}S^{\frac{1}{2}}\left(-c - a_{22}S + \frac{a_{21}X}{1+pX+qS} - \beta I\right) - \frac{\sigma_{21}^2S^{\frac{1}{2}}X}{8(1+pX+qS)} \\ &\quad + \frac{1}{2}I^{\frac{1}{2}}(-d - a_{33}I + \beta S) - \frac{1}{8}\sigma^2S^{\frac{1}{2}}I - \frac{1}{8}\sigma^2I^{\frac{1}{2}}S \\ &\le -\frac{1}{2}a_{11}X^{\frac{3}{2}} + \frac{1}{2}bX^{\frac{1}{2}} - \frac{1}{2}a_{22}S^{\frac{3}{2}} + \frac{a_{21}}{2p}S^{\frac{1}{2}} - \frac{1}{2}a_{33}I^{\frac{3}{2}} + \frac{1}{2}\beta I^{\frac{1}{2}}S. \end{aligned}$$

Applying the Hölder inequality $ab \le \frac{a^p}{p} + \frac{b^q}{q}$, $\frac{1}{p} + \frac{1}{q} = 1$ $(p,q>1)$, we have

$$I^{\frac{1}{2}}S \le \frac{1}{3}I^{\frac{3}{2}} + \frac{2}{3}S^{\frac{3}{2}}.$$

Therefore,

$$\begin{aligned} LV &\le -\frac{1}{2}a_{11}X^{\frac{3}{2}} + \frac{1}{2}(b+2)X^{\frac{1}{2}} - \frac{1}{2}\left(a_{22} - \frac{2}{3}\beta\right)S^{\frac{3}{2}} + \frac{1}{2}\left(\frac{a_{21}}{p} + 2\right)S^{\frac{1}{2}} \\ &\quad - \frac{1}{2}\left(a_{33} - \frac{1}{3}\beta\right)I^{\frac{3}{2}} + I^{\frac{1}{2}} - \left(X^{\frac{1}{2}} + S^{\frac{1}{2}} + I^{\frac{1}{2}}\right) \\ &\le H_0 - V(X,S,I), \end{aligned}$$

where $H_0 > 0$ is a positive constant.

Thus

$$\begin{aligned} dV &\le \left[H_0 - V(X,S,I)\right]dt - \frac{\sigma_{12}X^{\frac{1}{2}}S}{2(1+pX+qS)}\,dB_1(t) + \frac{\sigma_{21}S^{\frac{1}{2}}X}{2(1+pX+qS)}\,dB_1(t) \\ &\quad - \frac{1}{2}\sigma S^{\frac{1}{2}}I\,dB_2(t) + \frac{1}{2}\sigma I^{\frac{1}{2}}S\,dB_2(t). \end{aligned}$$

Applying Itô's formula to $e^tV(X,S,I)$ yields

$$\begin{aligned} d\left(e^tV(X,S,I)\right) &= e^t\left[V(X,S,I)\,dt + dV(X,S,I)\right] \\ &\le e^tH_0\,dt + e^t\left[-\frac{\sigma_{12}X^{\frac{1}{2}}S}{2(1+pX+qS)}\,dB_1(t) + \frac{\sigma_{21}S^{\frac{1}{2}}X}{2(1+pX+qS)}\,dB_1(t) \right. \\ &\quad \left. - \frac{1}{2}\sigma S^{\frac{1}{2}}I\,dB_2(t) + \frac{1}{2}\sigma I^{\frac{1}{2}}S\,dB_2(t)\right]. \end{aligned}$$

So we have

$$e^t EV(X,S,I) \le V\bigl(X(0),S(0),I(0)\bigr) + H_0\bigl(e^t - 1\bigr)$$

and

$$\limsup_{t\to+\infty} EV(X,S,I) \le H_0.$$

Applying the second inequality of Lemma 2.2 and letting $n = 3, p = \frac{1}{2}$, we have

$$3^{\frac{3}{4}\wedge 0}\bigl|\bigl(X(t),S(t),I(t)\bigr)\bigr|^{\frac{1}{2}} \le V(X,S,I).$$

Thus, we obtain

$$\limsup_{t\to+\infty} E\bigl|\bigl(X(t),S(t),I(t)\bigr)\bigr|^{\frac{1}{2}} \le H.$$

Therefore, for any $\varepsilon > 0$, set $\chi = \frac{H^2}{\varepsilon^2}$, applying the Chebyshev inequality, we have

$$P\bigl\{\bigl|\bigl(X(t),S(t),I(t)\bigr)\bigr| > \chi\bigr\} \le \frac{E|(X(t),S(t),I(t))|^{\frac{1}{2}}}{\sqrt{\chi}},$$

that is,

$$\limsup_{t\to\infty} P\bigl\{\bigl|\bigl(X(t),S(t),I(t)\bigr)\bigr| > \chi\bigr\} \le \varepsilon.$$ □

3 Asymptotic behaviors

System (1) has three equilibrium points [32]: (i) when $R_0 = \frac{a_{21}b}{c(a_{11}+pb)} < 1$, system (1) has an equilibrium point $E_1(K,0,0)$; (ii) when $R_0 = \frac{a_{21}b}{c(a_{11}+pb)} > 1$ and $R_1 = \frac{a_{21}b}{(c+\frac{da_{22}}{\beta})(a_{11}+pb+\frac{qda_{11}}{\beta})} < 1$, system (1) has another disease free equilibrium point $E_2(\overline{X},\overline{S},0)$; (iii) when $R_1 = \frac{a_{21}b}{(c+\frac{da_{22}}{\beta})(a_{11}+pb+\frac{qda_{11}}{\beta})} > 1$, system (1) has a positive equilibrium point $E_3(X^*,S^*,I^*)$. For its stochastic system (2), however, these equilibrium points do not exist.

In this section, we study the asymptotic behaviors of model (2) around the three equilibrium points $E_1(K,0,0)$, $E_2(\overline{X},\overline{S},0)$, and $E_3(X^*,S^*,I^*)$ of its deterministic model (1), respectively.

3.1 Asymptotic behaviors around the equilibrium point E_1 of system (1)

When $R_0 < 1$, system (1) has an equilibrium point $E_1(K,0,0) = (\frac{b}{a_{11}},0,0)$, but it is not the equilibrium point of system (2). In this subsection, we study the asymptotic behaviors of system (2) around $E_1(K,0,0)$.

Theorem 3.1 *Let $(X(t),S(t),I(t))$ be the solution of model* (2) *with initial value* $(X(0),S(0),I(0)) \in R^3_+$. *If $R_0 < 1$ and $K = \frac{b}{a_{11}} \le \frac{c}{a_{21}}$, then*

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \bigl[\bigl(X(\theta)-K\bigr)^2 + S(\theta)^2 + I^2(\theta)\bigr]\,d\theta \le \frac{\sigma_{12}^2 K}{2q^2 W_1},$$

where $W_1 = \min\{a_{11}, \frac{a_{12}a_{22}}{a_{21}}, \frac{a_{12}a_{33}}{a_{21}}\}$.

Proof Note that $(K,0,0)$ is the equilibrium point of system (1), where $K=\frac{b}{a_{11}}$.

Define

$$V(X,S,I)=\left(X-K-K\ln\frac{X}{K}\right)+\frac{a_{12}}{a_{21}}(S+I).$$

Applying Itô's formula to stochastic differential system (2) yields

$$dV=LV\,dt-\frac{\sigma_{12}(X-K)S}{1+pX+qS}\,dB_1(t)+\frac{a_{12}\sigma_{21}SX}{a_{21}(1+pX+qS)}\,dB_1(t), \tag{8}$$

where

$$\begin{aligned}LV={}&(X-K)\left[b-a_{11}X-\frac{a_{12}S}{1+pX+qS}\right]+\frac{\sigma_{12}^2KS^2}{2(1+pX+qS)^2}\\&+\frac{a_{12}}{a_{21}}\left[S\left(-c-a_{22}S+\frac{a_{21}X}{1+pX+qS}-\beta I\right)-I(d+a_{33}I-\beta S)\right]\\={}&(X-K)\left[b-a_{11}(X-K)-a_{11}K-\frac{a_{12}S}{1+pX+qS}\right]+\frac{\sigma_{12}^2KS^2}{2(1+pX+qS)^2}\\&+\frac{a_{12}}{a_{21}}\left[S\left(-c-a_{22}S+\frac{a_{21}X}{1+pX+qS}-\beta I\right)-I(d+a_{33}I-\beta S)\right]\\\le{}&-a_{11}(X-K)^2+\frac{a_{12}KS}{1+pX+qS}+\frac{\sigma_{12}^2KS^2}{2(1+pX+qS)^2}-\frac{a_{12}}{a_{21}}\left(cS+a_{22}S^2+a_{33}I^2\right)\\\le{}&-a_{11}(X-K)^2+a_{12}\left(K-\frac{c}{a_{21}}\right)S+\frac{\sigma_{12}^2K}{2q^2}-\frac{a_{12}a_{22}}{a_{21}}S^2-\frac{a_{12}a_{33}}{a_{21}}I^2\\\le{}&-a_{11}(X-K)^2-\frac{a_{12}a_{22}}{a_{21}}S^2-\frac{a_{12}a_{33}}{a_{21}}I^2+\frac{\sigma_{12}^2K}{2q^2}.\end{aligned}$$

Integrating equation (8) from 0 to t, we obtain

$$\begin{aligned}V(t)-V(0)\le{}&-\int_0^t a_{11}\bigl(X(\theta)-K\bigr)^2\,d\theta-\frac{a_{12}a_{22}}{a_{21}}\int_0^t S^2(\theta)\,d\theta\\&-\frac{a_{12}a_{33}}{a_{21}}\int_0^t I^2(\theta)\,d\theta+\frac{\sigma_{12}^2K}{2q^2}t+M_1(t),\end{aligned} \tag{9}$$

where

$$M_1(t)=\int_0^t\left[-\frac{\sigma_{12}(X(\theta)-K)S(\theta)}{1+pX(\theta)+qS(\theta)}+\frac{a_{12}\sigma_{21}S(\theta)X(\theta)}{a_{21}(1+pX(\theta)+qS(\theta))}\right]dB_1(\theta)$$

is a real-valued continuous local martingale.

