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On the bounds for the spectral norms of geometric circulant matrices

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Abstract
In this paper, we define a geometric circulant matrix whose entries are the generalized Fibonacci numbers and hyperharmonic Fibonacci numbers. Then we give upper and lower bounds for the spectral norms of these matrices.

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Keywords: generalized Fibonacci number; hyperharmonic Fibonacci number; geometric circulant matrix; spectral norms

1 Introduction

The circulant and r-circulant matrices have important applications in numerical analysis, probability, coding theory, and so on. An $n \times n$ matrix C_r is called an r-circulant matrix if it is defined as follows:

$$C_r = \begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \dots & c_{n-3} & c_{n-2} \\ rc_{n-2} & rc_{n-1} & c_0 & \dots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ rc_1 & rc_2 & rc_3 & \dots & rc_{n-1} & c_0 \end{pmatrix}.$$

The matrix C_r is determined by its first row elements and r, we denote $C_r = \operatorname{Circ}_r(c_0, c_1, c_2, \dots, c_{n-1})$. In particular for $r = 1$

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \dots & c_{n-3} & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \dots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ c_1 & c_2 & c_3 & \dots & c_{n-1} & c_0 \end{pmatrix}$$

is called a circulant matrix.

Circulant and r-circulant matrices with the special numbers have been studied by many researchers in last decade. For example, in [1], Solak has studied the spectral norms of circulant matrices with the Fibonacci and Lucas numbers. In [2], Kocer *et al.* obtained norms of circulant and semicirculant matrices with Horadam numbers. In [3], Shen and

Cen have given upper and lower bounds for the spectral norms of r-circulant matrices with the Fibonacci and Lucas numbers. In [4], Bahsi computed the spectral norms of circulant and r-circulant matrices with the hyperharmonic numbers. Moreover, in [5], Bahsi and Solak studied norms of circulant and r-circulant matrices with the hyper-Fibonacci and hyper-Lucas numbers. In [6], Jiang and Zhou studied spectral norms of even order r-circulant matrices. In [7, 8], Tuglu and Kızılateş have calculated Euclidean norm by using the finite difference operator and given spectral norms of circulant, r-circulant and some special matrices with the harmonic Fibonacci and hyperharmonic Fibonacci numbers. In [9], Yazlık and Taskara have presented new upper and lower bounds for the spectral norms of an r-circulant matrix with the generalized k-Horadam numbers. In [10], He *et al.* gave the upper bound estimation of the spectral norm for r-circulant matrices with Fibonacci and Lucas numbers.

In view of the above papers, we define a new circulant matrix which is called geometric circulant matrix and give upper and lower bounds for the spectral norms of this matrix with the generalized Fibonacci and hyperharmonic Fibonacci numbers by using the same method given in [3].

2 Preliminaries

The well-known Fibonacci and Lucas sequences are defined by the following recurrence relations: for $n \geq 0$,

$$F_{n+2} = F_{n+1} + F_n$$

and

$$L_{n+2} = L_{n+1} + L_n,$$

where $F_0 = 0$, $F_1 = 1$, $L_0 = 2$ and $L_1 = 1$, respectively. The generalized Fibonacci and Lucas sequences, $\{U_n\}$ and $\{V_n\}$, are defined by the following recurrence relations: for $n \geq 0$, and any non-zero integer p,

$$U_{n+2} = pU_{n+1} + U_n$$

and

$$V_{n+2} = pV_{n+1} + V_n,$$

where $U_0 = 0$, $U_1 = 1$, $V_0 = 2$ and $V_1 = p$. If we take $p = 1$, then $U_n = F_n$ and $V_n = L_n$. Let α and β be the roots of the characteristic equation $x^2 - px - 1 = 0$. Then the Binet formulas for the sequences $\{U_n\}$ and $\{V_n\}$ are given by

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

and

$$V_n = \alpha^n + \beta^n,$$

where $\alpha = \frac{p+\sqrt{p^2+4}}{2}$ and $\beta = \frac{p-\sqrt{p^2+4}}{2}$.