Thus

$$\begin{aligned}\limsup_{t\to+\infty}\frac{\langle M_1,M_1\rangle_t}{t}&=\limsup_{t\to+\infty}\frac{1}{t}\int_0^t\left[-\frac{\sigma_{12}(X(\theta)-K)S(\theta)}{1+pX(\theta)+qS(\theta)}+\frac{a_{12}\sigma_{21}S(\theta)X(\theta)}{a_{21}(1+pX(\theta)+qS(\theta))}\right]^2d\theta\\&\le 2C^2\left[\frac{\sigma_{12}^2}{q^2}+\frac{a_{12}^2\sigma_{21}^2}{a_{21}^2q^2}\right]<+\infty.\end{aligned}$$

Applying the strong law of large numbers, we obtain $\lim_{t\to+\infty}\frac{M_1(t)}{t}=0$.

Dividing equation (9) by t and taking the limit superior, we have

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t\left[a_{11}\big(X(\theta)-K\big)^2+\frac{a_{12}a_{22}}{a_{21}}S^2(\theta)+\frac{a_{12}a_{33}}{a_{21}}I^2(\theta)\right]d\theta\le\frac{\sigma_{12}^2K}{2q^2},$$

thus

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t\left[a_{11}\big(X(\theta)-K\big)^2+S^2(\theta)+I^2(\theta)\right]d\theta\le\frac{\sigma_{12}^2K}{2q^2W_1}. \qquad \square$$

Corollary 3.1 *From Theorem* 3.1, *when* $\sigma_{12}=0$, *we have*

$$LV\le -a_{11}(X-K)^2-\frac{a_{12}a_{22}}{a_{21}}S^2-\frac{a_{12}a_{33}}{a_{21}}I^2\le 0,$$

thus when $R_0<1$ *and* $K=\frac{b}{a_{11}}\le\frac{c}{a_{21}}$ *hold, the equilibrium point* $E_1(K,0,0)$ *of system* (1) *is globally asymptotically stable.*

Remark 3.1 From Theorem 3.1, if the interference intensity is sufficiently small, the solution of model (2) will fluctuate around the equilibrium point $E_1(K,0,0)$. Moreover, the fluctuation intensity is related with the disturbance intensity: the fluctuation intensity is positively correlated with the value of σ_{12}.

3.2 Asymptotic behaviors around the equilibrium point E_2 of system (1)

When $R_0>1$ and $R_1<1$, system (1) has an equilibrium point $E_2(\overline{X},\overline{S},0)$, but it is not the equilibrium point of system (2). In this subsection, we study the asymptotic behaviors of system (2) around $E_2(\overline{X},\overline{S},0)$.

Theorem 3.2 *Let* $(X(t),S(t),I(t))$ *be the solution of model* (2) *with initial value* $(X(0),S(0),I(0))\in R_+^3$. *If* $R_0>1, R_1<1$ *and* $a_{11}q>a_{22}p$, *then we have*

$$\limsup_{t\to+\infty}\frac{1}{t}\int_0^t\left[\big(X(\theta)-\overline{X}\big)^2+\big(S(\theta)-\overline{S}\big)^2+I^2(\theta)\right]d\theta\le\frac{U_2}{W_2},$$

where

$$U_2=\frac{\sigma_{12}^2\overline{X}}{2q^2}+\frac{a_{12}(1+p\overline{X})}{a_{21}(1+q\overline{S})}\left(\frac{\sigma_{21}^2\overline{S}}{2p^2}+\frac{\sigma^2\overline{S}}{2}C_0^2\right)$$

and

$$W_2=\min\left\{a_{11}-\frac{a_{12}p}{q},\frac{a_{12}a_{22}(1+p\overline{X})}{a_{21}(1+q\overline{S})},\frac{a_{12}a_{33}(1+p\overline{X})}{a_{21}(1+q\overline{S})}\right\}.$$

Proof Noting that $(\overline{X},\overline{S},0)$ is the equilibrium point of system (1), thus

$$b-a_{11}\overline{X}-\frac{a_{12}\overline{S}}{1+p\overline{X}+q\overline{S}}=0,\qquad c+a_{22}\overline{S}-\frac{a_{21}\overline{X}}{1+p\overline{X}+q\overline{S}}=0.$$

Define

$$\begin{aligned} V(X,S,I) &= \left(X-\overline{X}-\overline{X}\ln\frac{X}{\overline{X}}\right)+\frac{a_{12}(1+p\overline{X})}{a_{21}(1+q\overline{S})}\left(S-\overline{S}-\overline{S}\ln\frac{S}{\overline{S}}\right)+\frac{a_{12}(1+p\overline{X})}{a_{21}(1+q\overline{S})}I \\ &:= V_1+\frac{a_{12}(1+p\overline{X})}{a_{21}(1+q\overline{S})}V_2+\frac{a_{12}(1+p\overline{X})}{a_{21}(1+q\overline{S})}V_3. \end{aligned}$$

Applying Itô's formula to stochastic differential system (2) yields

$$dV_1 = LV_1\,dt-\frac{\sigma_{11}(X-\overline{X})S}{1+pX+qS}\,dB_1(t),$$

where

$$\begin{aligned} LV_1 &= (X-\overline{X})\left[b-a_{11}X-\frac{a_{12}S}{1+pX+qS}\right]+\frac{\sigma_{12}^2\overline{X}S^2}{2(1+pX+qS)^2} \\ &= (X-\overline{X})\left[b-a_{11}(X-\overline{X})-a_{11}\overline{X}-\frac{a_{12}S}{1+pX+qS}\right]+\frac{\sigma_{12}^2\overline{X}S^2}{2(1+pX+qS)^2} \\ &= (X-\overline{X})\left[-a_{11}(X-\overline{X})+a_{12}\frac{p\overline{S}(X-\overline{X})-(S-\overline{S})(1+p\overline{X})}{(1+p\overline{X}+q\overline{S})(1+pX+qS)}\right]+\frac{\sigma_{12}^2\overline{X}S^2}{2(1+pX+qS)^2} \\ &= -a_{11}(X-\overline{X})^2+\frac{a_{12}p\overline{S}(X-\overline{X})^2}{(1+p\overline{X}+q\overline{S})(1+pX+qS)}-\frac{a_{12}(1+p\overline{X})(S-\overline{S})(X-\overline{X})}{(1+p\overline{X}+q\overline{S})(1+pX+qS)} \\ &\quad +\frac{\sigma_{12}^2\overline{X}S^2}{2(1+pX+qS)^2}. \end{aligned}$$

Similarly,

$$dV_2 = LV_2\,dt+\frac{\sigma_{21}(S-\overline{S})X}{1+pX+qS}\,dB_1(t)-\sigma(S-\overline{S})I\,dB_2(t),$$

where

$$\begin{aligned} LV_2 &= (S-\overline{S})\left[-c-a_{22}S+\frac{a_{21}X}{1+pX+qS}-\beta I\right]+\frac{\sigma_{21}^2\overline{S}X^2}{2(1+pX+qS)^2}+\frac{\sigma^2\overline{S}}{2}I^2 \\ &= (S-\overline{S})\left[-c-a_{22}(S-\overline{S})-a_{22}\overline{S}+\frac{a_{21}X}{1+pX+qS}-\beta I\right]+\frac{\sigma_{21}^2\overline{S}X^2}{2(1+pX+qS)^2}+\frac{\sigma^2\overline{S}}{2}I^2 \\ &= (S-\overline{S})\left[-a_{22}(S-\overline{S})+a_{21}\frac{(X-\overline{X})(1+q\overline{S})-q\overline{X}(S-\overline{S})}{(1+p\overline{X}+q\overline{S})(1+pX+qS)}-\beta I\right] \\ &\quad +\frac{\sigma_{21}^2\overline{S}X^2}{2(1+pX+qS)^2}+\frac{\sigma^2\overline{S}}{2}I^2 \\ &= -a_{22}(S-\overline{S})^2+a_{21}\frac{(1+q\overline{S})(X-\overline{X})(S-\overline{S})}{(1+p\overline{X}+q\overline{S})(1+pX+qS)}-a_{21}\frac{q\overline{X}(S-\overline{S})^2}{(1+p\overline{X}+q\overline{S})(1+pX+qS)} \\ &\quad -\beta I(S-\overline{S})+\frac{\sigma_{21}^2\overline{S}X^2}{2(1+pX+qS)^2}+\frac{\sigma^2\overline{S}}{2}I^2. \end{aligned}$$

Also, we have

$$dV_3 = I(t)[-d - a_{33}I + \beta S]\,dt + \sigma SI\,dB_2(t).$$

Hence

$$\begin{aligned} dV = {} & LV\,dt - \frac{\sigma_{12}(X-\overline{X})S}{1+pX+qS}\,dB_1(t) + \frac{a_{12}(1+p\overline{X})}{a_{21}(1+q\overline{S})}\left[\frac{\sigma_{21}(S-\overline{S})X}{1+pX+qS}\,dB_1(t)\right. \\ & \left. - \sigma(S-\overline{S})I\,dB_2(t) + \sigma SI\,dB_2(t)\right], \end{aligned} \tag{10}$$

where

$$\begin{aligned} LV = {} & LV_1 + \frac{a_{12}(1+p\overline{X})}{a_{21}(1+q\overline{S})}LV_2 + \frac{a_{12}(1+p\overline{X})}{a_{21}(1+q\overline{S})}LV_3 \\ = {} & -a_{11}(X-\overline{X})^2 + \frac{a_{12}p\overline{S}(X-\overline{X})^2}{(1+p\overline{X}+q\overline{S})(1+pX+qS)} - \frac{a_{12}(1+p\overline{X})(S-\overline{S})(X-\overline{X})}{(1+p\overline{X}+q\overline{S})(1+pX+qS)} \\ & + \frac{\sigma_{12}^2\overline{X}S^2}{2(1+pX+qS)^2} + \frac{a_{12}(1+p\overline{X})}{a_{21}(1+q\overline{S})}\left[-a_{22}(S-\overline{S})^2 + a_{21}\frac{(1+q\overline{S})(X-\overline{X})(S-\overline{S})}{(1+p\overline{X}+q\overline{S})(1+pX+qS)}\right. \\ & - a_{21}\frac{q\overline{X}(S-\overline{S})^2}{(1+p\overline{X}+q\overline{S})(1+pX+qS)} - \beta I(S-\overline{S}) + \frac{\sigma_{21}^2\overline{S}X^2}{2(1+pX+qS)^2} + \frac{\sigma^2\overline{S}}{2}I^2 \\ & \left. + I(-d - a_{33}I + \beta S)\right] \\ \leq {} & -\left(a_{11} - \frac{a_{12}p}{q}\right)(X-\overline{X})^2 - \frac{a_{12}a_{22}(1+p\overline{X})}{a_{21}(1+q\overline{S})}(S-\overline{S})^2 - \frac{a_{12}a_{33}(1+p\overline{X})}{a_{21}(1+q\overline{S})}I^2 \\ & + \frac{a_{12}(1+p\overline{X})}{a_{21}(1+q\overline{S})}(\beta\overline{S} - d)I + \frac{\sigma_{12}^2\overline{X}S^2}{2(1+pX+qS)^2} \\ & + \frac{a_{12}(1+p\overline{X})}{a_{21}(1+q\overline{S})}\left(\frac{\sigma_{21}^2\overline{S}X^2}{2(1+pX+qS)^2} + \frac{\sigma^2\overline{S}}{2}I^2\right). \end{aligned}$$

Since $\beta\overline{S} < d$, thus

$$\begin{aligned} LV \leq {} & -\left(a_{11} - \frac{a_{12}p}{q}\right)(X-\overline{X})^2 - \frac{a_{12}a_{22}(1+p\overline{X})}{a_{21}(1+q\overline{S})}(S-\overline{S})^2 - \frac{a_{12}a_{33}(1+p\overline{X})}{a_{21}(1+q\overline{S})}I^2 \\ & + \frac{\sigma_{11}^2\overline{X}}{2q^2} + \frac{a_{12}(1+p\overline{X})}{a_{21}(1+q\overline{S})}\left(\frac{\sigma_{12}^2\overline{S}}{2p^2} + \frac{\sigma_2^2\overline{S}}{2}C_0^2\right). \end{aligned}$$

Integrating both sides of equation (10) from 0 to t yields

$$\begin{aligned} V(t) - V(0) \leq {} & \int_0^t \left[-\left(a_{11} - \frac{a_{12}p}{q}\right)\bigl(X(\theta)-\overline{X}\bigr)^2\right. \\ & \left. - \frac{a_{12}a_{22}(1+p\overline{X})}{a_{21}(1+q\overline{S})}\bigl(S(\theta)-\overline{S}\bigr)^2 - \frac{a_{12}a_{33}(1+p\overline{X})}{a_{21}(1+q\overline{S})}I^2(\theta)\right]d\theta \\ & + \left[\frac{\sigma_{12}^2\overline{X}}{2q^2} + \frac{a_{12}(1+p\overline{X})}{a_{21}(1+q\overline{S})}\left(\frac{\sigma_{21}^2\overline{S}}{2p^2} + \frac{\sigma^2\overline{S}}{2}C_0^2\right)\right]t + M_2(t) + M_3(t), \end{aligned} \tag{11}$$

where

$$M_2(t) = \int_0^t \left[-\frac{\sigma_{12}(X(\theta) - \overline{X})S(\theta)}{1 + pX(\theta) + qS(\theta)} + \frac{a_{12}(1 + p\overline{X})}{a_{21}(1 + q\overline{S})} \frac{\sigma_{21}(S(\theta) - \overline{S})X(\theta)}{1 + pX(\theta) + qS(\theta)} \right] dB_1(\theta)$$

and

$$M_3(t) = \int_0^t \frac{a_{12}\sigma(1 + p\overline{X})\overline{S}I(\theta)}{a_{21}(1 + q\overline{S})} \, dB_2(\theta)$$

are real-valued continuous local martingales.

Thus

$$\begin{aligned}
&\limsup_{t\to+\infty} \frac{\langle M_2, M_2\rangle_t}{t} \\
&\quad = \limsup_{t\to+\infty} \frac{1}{t} \int_0^t \left[-\frac{\sigma_{12}(X(\theta) - \overline{X})S(\theta)}{1 + pX(\theta) + qS(\theta)} + \frac{a_{12}(1 + p\overline{X})}{a_{21}(1 + q\overline{S})} \frac{\sigma_{21}(S(\theta) - \overline{S})X(\theta)}{1 + pX(\theta) + qS(\theta)} \right]^2 d\theta \\
&\quad \leq 2C_0^2 \left[\frac{\sigma_{12}^2}{q^2} + \frac{a_{12}^4\sigma_{21}^2(1 + p\overline{X})^2}{a_{21}^2 p^2(1 + q\overline{S})^2} \right] < +\infty
\end{aligned}$$

and

$$\begin{aligned}
\limsup_{t\to+\infty} \frac{\langle M_3, M_3\rangle_t}{t} &= \limsup_{t\to+\infty} \frac{1}{t} \int_0^t \left[\frac{a_{12}\sigma(1 + p\overline{X})\overline{S}I(\theta)}{a_{21}(1 + q\overline{S})} \right]^2 d\theta \\
&\leq C_0^2 \left[\frac{a_{12}^2\sigma^2\overline{S}^2(1 + p\overline{X})^2}{(1 + q\overline{S})^2} \right] < +\infty.
\end{aligned}$$

Applying the strong law of large numbers, we have $\lim_{t\to+\infty} \frac{M_i(t)}{t} = 0$ $(i = 2, 3)$.

Dividing equation (11) by t and taking the limit superior, we have

$$\begin{aligned}
&\limsup_{t\to+\infty} \frac{1}{t} \int_0^t \left[\left(a_{11} - \frac{a_{12}p}{q} \right) \left(X(\theta) - \overline{X} \right)^2 + \frac{a_{12}a_{22}(1 + p\overline{X})}{a_{21}(1 + q\overline{S})} \left(S(\theta) - \overline{S} \right)^2 \right. \\
&\quad \left. + \frac{a_{12}a_{33}(1 + p\overline{X})}{a_{21}(1 + q\overline{S})} I^2(\theta) \right] d\theta \leq \frac{\sigma_{12}^2\overline{X}}{2q^2} + \frac{a_{12}(1 + p\overline{X})}{a_{21}(1 + q\overline{S})} \left(\frac{\sigma_{21}^2\overline{S}}{2p^2} + \frac{\sigma^2\overline{S}}{2} C_0^2 \right).
\end{aligned}$$

Thus

$$\limsup_{t\to+\infty} \frac{1}{t} \int_0^t \left[\left(X(\theta) - \overline{X} \right)^2 + \left(S(\theta) - \overline{S} \right)^2 + I^2(\theta) \right] d\theta \leq \frac{U_2}{W_2}. \qquad \square$$

Corollary 3.2 *From Theorem* 3.2, *when* $\sigma_{12} = \sigma_{21} = \sigma = 0$, *we have*

$$LV \leq -\left(a_{11} - \frac{a_{12}p}{q} \right)(X - \overline{X})^2 - \frac{a_{12}a_{22}(1 + p\overline{X})}{a_{21}(1 + q\overline{S})}(S - \overline{S})^2 - \frac{a_{12}a_{33}(1 + p\overline{X})}{a_{21}(1 + q\overline{S})} I^2 \leq 0,$$

thus when $a_{11}q > a_{22}p$, $R_0 > 1$ *and* $R_1 < 1$ *hold, the equilibrium point* $E_2(\overline{X}, \overline{S}, 0)$ *of system* (1) *is globally asymptotically stable*.

Remark 3.2 From Theorem 3.2, if the interference intensity is sufficiently small, the solution of model (2) will fluctuates around the equilibrium point $E_2(\overline{X},\overline{S},0)$. Moreover, the fluctuation intensity is related with the disturbance intensity: the fluctuation intensity is positively correlated with the value of σ_{12},σ_{21} and σ.

3.3 Asymptotic behaviors around the positive equilibrium point E_3 of system (1)

When $R_1 > 1$, system (1) has a positive equilibrium point $E_3(X^*,S^*,I^*)$, but it is not the equilibrium point of model (2). Now, we explore the asymptotic behaviors of system (2) around $E_3(X^*,S^*,I^*)$.

$X(t)$ is a temporally homogeneous Markov process in E_l, which is given by the stochastic differential equation

$$dX(t) = b(X)\,dt + \sum_{m=1}^{k}\sigma_m(x)\,dB_m(t),$$

where $E_l \subset R^l$ represents a l-dimensional Euclidean space.

The diffusion matrix of $X(t)$ is given by

$$\Lambda(x) = \bigl(a_{i,j}(x)\bigr), a_{i,j}(x) = \sum_{m=1}^{k}\sigma_m^i(x)\sigma_m^j(x).$$

Assumption 3.1 ([33]) Assume that there is a bounded domain $U \subset E_l$ with regular boundary, satisfying the following conditions:

(1) In the domain U and some of its neighbors, the minimum eigenvalue of the diffusion matrix $A(x)$ is nonzero.

(2) When $x \in E_l\backslash U$, the mean time τ at which a path starting from x to the set U is limited, and $\sup_{x\in H} E_x\tau < \infty$ for every compact subset $H \subset E_l$.