On the other hand, Yazlık and Taskara examined the generalized k-Horadam numbers via the following recurrence relations:

$$H_{k,n+2} = f(k)H_{k,n+1} + g(k)H_{k,n} \tag{1}$$

with the initial values $H_{k,0} = a$, $H_{k,1} = b$. Moreover they calculated sum of squares of k-Horadam numbers (see [9]). If we take $f(k) = p$, $g(k) = 1$, $a = 0$ and $b = 1$ in (1), we get

$$\sum_{i=0}^{n-1} U_i^2 = \frac{U_n^2 - U_{n-1}^2 + (-1)^n}{p^2} \tag{2}$$

and if we take $f(k) = p$, $g(k) = 1$, $a = 2$ and $b = p$ in (1), we have

$$\sum_{i=0}^{n-1} V_i^2 = \frac{V_n^2 - V_{n-1}^2 + p^2 - 4 + (1-(-1)^n)(p^2+4)}{p^2}. \tag{3}$$

In [11], Tuglu *et al.* defined hyperharmonic Fibonacci numbers for $n, r \geq 1$,

$$\mathbb{F}_n^{(r)} = \sum_{k=1}^{n} \mathbb{F}_k^{(r-1)},$$

where $\mathbb{F}_n^{(0)} = \frac{1}{F_n}$ and $\mathbb{F}_0 = 0$. Then they gave for the sum of the squares of hyperharmonic Fibonacci numbers as follows:

$$\frac{1}{\sqrt{n}}\mathbb{F}_{n-1}^{(r+1)} \leq \sqrt{\sum_{k=0}^{n-1}\left(\mathbb{F}_k^{(r)}\right)^2} \leq \mathbb{F}_{n-1}^{(r+1)}. \tag{4}$$

Now we give some definitions and Lemmas related to our study.

Definition 1 An $n \times n$ matrix C_{r^*} is called a geometric circulant matrix if it is of the form

$$C_{r^*} = \begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \dots & c_{n-3} & c_{n-2} \\ r^2c_{n-2} & rc_{n-1} & c_0 & \dots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ r^{n-1}c_1 & r^{n-2}c_2 & r^{n-3}c_3 & \dots & rc_{n-1} & c_0 \end{pmatrix}.$$

We denote it for brevity by $C_{r^*} = \mathrm{Circ}_{r^*}(c_0, c_1, c_2, \dots, c_{n-1})$. Note that, for $r = 1$, geometric circulant matrix turns into circulant matrix given in [11, 12]. In fact, in [11, 12], the authors calculated the spectral norms of the circulant matrices with the generalized Fibonacci and hyperharmonic Fibonacci numbers.

Definition 2 Let $A = (a_{ij})$ be any $m \times n$ matrix. The Euclidean norm of matrix A is

$$\|A\|_E = \sqrt{\left(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\right)}.$$

Definition 3 Let $A = (a_{ij})$ be any $m \times n$ matrix. The spectral norm of matrix A is

$$\|A\|_2 = \sqrt{\max_{1\leq i\leq n} \lambda_i\bigl(A^H A\bigr)},$$

where $\lambda_i(A^H A)$ is eigenvalue of $A^H A$ and A^H is conjugate transpose of matrix A.

Then the following inequalities hold between the Euclidean norm and the spectral norm:

$$\frac{1}{\sqrt{n}}\|A\|_E \leq \|A\|_2 \leq \|A\|_E, \tag{5}$$

$$\|A\|_2 \leq \|A\|_E \leq \sqrt{n}\|A\|_2. \tag{6}$$

Definition 4 Let $A = (a_{ij})$ and $B = (b_{ij})$ are each $m \times n$ matrices, then their Hadamard product is the $m \times n$ matrix of elementwise products

$$A \circ B = (a_{ij}b_{ij}).$$

Lemma 1 [13] *Let A and B be two $m \times n$ matrices. We have*

$$\|A \circ B\|_2 \leq r_1(A)c_1(B),$$

where

$$r_1(A) = \max_{1\leq i\leq m} \sqrt{\sum_{j=1}^{n} |a_{ij}|^2},$$

$$c_1(B) = \max_{1\leq j\leq n} \sqrt{\sum_{i=1}^{m} |b_{ij}|^2}.$$

3 Main results

Theorem 1 *Let $U_{r^*} = \mathrm{Circ}_{r^*}(U_0, U_1, U_2, \ldots, U_{n-1})$ be an $n \times n$ geometric circulant matrix.*

(i) *If $|r| > 1$, then*

$$\sqrt{\frac{U_n^2 - U_{n-1}^2 + (-1)^n}{p^2}} \leq \|U_{r^*}\|_2 \leq \sqrt{\frac{(|r|^2 - |r|^{2n})(U_n^2 - U_{n-1}^2 + (-1)^n)}{(1-|r|^2)p^2}}.$$