Lemma 3.1 ([33]) *When Assumption* 3.1 *holds, the Markov process* $X(t)$ *has a stationary distribution* $\mu(\cdot)$ *with density in* E_l. *Let* $f(x)$ *be a function integrable with respect to the measure* μ, *where* $x \in E_l$, *then, for any Borel set* $B \subset E_l$, *we have*

$$\lim_{t\to\infty} P(t,x,B) = \mu(B)$$

and

$$P_x\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T f\bigl(x(t)\bigr)\,dt = \int_{E_l} f(x)\mu(dx)\right\} = 1.$$

Theorem 3.3 *Let* $(X(t),S(t),I(t))$ *be the solution of model* (2) *with initial value* $(X(0),S(0),I(0)) \in R_+^3$. *If* $a_{11}q > a_{12}p$ *and* $R_1 > 1$ *hold, then*

$$\limsup_{t\to+\infty}\frac{1}{t}\int_0^t\bigl[\bigl(X(\theta)-X^*\bigr)^2 + \bigl(S(\theta)-S^*\bigr)^2 + \bigl(I(\theta)-I^*\bigr)^2\bigr]\,d\theta \le \frac{U_3}{W_3},$$

where

$$U_3 = \frac{\sigma_{12}^2X^*}{2q^2} + \frac{a_{12}(1+pX^*)}{a_{21}(1+qS^*)}\left(\frac{\sigma_{21}^2S^*}{2p^2} + \frac{C_0^2\sigma^2}{2}\bigl(S^*+I^*\bigr)\right)$$

and

$$W_3 = \min\left\{a_{11} - \frac{a_{12}p}{q}, \frac{a_{12}a_{22}(1+pX^*)}{a_{21}(1+qS^*)}, \frac{a_{12}a_{33}(1+pX^*)}{a_{21}(1+qS^*)}\right\}.$$

Proof Noting that (X^*, S^*, I^*) is the equilibrium point of system (1), thus

$$\begin{cases} b - a_{11}X^* - \frac{a_{12}S^*}{1+pX^*+qS^*} = 0, \\ -c - a_{22}S^* + \frac{a_{21}X^*}{1+pX^*+qS^*} - \beta I^* = 0, \\ \beta S^* - d - a_{33}I^* = 0. \end{cases}$$

Define

$$\begin{aligned} V(X,S,I) &= \left(X - X^* - X^* \ln\frac{X}{X^*}\right) + \frac{a_{12}(1+pX^*)}{a_{21}(1+qS^*)}\left(S - S^* - S^* \ln\frac{S}{S^*}\right) \\ &\quad + \frac{a_{12}(1+pX^*)}{a_{21}(1+qS^*)}\left(I - I^* - I^* \ln\frac{I}{I^*}\right) \\ &:= V_1 + \frac{a_{12}(1+pX^*)}{a_{21}(1+qS^*)}V_2 + \frac{a_{12}(1+pX^*)}{a_{21}(1+qS^*)}V_3. \end{aligned}$$

Applying Itô's formula to the stochastic differential system (2) yields

$$dV_1 = LV_1\,dt - \frac{\sigma_{12}(X-X^*)S}{1+pX+qS}\,dB_1(t),$$

where

$$\begin{aligned} LV_1 &= (X - X^*)\left[b - a_{11}X - \frac{a_{12}S}{1+pX+qS}\right] + \frac{\sigma_{12}^2 X^* S^2}{2(1+pX+qS)^2} \\ &= (X - X^*)\left[b - a_{11}(X - X^*) - a_{11}X^* - \frac{a_{12}S}{1+pX+qS}\right] + \frac{\sigma_{12}^2 X^* S^2}{2(1+pX+qS)^2} \\ &= (X - X^*)\left[-a_{11}(X - X^*) + a_{12}\frac{pS^*(X-X^*) - (S-S^*)(1+pX^*)}{(1+pX^*+qS^*)(1+pX+qS)}\right] \\ &\quad + \frac{\sigma_{12}^2 X^* S^2}{2(1+pX+qS)^2} \\ &= -a_{11}(X - X^*)^2 + \frac{a_{12}pS^*(X-X^*)^2}{(1+pX^*+qS^*)(1+pX+qS)} + \frac{\sigma_{12}^2 X^* S^2}{2(1+pX+qS)^2} \\ &\quad - \frac{a_{12}(1+pX^*)(S-S^*)(X-X^*)}{(1+pX^*+qS^*)(1+pX+qS)}. \end{aligned}$$

Similarly,

$$dV_2 = LV_2\,dt + \frac{\sigma_{21}(S-S^*)X}{1+pX+qS}\,dB_1(t) - \sigma(S - S^*)I\,dB_2(t),$$

where

$$\begin{aligned}
LV_2 &= (S-S^*)\left[-c-a_{22}S+\frac{a_{21}X}{1+pX+qS}-\beta I\right]+\frac{\sigma_{21}^2S^*X^2}{2(1+pX+qS)^2}+\frac{1}{2}\sigma^2S^*I^2\\
&= (S-S^*)\left[-c-a_{22}(S-S^*)-a_{22}S^*+\frac{a_{21}X}{1+pX+qS}-\beta(I-I^*)-\beta I^*\right]\\
&\quad+\frac{\sigma_{21}^2S^*X^2}{2(1+pX+qS)^2}+\frac{1}{2}\sigma^2S^*I^2\\
&= (S-S^*)\left[-a_{22}(S-S^*)+a_{21}\frac{(X-X^*)(1+qS^*)-qX^*(S-S^*)}{(1+pX^*+qS^*)(1+pX+qS)}-\beta(I-I^*)\right]\\
&\quad+\frac{\sigma_{21}^2S^*X^2}{2(1+pX+qS)^2}+\frac{1}{2}\sigma^2S^*I^2\\
&= -a_{22}(S-S^*)^2+a_{21}\frac{(1+qS^*)(X-X^*)(S-S^*)}{(1+pX^*+qS^*)(1+pX+qS)}-\beta(I-I^*)(S-S^*)\\
&\quad-a_{21}\frac{qX^*(S-S^*)^2}{(1+pX^*+qS^*)(1+pX+qS)}+\frac{\sigma_{21}^2S^*X^2}{2(1+pX+qS)^2}+\frac{1}{2}\sigma^2S^*I^2.
\end{aligned}$$

Also, we have

$$dV_3 = LV_3\,dt+\sigma(I-I^*)S\,dB_2(t),$$

where

$$\begin{aligned}
LV_3 &= (I-I^*)[-d-a_{33}I+\beta S]+\frac{1}{2}\sigma^2I^*S^2\\
&= (I-I^*)\left[-d-a_{33}(I-I^*)-a_{33}I^*+\beta(S-S^*)+\beta S^*\right]+\frac{1}{2}\sigma^2I^*S^2\\
&= -a_{33}(I-I^*)^2+\beta(S-S^*)(I-I^*)+\frac{1}{2}\sigma^2I^*S^2.
\end{aligned}$$

Then we have

$$\begin{aligned}
dV &= LV\,dt-\frac{\sigma_{12}(X-X^*)S}{1+pX+qS}\,dB_1(t)+\frac{a_{12}(1+pX^*)}{a_{21}(1+qS^*)}\left[\frac{\sigma_{21}(S-S^*)X}{1+pX+qS}\,dB_1(t)\right.\\
&\quad\left.-\sigma(S-S^*)I\,dB_2(t)+\sigma(I-I^*)S\,dB_2(t)\right],
\end{aligned}\tag{12}$$

where

$$\begin{aligned}
LV &= LV_1+\frac{a_{12}(1+pX^*)}{a_{21}(1+qS^*)}LV_2+\frac{a_{12}(1+pX^*)}{a_{21}(1+qS^*)}LV_3\\
&= -a_{11}(X-X^*)^2+\frac{a_{12}pS^*(X-X^*)^2}{(1+pX^*+qS^*)(1+pX+qS)}+\frac{\sigma_{12}^2X^*S^2}{2(1+pX+qS)^2}\\
&\quad-\frac{a_{12}(1+pX^*)(S-S^*)(X-X^*)}{(1+pX^*+qS^*)(1+pX+qS)}+\frac{a_{12}(1+pX^*)}{a_{21}(1+qS^*)}\left[-a_{22}(S-S^*)^2-a_{33}(I-I^*)^2\right.\\
&\quad+a_{21}\frac{(1+qS^*)(X-X^*)(S-S^*)}{(1+pX^*+qS^*)(1+pX+qS)}-a_{21}\frac{qX^*(S-S^*)^2}{(1+pX^*+qS^*)(1+pX+qS)}\\
&\quad\left.+\frac{\sigma_{21}^2S^*X^2}{2(1+pX+qS)^2}+\frac{\sigma^2S^*}{2}I^2+\frac{\sigma^2I^*}{2}S^2\right]
\end{aligned}$$

$$\begin{aligned}&\le -\left(a_{11}-\frac{a_{12}p}{q}\right)\left(X-X^*\right)^2-\frac{a_{12}a_{22}(1+pX^*)}{a_{21}(1+qS^*)}\left(S-S^*\right)^2-\frac{a_{12}a_{33}(1+pX^*)}{a_{21}(1+qS^*)}\left(I-I^*\right)^2\\&\quad+\frac{\sigma_{12}^2X^*}{2q^2}+\frac{a_{12}(1+pX^*)}{a_{21}(1+qS^*)}\left[\frac{\sigma_{21}^2S^*}{2p^2}+\frac{C^2\sigma^2}{2}\left(S^*+I^*\right)\right].\end{aligned}$$

It is easy to see that, for any

$$\phi<\min\left\{\left(a_{11}-\frac{a_{12}p}{q}\right)X^*,\frac{a_{12}a_{22}(1+pX^*)}{a_{21}(1+qS^*)}S^*,\frac{a_{12}a_{33}(1+pX^*)}{a_{21}(1+qS^*)}I^*\right\},$$

the ellipsoid

$$\begin{aligned}&-\left(a_{11}-\frac{a_{12}p}{q}\right)\left(X-X^*\right)^2-\frac{a_{12}a_{22}(1+pX^*)}{a_{21}(1+qS^*)}\left(S-S^*\right)^2\\&\quad-\frac{a_{12}a_{33}(1+pX^*)}{a_{21}(1+qS^*)}\left(I-I^*\right)^2+\phi=0\end{aligned}$$

lies entirely in R^3_+. Let U to be any neighborhood of the ellipsoid with $\bar{U}\subseteq E_3=R^3_+$, thus for any $x\in U\backslash E_l$, we have $LV\le -\overline{M}$ ($\overline{M}$ is a positive constant). Therefore, condition (2) in Assumption 3.1 is satisfied. Moreover, there exists a $G=\min\{\sigma_1^2x_1^2,\sigma_2^2x_2^2,\sigma_3^2x_3^2,(x_1,x_2,x_3)\in\overline{U}\}>0$ such that

$$\sum_{i,j=1}^3\left(\sum_{k=1}^3a_{ik}(x)a_{jk}(x)\right)\xi_i\xi_j=\sigma_1^2x_1^2\xi_1^2+\sigma_2^2x_2^2\xi_2^2+\sigma_3^2x_3^2\xi_3^2\ge G\|\xi\|^2$$

for all $x\in\bar{U},\xi\in R^3$, which means condition (1) in Assumption 3.1 is satisfied. Therefore, the stochastic model (2) has a unique stationary distribution $\mu(\cdot)$, it also has the ergodic property.