(ii) *If $|r| < 1$, then*

$$\frac{|r|}{\sqrt{p^2+4}}\sqrt{\frac{2|r|^{2n+2} - |r|^{2n}(p^2+2) - |r|^2 V_{2n} + V_{2n-2}}{|r|^4 - |r|^2(p^2+2)+1} - 2\frac{|r|^{2n} - (-1)^n}{|r|^2+1}}$$
$$\leq \|U_{r^*}\|_2 \leq \sqrt{\frac{(n-1)(U_n^2 - U_{n-1}^2 + (-1)^n)}{p^2}}.$$

Proof We have the matrix

$$U_{r^*} = \begin{pmatrix} U_0 & U_1 & U_2 & \dots & U_{n-2} & U_{n-1} \\ rU_{n-1} & U_0 & U_1 & \dots & U_{n-3} & U_{n-2} \\ r^2U_{n-2} & rU_{n-1} & U_0 & \dots & U_{n-4} & U_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ r^{n-1}U_1 & r^{n-2}U_2 & r^{n-3}U_3 & \dots & rU_{n-1} & U_0 \end{pmatrix}.$$

(i) From $|r| > 1$ and definition of Euclidean norm, we have

$$\begin{aligned} \|U_{r^*}\|_E^2 &= \sum_{k=0}^{n-1}(n-k)U_k^2 + \sum_{k=1}^{n-1} k\left|r^{n-k}\right|^2 U_k^2 \\ &\geq \sum_{k=0}^{n-1}(n-k)U_k^2 + \sum_{k=1}^{n-1} kU_k^2 \\ &= n\sum_{k=0}^{n-1} U_k^2 \\ &= n\frac{U_n^2 - U_{n-1}^2 + (-1)^n}{p^2}, \end{aligned}$$

that is,

$$\frac{1}{\sqrt{n}}\|U_{r^*}\|_E \geq \sqrt{\frac{U_n^2 - U_{n-1}^2 + (-1)^n}{p^2}};$$

from (5), we have

$$\sqrt{\frac{U_n^2 - U_{n-1}^2 + (-1)^n}{p^2}} \leq \|U_{r^*}\|_2.$$

On the other hand, let the matrices A and B be defined by

$$A = \begin{pmatrix} U_0 & 1 & 1 & \dots & 1 & 1 \\ r & U_0 & 1 & \dots & 1 & 1 \\ r^2 & r & U_0 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ r^{n-1} & r^{n-2} & r^{n-3} & \dots & r & U_0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} U_0 & U_1 & U_2 & \dots & U_{n-2} & U_{n-1} \\ U_{n-1} & U_0 & U_1 & \dots & U_{n-3} & U_{n-2} \\ U_{n-2} & U_{n-1} & U_0 & \dots & U_{n-4} & U_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ U_1 & U_2 & U_3 & \dots & U_{n-1} & U_0 \end{pmatrix}.$$

That is, $U_{r^*} = A \circ B$. Then we obtain

$$\begin{aligned} r_1(A) &= \max_{1\le i\le n} \sqrt{\sum_{j=1}^{n} |a_{ij}|^2} \\ &= \sqrt{\left|r^1\right|^2 + \cdots + \left|r^{n-1}\right|^2} \\ &= \sqrt{\frac{|r|^2 - |r|^{2n}}{1-|r|^2}} \end{aligned}$$

and

$$\begin{aligned} c_1(B) &= \max_{1\le j\le n} \sqrt{\sum_{i=1}^{n} |b_{ij}|^2} \\ &= \sqrt{\sum_{k=0}^{n-1} U_k^2} \\ &= \sqrt{\frac{U_n^2 - U_{n-1}^2 + (-1)^n}{p^2}}. \end{aligned}$$

From Lemma 1, we have

$$\|U_{r^*}\|_2 \le \sqrt{\frac{(|r|^2 - |r|^{2n})(U_n^2 - U_{n-1}^2 + (-1)^n)}{(1-|r|^2)p^2}}.$$

Thus, we have

$$\sqrt{\frac{U_n^2 - U_{n-1}^2 + (-1)^n}{p^2}} \le \|U_{r^*}\|_2 \le \sqrt{\frac{(|r|^2 - |r|^{2n})(U_n^2 - U_{n-1}^2 + (-1)^n)}{(1-|r|^2)p^2}}.$$

(ii) From $|r| < 1$, we have

$$\begin{aligned} \|U_{r^*}\|_E^2 &= \sum_{k=0}^{n-1} (n-k)U_k^2 + \sum_{k=1}^{n-1} k\left|r^{n-k}\right|^2 U_k^2 \\ &\ge \sum_{k=0}^{n-1} (n-k)\left|r^{n-k}\right|^2 U_k^2 + \sum_{k=1}^{n-1} k\left|r^{n-k}\right|^2 U_k^2 \\ &= n|r|^{2n} \sum_{k=0}^{n-1} \left(\frac{U_k}{|r|^k}\right)^2 \\ &= \frac{n|r|^{2n}}{p^2+4} \sum_{k=0}^{n-1} \left(\frac{\alpha^k - \beta^k}{|r|^k}\right)^2 \\ &= \frac{n|r|^{2n}}{p^2+4} \left(\frac{1-(\frac{\alpha^2}{|r|^2})^n}{1-(\frac{\alpha^2}{|r|^2})} + \frac{1-(\frac{\beta^2}{|r|^2})^n}{1-(\frac{\beta^2}{|r|^2})} - 2\frac{1-(\frac{-1}{|r|^2})^n}{1+\frac{1}{|r|^2}}\right) \\ &= \frac{n|r|^2}{p^2+4} \sqrt{\frac{2|r|^{2n+2} - |r|^{2n}(p^2+2) - |r|^2 V_{2n} + V_{2n-2}}{|r|^4 - |r|^2(p^2+2) + 1} - 2\frac{|r|^{2n} - (-1)^n}{|r|^2+1}}, \end{aligned}$$

$$\frac{1}{\sqrt{n}}\|U_{r^*}\|_E \geq \frac{|r|}{\sqrt{p^2+4}}\sqrt{\frac{2|r|^{2n+2}-|r|^{2n}(p^2+2)-|r|^2V_{2n}+V_{2n-2}}{|r|^4-|r|^2(p^2+2)+1}-2\frac{|r|^{2n}-(-1)^n}{|r|^2+1}}.$$

From (5)

$$\frac{|r|}{\sqrt{p^2+4}}\sqrt{\frac{2|r|^{2n+2}-|r|^{2n}(p^2+2)-|r|^2V_{2n}+V_{2n-2}}{|r|^4-|r|^2(p^2+2)+1}-2\frac{|r|^{2n}-(-1)^n}{|r|^2+1}} \leq \|U_{r^*}\|_2.$$

On the other hand, let the matrices A and B be defined by

$$A = \begin{pmatrix} U_0 & 1 & 1 & \dots & 1 & 1 \\ r & U_0 & 1 & \dots & 1 & 1 \\ r^2 & r & U_0 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ r^{n-1} & r^{n-2} & r^{n-3} & \dots & r & U_0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} U_0 & U_1 & U_2 & \dots & U_{n-2} & U_{n-1} \\ U_{n-1} & U_0 & U_1 & \dots & U_{n-3} & U_{n-2} \\ U_{n-2} & U_{n-1} & U_0 & \dots & U_{n-4} & U_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ U_1 & U_2 & U_3 & \dots & U_{n-1} & U_0 \end{pmatrix}.$$

That is, $U_{r^*} = A \circ B$. Then we obtain

$$\begin{aligned} r_1(A) &= \max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n}|a_{ij}|^2} \\ &= \sqrt{U_0^2+n-1} \\ &= \sqrt{n-1} \end{aligned}$$

and

$$\begin{aligned} c_1(B) &= \max_{1\leq j\leq n}\sqrt{\sum_{i=1}^{n}|b_{ij}|^2} \\ &= \sqrt{\sum_{k=0}^{n-1}U_k^2} \\ &= \sqrt{\frac{U_n^2-U_{n-1}^2+(-1)^n}{p^2}}. \end{aligned}$$

Hence, from Lemma 1, we have

$$\|U_{r^*}\|_2 \leq \sqrt{\frac{(n-1)(U_n^2-U_{n-1}^2+(-1)^n)}{p^2}}.$$

Thus, we have

$$\frac{|r|}{\sqrt{p^2+4}}\sqrt{\frac{2|r|^{2n+2}-|r|^{2n}(p^2+2)-|r|^2V_{2n}+V_{2n-2}}{|r|^4-|r|^2(p^2+2)+1}-2\frac{|r|^{2n}-(-1)^n}{|r|^2+1}}$$
$$\leq \|U_{r^*}\|_2 \leq \sqrt{\frac{(n-1)(U_n^2-U_{n-1}^2+(-1)^n)}{p^2}}.$$ □

Theorem 2 *Let* $V_{r^*}=\mathrm{Circ}_{r^*}(V_0,V_1,V_2,\dots,V_{n-1})$ *be an* $n\times n$ *geometric circulant matrix.*