Integrating equation (12) from 0 to t on both sides yields

$$\begin{aligned}V(t)-V(0)\le{}&\int_0^t\left[-\left(a_{11}-\frac{a_{12}p}{q}\right)\left(X(\theta)-X^*\right)^2-\frac{a_{12}a_{22}(1+pX^*)}{a_{21}(1+qS^*)}\left(S(\theta)-S^*\right)^2\right.\\&\left.-\frac{a_{12}a_{33}(1+pX^*)}{a_{21}(1+qS^*)}\left(I(\theta)-I^*\right)^2\right]d\theta\\&+\left[\frac{\sigma_{12}^2X^*}{2q^2}+\frac{a_{12}(1+pX^*)}{a_{21}(1+qS^*)}\left(\frac{\sigma_{21}^2S^*}{2p^2}+\frac{C_0^2\sigma^2}{2}\left(S^*+I^*\right)\right)\right]\\&+M_4(t)+M_5(t),\end{aligned}\tag{13}$$

where

$$M_4(t)=\int_0^t\left[-\frac{\sigma_{12}(X(\theta)-X^*)S(\theta)}{1+pX(\theta)+qS(\theta)}+\frac{a_{12}(1+pX^*)}{a_{21}(1+qS^*)}\frac{\sigma_{21}(S(\theta)-S^*)X(\theta)}{1+pX(\theta)+qS(\theta)}\right]dB_1(\theta)$$

and

$$M_5(t)=\int_0^t\frac{a_{12}(1+pX^*)}{a_{21}(1+qS^*)}\left[-\sigma\left(S(\theta)-S^*\right)I(\theta)+\sigma\left(I(\theta)-I^*\right)S(\theta)\right]dB_2(\theta)$$

are real-valued continuous local martingales.

Thus

$$\begin{aligned}
&\limsup_{t\to+\infty}\frac{\langle M_4,M_4\rangle_t}{t}\\
&\quad=\limsup_{t\to+\infty}\frac{1}{t}\int_0^t\left[\frac{a_{12}(1+pX^*)}{a_{21}(1+qS^*)}\frac{\sigma_{21}(S(\theta)-S^*)X(\theta)}{1+pX(\theta)+qS(\theta)}-\frac{\sigma_{12}(X(\theta)-X^*)S(\theta)}{1+pX(\theta)+qS(\theta)}\right]^2d\theta\\
&\quad\leq 2C_0^2\left[\frac{\sigma_{12}^2}{q^2}+\frac{a_{12}^4\sigma_{21}^2(1+pX^*)^2}{a_{21}^2p^2(1+qS^*)^2}\right]<+\infty
\end{aligned}$$

and

$$\limsup_{t\to+\infty}\frac{\langle M_5,M_5\rangle_t}{t}=\limsup_{t\to+\infty}\frac{1}{t}\int_0^t\bigl(\sigma S^*I(\theta)-\sigma I^*S(\theta)\bigr)^2d\theta\leq 2C_0^2\sigma^2\bigl(S^*+I^*\bigr)<+\infty.$$

Applying the strong law of large numbers, we have $\lim_{t\to+\infty}\frac{M_i(t)}{t}=0$ ($i=4,5$).

Dividing equation (13) by t and taking the limit superior, we have

$$\begin{aligned}
&\limsup_{t\to+\infty}\frac{1}{t}\int_0^t\left[\left(a_{11}-\frac{a_{12}p}{q}\right)\bigl(X(\theta)-X^*\bigr)^2+\frac{a_{12}a_{22}(1+pX^*)}{a_{21}(1+qS^*)}\bigl(S(\theta)-S^*\bigr)^2\right.\\
&\qquad\left.+\frac{a_{12}a_{33}(1+pX^*)}{a_{21}(1+qS^*)}\bigl(I(\theta)-I^*\bigr)^2\right]d\theta\\
&\quad\leq\frac{\sigma_{12}^2X^*}{2q^2}+\frac{a_{12}(1+pX^*)}{a_{21}(1+qS^*)}\left[\frac{\sigma_{21}^2S^*}{2p^2}+\frac{C_0^2\sigma^2}{2}\bigl(S^*+I^*\bigr)\right],
\end{aligned}$$

thus

$$\limsup_{t\to+\infty}\frac{1}{t}\int_0^t\bigl[\bigl(X(\theta)-X^*\bigr)^2+\bigl(S(\theta)-S^*\bigr)^2+\bigl(I(\theta)-I^*\bigr)^2\bigr]d\theta\leq\frac{U_3}{W_3}.\qquad(14)$$

□

Corollary 3.3 *From Theorem* 3.3, *when* $\sigma_{12}=\sigma_{21}=\sigma=0$, *we have*

$$\begin{aligned}
LV&\leq-\left(a_{11}-\frac{a_{12}p}{q}\right)\bigl(X-X^*\bigr)^2-\frac{a_{12}a_{22}(1+pX^*)}{a_{21}(1+qS^*)}\bigl(S-S^*\bigr)^2-\frac{a_{12}a_{33}(1+pX^*)}{a_{21}(1+qS^*)}\bigl(I-I^*\bigr)^2\\
&\leq 0.
\end{aligned}$$

Thus when $a_{11}q>a_{22}p$ *and* $R_1>1$ *hold, the positive equilibrium point* $E_3(X^*,S^*,I^*)$ *of system* (1) *is globally asymptotically stable.*

Remark 3.3 From Theorem 3.3, if the interference intensity is sufficiently small, the solution of model (2) will fluctuates around the equilibrium point $E_3(X^*,S^*,I^*)$. Moreover, the fluctuation intensity is related with the disturbance intensity: the fluctuation intensity is positively correlated with the value of σ_{12},σ_{21} and σ.

Remark 3.4 If the conditions in Theorem 3.3 are hold, then the solution of model (2) has a unique stationary distribution, it also has the ergodic property.

4 Persistence in mean and extinction

When we consider a biological population system, persistence in mean and extinction are two very important properties. In this section, we investigate the persistence in mean and extinction of system (2).

Since there is no equilibrium point in system (2), we cannot determine the persistence of system (2) by proving the stability of the equilibrium point as a deterministic system.

Definition 4.1 ([5]) The definition of persistence in mean and extinction are given as follows:

(1) The species $X(t)$ is said to be in extinction if $\lim_{t\to+\infty} X(t) = 0$.

(2) The species $X(t)$ is said to be in persistence in mean if $\lim_{t\to+\infty}\langle X(t)\rangle_* > 0$.

Lemma 4.1 ([34]) *Let $X(t) \in C(\Omega \times [0,+\infty), R_+)$.*

(1) *If there exist $T > 0, \lambda_0 > 0, \lambda, n_i$, when $t \geq T$, we have*

$$\ln X(t) \leq \lambda t - \lambda_0 \int_0^t X(s)\,ds + \sum_{i=1}^{j} n_i B(t) \quad a.s.,$$

then

$$\begin{cases} \langle X\rangle^* \leq \frac{\lambda}{\lambda_0}\ a.s., & \text{if } \lambda \geq 0; \\ \lim_{t\to+\infty} X(t) = 0\ a.s., & \text{if } \lambda < 0. \end{cases}$$

(2) *If there exist $T > 0, \lambda_0 > 0, \lambda > 0, n_i$, when $t \geq T$, we have*

$$\ln X(t) \geq \lambda t - \lambda_0 \int_0^t X(s)\,ds + \sum_{i=1}^{j} n_i B(t) \quad a.s.,$$

then $\langle X\rangle_* \geq \frac{\lambda}{\lambda_0}$ a.s.

4.1 Persistence in mean

Theorem 4.1 *Let $(X(t), S(t), I(t))$ be the solution of model (2) with initial value $(X(0), S(0), I(0)) \in R_+^3$. Model (2) has persistence in mean if conditions $a_{11}q > a_{12}p, R_1 > 1$, and*

$$\varrho = \max\{\sigma_{12}, \sigma_{21}, \sigma\} < \min\left\{X^*\sqrt{\frac{W_3}{U_0}}, S^*\sqrt{\frac{W_3}{U_0}}, I^*\sqrt{\frac{W_3}{U_0}}\right\}$$

hold, that is,

$$\liminf_{t\to+\infty} \frac{1}{t}\int_0^t X(\theta)\,d\theta > 0, \qquad \liminf_{t\to+\infty} \frac{1}{t}\int_0^t S(\theta)\,d\theta > 0, \qquad \liminf_{t\to+\infty} \frac{1}{t}\int_0^t I(\theta)\,d\theta > 0,$$

where

$$U_0 = \frac{X^*}{2q^2} + \frac{a_{12}(1+pX^*)}{a_{21}(1+qS^*)}\left(\frac{S^*}{2p^2} + \frac{C_0^2(S^*+I^*)}{2}\right),$$

U_3 and W_3 are defined in Theorem 3.3.

Proof Applying equation (14) in Theorem 3.3 we have

$$\begin{cases} \limsup_{t\to+\infty} \frac{1}{t}\int_0^t (X(\theta)-X^*)^2 \le \frac{U_3}{W_3}, \\ \limsup_{t\to+\infty} \frac{1}{t}\int_0^t (S(\theta)-S^*)^2 \le \frac{U_3}{W_3}, \\ \limsup_{t\to+\infty} \frac{1}{t}\int_0^t (I(\theta)-I^*)^2 \le \frac{U_3}{W_3}. \end{cases} \tag{15}$$

Applying the inequality $2a^2 - 2ab \le a^2 + (a-b)^2$ to $X(t)$, we have

$$X \ge \frac{X^*}{2} - \frac{(X-X^*)^2}{2X^*}.$$

Therefore

$$\begin{aligned} U_3 &= \frac{\sigma_{12}^2 X^*}{2q^2} + \frac{a_{12}(1+pX^*)}{a_{21}(1+qS^*)}\left[\frac{\sigma_{21}^2 S^*}{2p^2} + \frac{C_0^2\sigma^2}{2}\left(S^* + I^*\right)\right] \\ &\le \varrho^2\left[\frac{X^*}{2q^2} + \frac{a_{12}(1+pX^*)}{a_{21}(1+qS^*)}\left(\frac{S^*}{2p^2} + \frac{C_0^2(S^*+I^*)}{2}\right)\right] \\ &= \varrho^2 U_0. \end{aligned}$$