(i) *If* $|r|>1$, *then*

$$\sqrt{\frac{V_n^2-V_{n-1}^2+p^2-4+(1-(-1)^n)(p^2+4)}{p^2}}$$
$$\leq \|V_{r^*}\|_2 \leq \sqrt{\frac{1-|r|^{2n}}{1-|r|^2}\frac{V_n^2-V_{n-1}^2+p^2-4+(1-(-1)^n)(p^2+4)}{p^2}}.$$

(ii) *If* $|r|<1$, *then*

$$|r|\sqrt{\frac{2|r|^{2n+2}-|r|^{2n}(p^2+2)-|r|^2V_{2n}+V_{2n-2}}{|r|^4-|r|^2(p^2+2)+1}+2\frac{|r|^{2n}-(-1)^n}{|r|^2+1}}$$
$$\leq \|V_{r^*}\|_2 \leq \sqrt{\frac{n(V_n^2-V_{n-1}^2+p^2-4+(1-(-1)^n)(p^2+4))}{p^2}}.$$

Proof We have the matrix

$$V_{r^*}=\begin{pmatrix} V_0 & V_1 & V_2 & \dots & V_{n-2} & V_{n-1}\\ rV_{n-1} & V_0 & V_1 & \dots & V_{n-3} & V_{n-2}\\ r^2V_{n-2} & rV_{n-1} & V_0 & \dots & V_{n-4} & V_{n-3}\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ r^{n-1}V_1 & r^{n-2}V_2 & r^{n-3}V_3 & \dots & rV_{n-1} & V_0 \end{pmatrix}.$$

(i) From $|r|>1$, we have

$$\begin{aligned}\|V_{r^*}\|_E^2 &= \sum_{k=0}^{n-1}(n-k)V_k^2+\sum_{k=1}^{n-1}k\left|r^{n-k}\right|^2V_k^2\\ &\geq n\sum_{k=0}^{n-1}V_k^2\\ &= \frac{n(V_n^2-V_{n-1}^2+p^2-4+(1-(-1)^n)(p^2+4))}{p^2},\end{aligned}$$

that is,

$$\frac{1}{\sqrt{n}}\|V_{r^*}\|_E \geq \sqrt{\frac{V_n^2-V_{n-1}^2+p^2-4+(1-(-1)^n)(p^2+4)}{p^2}},$$

from (5), we have

$$\sqrt{\frac{V_n^2 - V_{n-1}^2 + p^2 - 4 + (1-(-1)^n)(p^2+4)}{p^2}} \leq \|V_{r^*}\|_2.$$

On the other hand, let the matrices A and B be defined by

$$A = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ r & 1 & 1 & \dots & 1 & 1 \\ r^2 & r & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ r^{n-1} & r^{n-2} & r^{n-3} & \dots & r & 1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} V_0 & V_1 & V_2 & \dots & V_{n-2} & V_{n-1} \\ V_{n-1} & V_0 & V_1 & \dots & V_{n-3} & V_{n-2} \\ V_{n-2} & V_{n-1} & V_0 & \dots & V_{n-4} & V_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ V_1 & V_2 & V_3 & \dots & V_{n-1} & V_0 \end{pmatrix}.$$

That is, $V_{r^*} = A \circ B$. Then we obtain

$$\begin{aligned} r_1(A) &= \max_{1\leq i\leq n} \sqrt{\sum_{j=1}^{n} |a_{ij}|^2} \\ &= \sqrt{1 + \left|r^1\right|^2 + \cdots + \left|r^{n-1}\right|^2} \\ &= \sqrt{\frac{1-|r|^{2n}}{1-|r|^2}} \end{aligned}$$

and

$$\begin{aligned} c_1(B) &= \max_{1\leq j\leq n} \sqrt{\sum_{i=1}^{n} |b_{ij}|^2} \\ &= \sqrt{\sum_{k=0}^{n-1} V_k^2} \\ &= \sqrt{\frac{V_n^2 - V_{n-1}^2 + p^2 - 4 + (1-(-1)^n)(p^2+4)}{p^2}}. \end{aligned}$$

From Lemma 1, we have

$$\|V_{r^*}\|_2 \leq \sqrt{\frac{1-|r|^{2n}}{1-|r|^2} \frac{V_n^2 - V_{n-1}^2 + p^2 - 4 + (1-(-1)^n)(p^2+4)}{p^2}}.$$

Thus, we have

$$\begin{aligned}&\sqrt{\frac{V_n^2 - V_{n-1}^2 + p^2 - 4 + (1-(-1)^n)(p^2+4)}{p^2}} \\ &\quad \le \| V_{r^*} \|_2 \le \sqrt{\frac{1-|r|^{2n}}{1-|r|^2}\frac{V_n^2 - V_{n-1}^2 + p^2 - 4 + (1-(-1)^n)(p^2+4)}{p^2}}.\end{aligned}$$

(ii) From $|r| < 1$, we have

$$\begin{aligned}\| V_{r^*} \|_E^2 &= \sum_{k=0}^{n-1}(n-k)V_k^2 + \sum_{k=1}^{n-1} k\left|r^{n-k}\right|^2 V_k^2 \\ &\ge \sum_{k=0}^{n-1}(n-k)\left|r^{n-k}\right|^2 V_k^2 + \sum_{k=1}^{n-1} k\left|r^{n-k}\right|^2 V_k^2 \\ &= n|r|^{2n}\sum_{k=0}^{n-1}\left(\frac{V_k}{|r|^k}\right)^2 \\ &= n|r|^{2n}\left(\sum_{k=0}^{n-1}\frac{\alpha^{2k}}{|r|^{2k}} + \sum_{k=0}^{n-1}\frac{\beta^{2k}}{|r|^{2k}} + 2\sum_{k=0}^{n-1}\frac{(-1)^k}{|r|^{2k}}\right) \\ &= n|r|^{2n}\left(\frac{1-(\frac{\alpha^2}{|r|^2})^n}{1-(\frac{\alpha^2}{|r|^2})} + \frac{1-(\frac{\beta^2}{|r|^2})^n}{1-(\frac{\beta^2}{|r|^2})} + 2\frac{1-(\frac{-1}{|r|^2})^n}{1+\frac{1}{|r|^2}}\right) \\ &= n|r|^2\left(\frac{2|r|^{2n+2} - |r|^{2n}(p^2+2) - |r|^2 V_{2n} + V_{2n-2}}{|r|^4 - |r|^2(p^2+2)+1} + 2\frac{|r|^{2n}-(-1)^n}{|r|^2+1}\right), \\ \frac{1}{\sqrt{n}}\| V_{r^*} \|_E &\ge |r|\sqrt{\frac{2|r|^{2n+2} - |r|^{2n}(p^2+2) - |r|^2 V_{2n} + V_{2n-2}}{|r|^4 - |r|^2(p^2+2)+1} + 2\frac{|r|^{2n}-(-1)^n}{|r|^2+1}}.\end{aligned}$$

From (5),

$$|r|\sqrt{\frac{2|r|^{2n+2} - |r|^{2n}(p^2+2) - |r|^2 V_{2n} + V_{2n-2}}{|r|^4 - |r|^2(p^2+2)+1} + 2\frac{|r|^{2n}-(-1)^n}{|r|^2+1}} \le \| V_{r^*} \|_2.$$

On the other hand, let the matrices A and B be defined by

$$A = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 & 1 \\ r & 1 & 1 & \ldots & 1 & 1 \\ r^2 & r & 1 & \ldots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ r^{n-1} & r^{n-2} & r^{n-3} & \ldots & r & 1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} V_0 & V_1 & V_2 & \dots & V_{n-2} & V_{n-1} \\ V_{n-1} & V_0 & V_1 & \dots & V_{n-3} & V_{n-2} \\ V_{n-2} & V_{n-1} & V_0 & \dots & V_{n-4} & V_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ V_1 & V_2 & V_3 & \dots & V_{n-1} & V_0 \end{pmatrix}.$$

That is, $V_{r^*} = A \circ B$. Then we obtain

$$r_1(A) = \max_{1\leq i\leq n} \sqrt{\sum_{j=1}^{n} |a_{ij}|^2} = \sqrt{n}$$

and

$$\begin{aligned} c_1(B) &= \max_{1\leq j\leq n} \sqrt{\sum_{i=1}^{n} |b_{ij}|^2} \\ &= \sqrt{\sum_{k=0}^{n-1} V_k^2} \\ &= \sqrt{\frac{V_n^2 - V_{n-1}^2 + p^2 - 4 + (1-(-1)^n)(p^2+4)}{p^2}}. \end{aligned}$$

From Lemma 1, we have

$$\| V_{r^*} \|_2 \leq \sqrt{\frac{n(V_n^2 - V_{n-1}^2 + p^2 - 4 + (1-(-1)^n)(p^2+4))}{p^2}}.$$