When $\varrho < X^*\sqrt{\frac{W_3}{U_0}}$, we have

$$\begin{aligned} \liminf_{t\to+\infty} \frac{1}{t}\int_0^t X(\theta)\,d\theta &\ge \frac{X^*}{2} - \limsup_{t\to+\infty}\frac{1}{t}\int_0^t \frac{(X(\theta)-X^*)^2}{2X^*}\,d\theta \\ &\ge \frac{X^*}{2} - \frac{U_3}{2W_3X^*} \\ &\ge \frac{X^*}{2} - \frac{\sigma^2 U_0}{2W_3X^*} \\ &> 0. \end{aligned}$$

Similarly, when $\varrho < S^*\sqrt{\frac{W_3}{U_0}}$, we have

$$\begin{aligned} \liminf_{t\to+\infty} \frac{1}{t}\int_0^t S(\theta)\,d\theta &\ge \frac{S^*}{2} - \limsup_{t\to+\infty}\frac{1}{t}\int_0^t \frac{(S(\theta)-S^*)^2}{2S^*}\,d\theta \\ &\ge \frac{S^*}{2} - \frac{U_3}{2W_3S^*} \\ &\ge \frac{S^*}{2} - \frac{\sigma^2 U_0}{2W_3S^*} \\ &> 0. \end{aligned}$$

When $\varrho < I^*\sqrt{\frac{W_3}{U_0}}$, we have

$$\begin{aligned} \liminf_{t\to+\infty} \frac{1}{t}\int_0^t I(\theta)\,d\theta &\ge \frac{I^*}{2} - \limsup_{t\to+\infty}\frac{1}{t}\int_0^t \frac{(I(\theta)-I^*)^2}{2I^*}\,d\theta \\ &\ge \frac{I^*}{2} - \frac{U_3}{2W_3I^*} \end{aligned}$$

$$\geq \frac{I^*}{2} - \frac{\sigma^2 U_0}{2W_3 I^*}$$
$$> 0. \qquad \square$$

Remark 4.1 From Theorem 4.1, when $R_1 > 1, a_{11}q > a_{12}p$ and the intensity of random disturbance is sufficiently small, system (2) will persistence in mean. This shows that biological populations can resist a small environmental disturbance to maintain the original persistence.

4.2 Extinction

Extinction and persistence in mean are closely related, so we also concern ourselves with the situation of population extinction. In this subsection, we point out the conditions of predator extinction.

Theorem 4.2 *Let* $(X(t), S(t), I(t))$ *be the solution of model* (2) *with initial value* $(X(0), S(0), I(0)) \in R^3_+$. *If one of the following conditions holds*:

(1) $\sigma_{21} > \max\{\frac{a_{21}}{\sqrt{2c}}, \sqrt{a_{21}p}\}$,

(2) $R^* = \frac{a_{21}}{pc} - \frac{\sigma_{21}^2}{2p^2c} < 1, \sigma_{21} \leq \sqrt{a_{21p}}$,

then

$$\lim_{t\to+\infty} X(t) = \frac{b}{a_{11}}, \qquad \lim_{t\to+\infty} S(t) = 0, \qquad \lim_{t\to+\infty} I(t) = 0.$$

Proof Applying Itô's formula to the second equation of stochastic differential system (2) yields

$$\begin{aligned} d\ln S(t) &= \left[-c - \frac{\sigma^2 I^2}{2} - \frac{\sigma_{21}^2 X^2}{2(1+pX+qS)^2} - a_{22}S + \frac{a_{21}X}{1+pX+qS} - \beta I\right]dt \\ &\quad + \frac{\sigma_{21}X}{1+pX+qS}\,dB_1(t) - \sigma I\,dB_2(t) \\ &\leq \left[-c - a_{22}S - \frac{\sigma_{21}^2}{2}\left(\frac{X}{1+pX+qS} - \frac{a_{21}}{\sigma_{21}^2}\right)^2 + \frac{a_{21}^2}{2\sigma_{21}^2}\right]dt \\ &\quad + \frac{\sigma_{21}X}{1+pX+qS}\,dB_1(t) - \sigma I\,dB_2(t). \end{aligned} \tag{16}$$

Case I. When $\sigma_{21} > \max\{\frac{a_{21}}{\sqrt{2c}}, \sqrt{a_{21}p}\}$, inequality (16) takes its maximum value on the interval $[0, \frac{1}{p}]$ at $\frac{a_{21}}{\sigma_{21}^2}$, so we have

$$d\ln S(t) \leq \left[-c - a_{22}S + \frac{a_{21}^2}{2\sigma_{21}^2}\right]dt + \frac{\sigma_{21}X}{1+pX+qS}\,dB_1(t) - \sigma I\,dB_2(t).$$

Integrating (16) from 0 to t and dividing it by t, we get

$$\frac{1}{t}\ln\frac{S(t)}{S(0)} \leq \frac{a_{21}^2}{2\sigma_{21}^2} - c - a_{22}\langle S(t)\rangle + t^{-1}\int_0^t \frac{\sigma_{21}X(\theta)}{1+pX(\theta)+qS(\theta)}\,dB_1(\theta) - t^{-1}\int_0^t \sigma I(\theta)\,dB_2(\theta).$$

Applying Lemma 4.1, we obtain

$$\lim_{t\to+\infty} S(t) = 0.$$

Case II. When $R^* = \frac{a_{21}}{pc} - \frac{\sigma_{21}^2}{2p^2c} < 1$ and $\sigma_{21} \leq \sqrt{a_{21p}}$, inequality (16) takes its maximum value on the interval $[0, \frac{1}{p}]$ at $\frac{1}{p}$, so we have

$$d\ln S(t) \leq \left[\frac{a_{21}}{p} - \frac{\sigma_{21}^2}{2p^2} - c - a_{22}S\right]dt + \frac{\sigma_{21}X}{1+pX+qS}\,dB_1(t) - \sigma I\,dB_2(t).$$

Integrating (16) from 0 to t and dividing it by t, we obtain

$$\begin{aligned}\frac{1}{t}\ln\frac{S(t)}{S(0)} &\leq \frac{a_{21}}{p} - \frac{\sigma_{21}^2}{2p^2} - c - a_{22}\langle S(t)\rangle + t^{-1}\int_0^t \frac{\sigma_{21}X(\theta)}{1+pX(\theta)+qS(\theta)}\,dB_1(\theta) \\ &\quad - t^{-1}\int_0^t \sigma I(\theta)\,dB_2(\theta) \\ &= c\left(\frac{a_{21}}{cp} - \frac{\sigma_{21}^2}{2cp^2} - 1\right) - a_{22}\langle S(t)\rangle + t^{-1}\int_0^t \frac{\sigma_{21}X(\theta)}{1+pX(\theta)+qS(\theta)}\,dB_1(\theta) \\ &\quad - t^{-1}\int_0^t \sigma I(\theta)\,dB_2(\theta) \\ &= c\left(R^* - 1\right) - a_{22}\langle S(t)\rangle + t^{-1}\int_0^t \frac{\sigma_{21}X(\theta)}{1+pX(\theta)+qS(\theta)}\,dB_1(\theta) \\ &\quad - t^{-1}\int_0^t \sigma I(\theta)\,dB_2(\theta).\end{aligned}$$

Applying Lemma 4.1, we obtain

$$\lim_{t\to+\infty} S(t) = 0.$$

Applying Itô's formula to the third equation of stochastic differential system (2), one has

$$d\ln I(t) = \left[-d - \frac{\sigma^2S^2}{2} - a_{33}I + \beta S\right]dt + \sigma S\,dB_2(t).$$

Since $\lim_{t\to+\infty}S(t) = 0$, there is an arbitrarily small constant $\varepsilon > 0$ such that when $t > T$, we have $-\frac{\sigma^2S^2}{2} + \beta S < \varepsilon$, thus

$$\begin{aligned}\ln I(t) &= \left(-d - a_{33}I + \beta S - \frac{\sigma^2S^2}{2}\right)dt + \sigma S\,dB_2(t) \\ &\leq (\varepsilon - d - a_{33}I)\,dt + \sigma S\,dB_2(t).\end{aligned} \tag{17}$$

Integrating equation (17) from 0 to t and dividing it by t yields

$$\frac{1}{t}\ln\frac{I(t)}{I(0)} \leq \varepsilon - d - a_{33}\langle I(t)\rangle + t^{-1}\int_0^t \sigma S(\theta)\,dB_2(\theta).$$

Applying Lemma 4.1 and the arbitrariness of ε, we obtain

$$\lim_{t\to+\infty} I(t) = 0.$$

Similarly,

$$d\ln X(t) = \left(b - a_{11}X - \frac{a_{12}S}{1+pX+qS} - \frac{\sigma_{12}^2 S^2}{2(1+pX+qS)^2}\right)dt - \frac{\sigma_{12}S}{1+pX+qS}\,dB_1(t).$$

Since $\lim_{t\to+\infty} S(t) = 0$, there is an arbitrarily small constant $\varepsilon > 0$ such that when $t > T$, we have $\frac{S}{1+pX+qS} < \varepsilon$, thus

$$d\ln X(t) \geq \left(b - a_{11}X - a_{12}\varepsilon - \sigma_{12}^2\varepsilon^2\right)dt - \frac{\sigma_{12}S}{1+pX+qS}\,dB_1(t).$$

Integrating the above equation from 0 to t and dividing it by t, one has

$$\frac{1}{t}\ln\frac{X(t)}{X(0)} \geq b - a_{12}\varepsilon - \sigma_{12}^2\varepsilon^2 - a_{11}\langle X(t)\rangle - t^{-1}\int_0^t \frac{\sigma_{12}S(\theta)}{1+pX(\theta)+qS(\theta)}\,dB_1(\theta).$$

Applying Lemma 4.1 and the arbitrariness of ε, we obtain

$$\lim_{t\to+\infty} X(t) \geq \frac{b}{a_{11}}. \tag{18}$$

On the other hand,

$$\begin{aligned} d\ln X(t) &= \left(b - a_{11}X - \frac{a_{12}S}{1+pX+qS} - \frac{\sigma_{12}^2 S^2}{2(1+pX+qS)^2}\right)dt - \frac{\sigma_{12}S}{1+pX+qS}\,dB_1(t) \\ &\leq (b - a_{11}X)\,dt - \frac{\sigma_{12}S}{1+pX+qS}\,dB_1(t). \end{aligned} \tag{19}$$

Integrating equation (19) from 0 to t and dividing it by t, we have

$$\frac{1}{t}\ln\frac{X(t)}{X(0)} \leq b - a_{11}\langle X(t)\rangle - t^{-1}\int_0^t \frac{\sigma_{12}S(\theta)}{1+pX(\theta)+qS(\theta)}\,dB_1(\theta).$$

Applying Lemma 4.1, we obtain

$$\lim_{t\to+\infty} X(t) \leq \frac{b}{a_{11}}. \tag{20}$$

From (18) and (20), we have

$$\lim_{t\to+\infty} X(t) = \frac{b}{a_{11}}. \qquad \square$$

Remark 4.2 From Theorem 4.2, if the intensity of random disturbance is sufficiently large or $R^* < 1$ and $\sigma_{12} \leq \sqrt{a_{21}p}$, the predator population will be extinct.

5 Conclusions and numerical simulations

This paper investigates a stochastic infected predator-prey model with Beddington-DeAngelis functional response. The existence of a global positive solution of model (2) is first proved, then we show the stochastically ultimate boundedness of the solution. In addition, by using the Lyapunov method and Itô's formula, we study the asymptotic properties

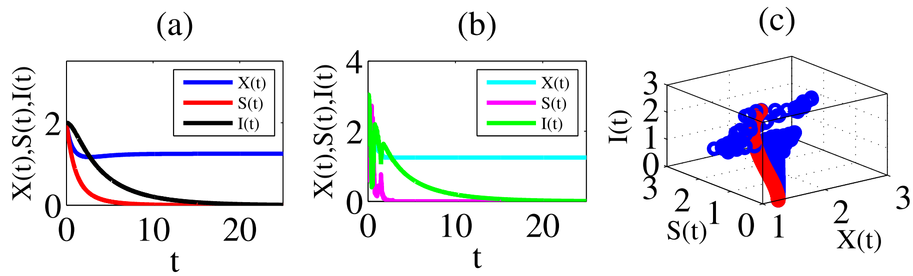

Figure 1 Time sequence diagram and phase portrait of model (2). (a) The deterministic model; **(b)** the Brownian motion model with $\sigma_{12} = \sigma_{21} = \sigma = 0.5$; **(c)** phase portrait: the red ∘ is corresponding to the deterministic model, while the blue ∘ represents the stochastic model.

and stationary distribution of the solution of model (2) around the equilibrium points of its deterministic. At last, we discuss the persistence in mean and extinction of model (2). The biological significance of the result shows that the external environment disturbance may have a certain effect on the stability of the biological system: the population's ability to adapt to the environment is limited. If the disturbance in the environment is small enough, the stability of the population will not be destroyed; if large disturbances occur in the environment, it may lead to the extinction of species.

We next give some numerical simulations to support our results. We consider the following discrete equations:

$$\begin{cases} X_{n+1} = X_n + X_n[b - a_{11}X_n - \frac{a_{12}S_n}{1+pX_n+qS_n}]\Delta t - \frac{\sigma_{12}S_nX_n}{1+pX_n+qS_n}\Delta W_{1k}, \\ S_{n+1} = S_n + S_n[-c - a_{22}S_n + \frac{a_{21}X_n}{1+pX_n+qS_n} - \beta I_n]\Delta t + \frac{\sigma_{21}X_nS_n}{1+pX_n+qS_n}\Delta W_{1k} - \sigma S_nI_n\Delta W_{2k}, \\ I_{n+1} = I_n + I_n[\beta S_n - d - a_{33}I_n]\Delta t + \sigma S_nI_n\Delta W_{2k}, \end{cases}$$

where $\Delta t = 0.01$, $\Delta W_{ik} \triangleq W(t_{k+1}) - W(t_k)$ obeys the Gaussian distribution $N(0,\Delta t)$.

In Figure 1, we choose $X(0) = 2, S(0) = 2, I(0) = 2, b = 1, c = 0.4, d = 0.2, a_{11} = 0.8, a_{12} = 0.55, a_{21} = 0.2, a_{22} = 0.1, a_{33} = 0.1, \beta = 0.2, p = 1, q = 1$, and step size $\triangle t = 0.001$.

Under this condition,

$$E_1 = (K,0,0) = (1.25,0,0), \qquad R_0 = 0.76, \qquad K = \frac{b}{a_{11}} = 1.25 \leq \frac{c}{a_{21}} = 2.$$

The numerical simulation of Figure 1 is consistent with our conclusion in Theorem 3.1.

In Figure 2, we choose $X(0) = 2, S(0) = 2, I(0) = 2, b = 0.6, c = 0.2, d = 0.3, a_{11} = 0.8, a_{12} = 0.6, a_{21} = 0.8, a_{22} = 0.1, a_{33} = 0.4, \beta = 0.2, p = 1, q = 1$, and step size $\triangle t = 0.001$.

Under this condition,

$$E_2 = (\overline{X},\overline{S},0) = (0.6,0.4,0), \qquad a_{11}q = 0.8 > a_{22}p = 0.1,$$

$$R_0 = 5 > 1, \qquad R_1 = 0.53 < 1.$$

In Figure 2(a), we choose $\sigma_{12} = \sigma_{21} = \sigma = 0.1$, thus

$$\limsup_{t\to+\infty} \frac{1}{t}\int_0^t \bigl[\bigl(X(\theta) - \overline{X}\bigr)^2 + \bigl(S(\theta) - \overline{S}\bigr)^2 + I^2(\theta)\bigr]\,d\theta \leq \frac{U_2}{W_2} = 11.8648.$$

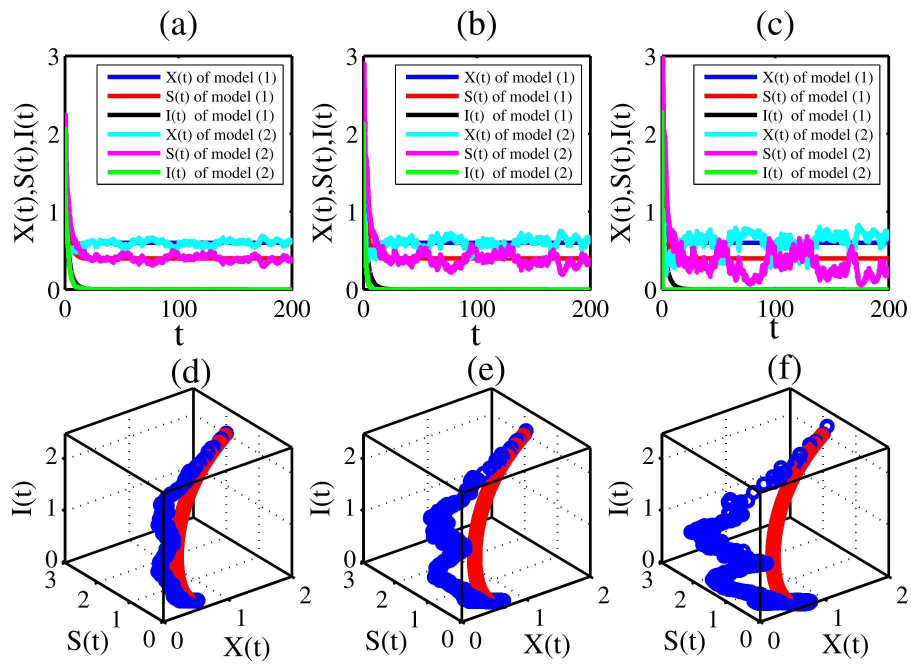

Figure 2 Time sequence diagram and phase portrait of model (2). (a)-(c) are a Brownian motion model with $\sigma_{12} = \sigma_{21} = \sigma = 0.1, 0.2, 0.4$, respectively. **(d)-(f)** are phase portraits of **(a)-(c)**, respectively. The red ∘ is corresponding to the deterministic model, while the blue ∘ represents the stochastic model.

In Figure 2(b), we choose $\sigma_{12} = \sigma_{21} = \sigma = 0.2$, thus

$$\limsup_{t\to+\infty} \frac{1}{t} \int_0^t \left[\left(X(\theta) - \overline{X}\right)^2 + \left(S(\theta) - \overline{S}\right)^2 + I^2(\theta) \right] d\theta \leq \frac{U_2}{W_2} = 47.4592.$$

In Figure 2(c), we choose $\sigma_{12} = \sigma_{21} = \sigma = 0.4$, thus

$$\limsup_{t\to+\infty} \frac{1}{t} \int_0^t \left[\left(X(\theta) - \overline{X}\right)^2 + \left(S(\theta) - \overline{S}\right)^2 + I^2(\theta) \right] d\theta \leq \frac{U_2}{W_2} = 189.8369.$$

Figure 2 shows that the solution of model (2) fluctuates around the equilibrium $E_2(0.6, 0.4, 0)$. In addition, the fluctuation intensity is related with the disturbance intensity: with the increase of $\sigma_{12}, \sigma_{21}, \sigma$, the fluctuation intensity is also increasing. These all meet the conditions of Theorem 3.2.

In Figure 3, we choose $X(0) = 2, S(0) = 2, I(0) = 2, b = 1, c = 0.1, d = 0.1, a_{11} = 0.5, a_{12} = 0.3, a_{21} = 1, a_{22} = 0.1, a_{33} = 0.2, \beta = 0.5, p = 1, q = 1$, and step size $\triangle t = 0.001$.