Thus we have

$$\begin{aligned} &|r| \sqrt{\frac{2|r|^{2n+2} - |r|^{2n}(p^2+2) - |r|^2 V_{2n} + V_{2n-2}}{|r|^4 - |r|^2(p^2+2) + 1} + 2\frac{|r|^{2n} - (-1)^n}{|r|^2+1}} \\ &\quad \leq \| V_{r^*} \|_2 \leq \sqrt{\frac{n(V_n^2 - V_{n-1}^2 + p^2 - 4 + (1-(-1)^n)(p^2+4))}{p^2}}. \end{aligned}$$

□

Theorem 3 *Let* $\mathbb{F}_{r^*}^{(k)} = \mathrm{Circ}_{r^*}(\mathbb{F}_0^{(k)}, \mathbb{F}_1^{(k)}, \mathbb{F}_2^{(k)}, \dots, \mathbb{F}_{n-1}^{(k)})$ *be an* $n \times n$ *geometric circulant matrix.*

(i) *If* $|r| > 1$, *then*

$$\frac{1}{\sqrt{n}} \mathbb{F}_{n-1}^{(k+1)} \leq \left\| \mathbb{F}_{r^*}^{(k)} \right\|_2 \leq \sqrt{\frac{|r|^2 - |r|^{2n}}{1 - |r|^2}} \mathbb{F}_{n-1}^{(k+1)}.$$

(ii) *If* $|r| < 1$, *then*

$$\frac{|r|^n}{\sqrt{n}} \mathbb{F}_{n-1}^{(k+1)} \leq \left\| \mathbb{F}_{r^*}^{(k)} \right\|_2 \leq \sqrt{n-1} \mathbb{F}_{n-1}^{(k+1)}.$$

Proof Since the $\mathbb{F}_{r^*}^{(k)}$ is of the form

$$\mathbb{F}_{r^*}^{(k)} = \begin{pmatrix} \mathbb{F}_0^{(k)} & \mathbb{F}_1^{(k)} & \mathbb{F}_2^{(k)} & \dots & \mathbb{F}_{n-2}^{(k)} & \mathbb{F}_{n-1}^{(k)} \\ r\mathbb{F}_{n-1}^{(k)} & \mathbb{F}_0^{(k)} & \mathbb{F}_1^{(k)} & \dots & \mathbb{F}_{n-3}^{(k)} & \mathbb{F}_{n-2}^{(k)} \\ r^2\mathbb{F}_{n-2}^{(k)} & r\mathbb{F}_{n-1}^{(k)} & \mathbb{F}_0^{(k)} & \dots & \mathbb{F}_{n-4}^{(k)} & \mathbb{F}_{n-3}^{(k)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ r^{n-1}\mathbb{F}_1^{(k)} & r^{n-2}\mathbb{F}_2^{(k)} & r^{n-3}\mathbb{F}_3^{(k)} & \dots & r\mathbb{F}_{n-1}^{(k)} & \mathbb{F}_0^{(k)} \end{pmatrix}$$

and from the definition of Euclidean norm, we have

$$\left\|\mathbb{F}_{r^*}^{(k)}\right\|_E^2 = \sum_{s=0}^{n-1}(n-s)\left(\mathbb{F}_s^{(k)}\right)^2 + \sum_{s=1}^{n-1} s\left|r^{n-s}\right|^2\left(\mathbb{F}_s^{(k)}\right)^2.$$

(i) From $|r| > 1$, we have

$$\begin{aligned}\left\|\mathbb{F}_{r^*}^{(k)}\right\|_E^2 &\geq \sum_{s=0}^{n-1}(n-s)\left(\mathbb{F}_s^{(k)}\right)^2 + \sum_{s=1}^{n-1} s\left(\mathbb{F}_s^{(k)}\right)^2 \\ &= n\sum_{s=0}^{n-1}\left(\mathbb{F}_s^{(k)}\right)^2.\end{aligned}$$

Thus from (5) and (4),

$$\frac{1}{\sqrt{n}}\mathbb{F}_{n-1}^{(k+1)} \leq \left\|\mathbb{F}_{r^*}^{(k)}\right\|_2.$$