Under this condition,

$$E_3 = \left(X^*, S^*, I^*\right) = (1.9085, 0.5231, 0.8071),$$
$$a_{11}q = 0.5 > a_{22}p = 0.3, \qquad R_1 = 5.21 > 1.$$

In Figure 3(a), we choose $\sigma_{12} = \sigma_{21} = \sigma = 0.03$, thus

$$\limsup_{t\to+\infty} \frac{1}{t} \int_0^t \left[\left(X(\theta) - X^*\right)^2 + \left(S(\theta) - S^*\right)^2 + \left(I(\theta) - I^*\right)^2 \right] d\theta \leq \frac{U_3}{W_3} = 8.1131.$$

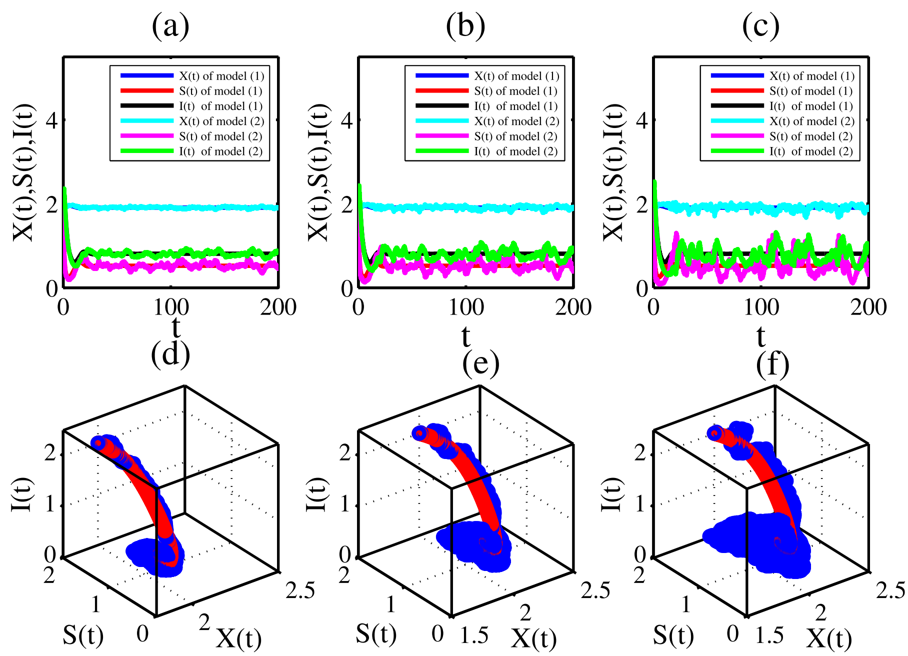

Figure 3 Time sequence diagram and phase portrait of model (2). (a) The Brownian motion model with $\sigma_{12} = \sigma_{21} = \sigma = 0.03, 0.06, 0.1$, respectively. **(d)**-**(f)** are phase portraits of **(a)**-**(c)**, respectively. The red ∘ is corresponding to the deterministic model, while the blue ∘ represents the stochastic model.

In Figure 3(b), we choose $\sigma_{12} = \sigma_{21} = \sigma = 0.06$, thus

$$\limsup_{t\to+\infty} \frac{1}{t}\int_0^t \left[\left(X(\theta) - X^*\right)^2 + \left(S(\theta) - S^*\right)^2 + \left(I(\theta) - I^*\right)^2\right] d\theta \le \frac{U_3}{W_3} = 22.4523.$$

In Figure 3(c), we choose $\sigma_{12} = \sigma_{21} = \sigma = 0.1$, thus

$$\limsup_{t\to+\infty} \frac{1}{t}\int_0^t \left[\left(X(\theta) - X^*\right)^2 + \left(S(\theta) - S^*\right)^2 + \left(I(\theta) - I^*\right)^2\right] d\theta \le \frac{U_3}{W_3} = 90.1453.$$

Figure 3 shows that the solution of model (2) fluctuates around $E_3(1.9085, 0.5231, 0.8071)$. In addition, the fluctuation intensity is related with the disturbance intensity: with the increase of σ_{12}, σ_{21} and σ, the fluctuation intensity is also increasing. These all meet the conditions of Theorem 3.2.

In Figure 4, we choose $X(0) = 2, S(0) = 2, I(0) = 2, b = 1, c = 0.1, d = 0.1, a_{11} = 0.5, a_{12} = 0.3, a_{21} = 1, a_{22} = 0.1, a_{33} = 0.2, \beta = 0.5, p = 1, q = 1$, and step size $\triangle t = 0.001$. Figure 4 shows that the solution of model (2) fluctuates up and down in a small neighborhood. According to the density functions in Figure 4(b)-(d), we see that there is a stationary distribution. This is in line with our conclusions.

In Figure 5, we choose $X(0) = 2, S(0) = 2, I(0) = 2, b = 1, c = 0.1, d = 0.1, a_{11} = 0.5, a_{12} = 0.3, a_{21} = 0.1, a_{22} = 0.1, a_{33} = 0.2, \beta = 0.5, p = 0.5, q = 0.5$, and step size $\triangle t = 0.001$.

In this condition,

$$E_3 = \left(X^*, S^*, I^*\right) = (1.8793, 0.4339, 0.5847),$$

$$a_{11}q = 0.25 > a_{22}p = 0.15, \qquad R_1 = 3.4 > 1.$$

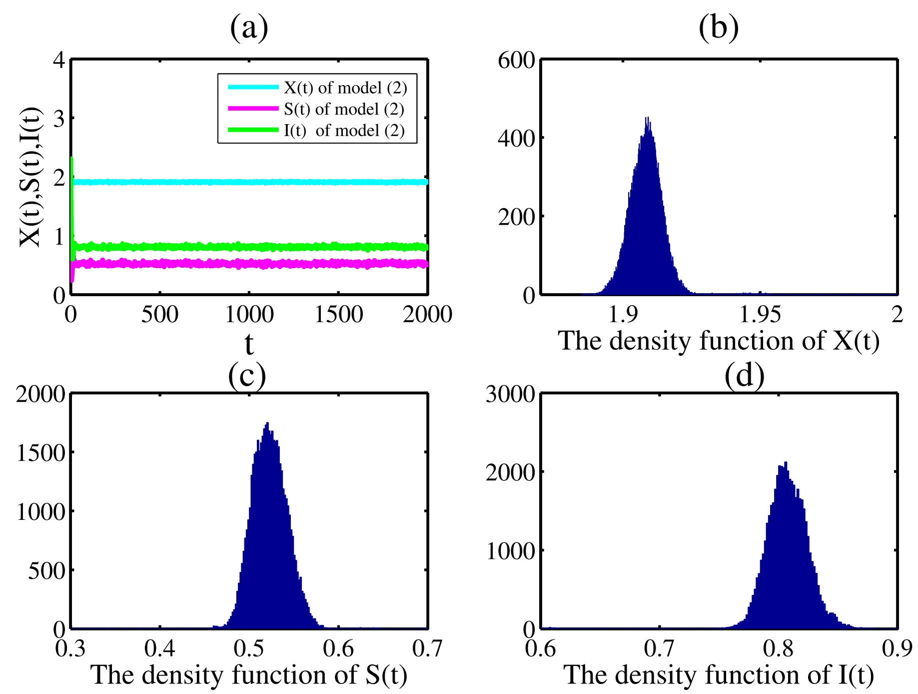

Figure 4 Time sequence diagram and density function of model (2) with $\sigma_{12} = \sigma_{21} = \sigma = 0.01$. (a) Time sequence diagram; **(b)-(d)** the density functions of $X(t)$, $S(t)$, $I(t)$, respectively.

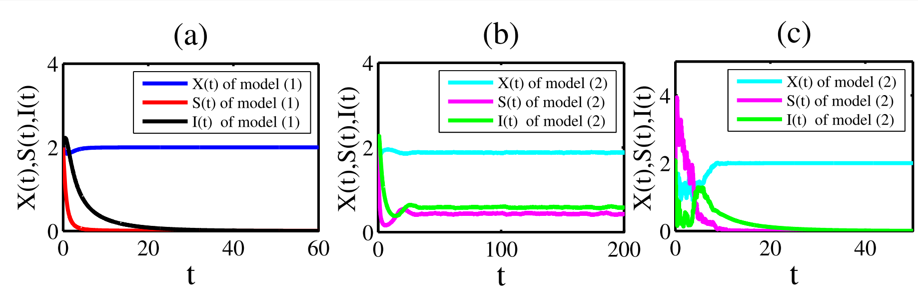

Figure 5 Persistence in mean and extinction of model (2). (a) The deterministic model; **(b)** persistence in mean of model (2); **(c)** extinction of model (2).

In Figure 5(b), we choose $\sigma_{12} = \sigma_{21} = \sigma = 0.006$. In this case,

$$\varrho = \max\{\sigma_{12}, \sigma_{21}, \sigma\} = 0.006 < \min\left\{X^*\sqrt{\frac{W_3}{U_0}}, S^*\sqrt{\frac{W_3}{U_0}}, I^*\sqrt{\frac{W_3}{U_0}}\right\} = 0.0064,$$

which satisfies the conditions in Theorem 4.1. Figure 5(b) shows that $X(t), S(t), I(t)$ have persistence in mean, this is in line with our conclusion in Theorem 4.1.

In Figure 5(c), we choose $\sigma_{12} = \sigma = 0.5$ and

$$\sigma_{21} = 1.2 > \max\left\{\frac{a_{12}}{\sqrt{2c}}, \sqrt{a_{21}p}\right\} = 0.223,$$

which satisfies the conditions in Theorem 4.2. Figure 5(c) shows that $S(t), I(t)$ are extinct and

$$\lim_{t\to+\infty} X(t) = \frac{b}{a_{11}} = 2,$$

this is in line with our conclusion in Theorem 4.2.

To sum up, we have the following conclusions:

I. Asymptotic behaviors

(1) When $R_0 < 1$ and $Ka_{21} < c$, the solution of model (2) is fluctuating around E_1. Therefore, the intensity of the fluctuation is positively correlated with σ_{12}.

(2) When $R_0 > 1, R_1 < 1$ and $a_{11}q > a_{22}p$, the solution of model (2) is fluctuating around E_2. Therefore, the intensity of the fluctuation is positively correlated with σ_{12}, σ_{21} and σ.

(3) When $R_1 > 1$ and $a_{11}q > a_{22}p$, the solution of model (2) is fluctuating around E_3. Therefore, the intensity of the fluctuation is positively correlated with σ_{12}, σ_{21} and σ. When the interference intensity is sufficient small, the solution of model (2) has a unique stationary distribution, it also has the ergodic property.

II. Persistence in mean and extinction

(1) When $R_1 > 1, a_{11}q > a_{12}p$ and $\varrho = \max\{\sigma_{12}, \sigma_{21}, \sigma\} < \min\{X^*\sqrt{\frac{W_3}{U_0}}, S^*\sqrt{\frac{W_3}{U_0}}, I^*\sqrt{\frac{W_3}{U_0}}\}$, the solution of model (2) can have persistence in mean.

(2) When $\sigma_{21} > \max\{\frac{a_{21}}{\sqrt{2c}}, \sqrt{a_{21}p}\}$ or $R^* < 1$ and $\sigma_{21} \leq \sqrt{a_{21p}}$, the predator of model (2) can be extinct.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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