On the other hand let the matrices $A^{(k)}$ and $B^{(k)}$ be defined by

$$A^{(k)} = \begin{pmatrix} \mathbb{F}_0^{(k)} & 1 & 1 & \dots & 1 & 1 \\ r & \mathbb{F}_0^{(k)} & 1 & \dots & 1 & 1 \\ r^2 & r & \mathbb{F}_0^{(k)} & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ r^{n-1} & r^{n-2} & r^{n-3} & \dots & r & \mathbb{F}_0^{(k)} \end{pmatrix}$$

and

$$B^{(k)} = \begin{pmatrix} \mathbb{F}_0^{(k)} & \mathbb{F}_1^{(k)} & \mathbb{F}_2^{(k)} & \dots & \mathbb{F}_{n-2}^{(k)} & \mathbb{F}_{n-1}^{(k)} \\ \mathbb{F}_{n-1}^{(k)} & \mathbb{F}_0^{(k)} & \mathbb{F}_1^{(k)} & \dots & \mathbb{F}_{n-3}^{(k)} & \mathbb{F}_{n-2}^{(k)} \\ \mathbb{F}_{n-2}^{(k)} & \mathbb{F}_{n-1}^{(k)} & \mathbb{F}_0^{(k)} & \dots & \mathbb{F}_{n-4}^{(k)} & \mathbb{F}_{n-3}^{(k)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \mathbb{F}_1^{(k)} & \mathbb{F}_2^{(k)} & \mathbb{F}_3^{(k)} & \dots & \mathbb{F}_{n-1}^{(k)} & \mathbb{F}_0^{(k)} \end{pmatrix}.$$

That is, $\mathbb{F}_{r^*}^{(k)} = A^{(k)} \circ B^{(k)}$. Then we obtain

$$r_1\left(A^{(k)}\right) = \max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n}\left|a_{ij}^{(k)}\right|^2} = \sqrt{\frac{|r|^2 - |r|^{2n}}{1 - |r|^2}}$$

and

$$c_1\left(B^{(k)}\right) = \max_{1\le j\le n}\sqrt{\sum_{i=1}^{n}\left|b_{ij}^{(k)}\right|^2} = \sqrt{\sum_{s=0}^{n-1}\left(\mathbb{F}_s^{(k)}\right)^2}.$$

From Lemma 1 and (4), we have

$$\left\|\mathbb{F}_{r^*}^{(k)}\right\|_2 \le \sqrt{\frac{|r|^2-|r|^{2n}}{1-|r|^2}}\mathbb{F}_{n-1}^{(k+1)},$$

which is desired result.

(ii) From $|r|<1$, we have

$$\begin{aligned}\left\|\mathbb{F}_{r^*}^{(k)}\right\|_E^2 &\ge \sum_{s=0}^{n-1}(n-s)\left|r^{n-s}\right|^2\left(\mathbb{F}_s^{(k)}\right)^2 + \sum_{s=1}^{n-1}s\left|r^{n-s}\right|^2\left(\mathbb{F}_s^{(k)}\right)^2\\ &\ge n|r|^{2n}\sum_{s=0}^{n-1}\left(\mathbb{F}_s^{(k)}\right)^2.\end{aligned}$$

From (5) and (4),

$$\frac{|r|^n}{\sqrt{n}}\mathbb{F}_{n-1}^{(k+1)} \le \left\|\mathbb{F}_{r^*}^{(k)}\right\|_2.$$

On the other hand, let the matrices $A^{(k)}$ and $B^{(k)}$ be defined by

$$A^{(k)} = \begin{pmatrix} \mathbb{F}_0^{(k)} & 1 & 1 & \dots & 1 & 1\\ r & \mathbb{F}_0^{(k)} & 1 & \dots & 1 & 1\\ r^2 & r & \mathbb{F}_0^{(k)} & \dots & 1 & 1\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ r^{n-1} & r^{n-2} & r^{n-3} & \dots & r & \mathbb{F}_0^{(k)}\end{pmatrix}$$

and

$$B^{(k)} = \begin{pmatrix} \mathbb{F}_0^{(k)} & \mathbb{F}_1^{(k)} & \mathbb{F}_2^{(k)} & \dots & \mathbb{F}_{n-2}^{(k)} & \mathbb{F}_{n-1}^{(k)}\\ \mathbb{F}_{n-1}^{(k)} & \mathbb{F}_0^{(k)} & \mathbb{F}_1^{(k)} & \dots & \mathbb{F}_{n-3}^{(k)} & \mathbb{F}_{n-2}^{(k)}\\ \mathbb{F}_{n-2}^{(k)} & \mathbb{F}_{n-1}^{(k)} & \mathbb{F}_0^{(k)} & \dots & \mathbb{F}_{n-4}^{(k)} & \mathbb{F}_{n-3}^{(k)}\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ \mathbb{F}_1^{(k)} & \mathbb{F}_2^{(k)} & \mathbb{F}_3^{(k)} & \dots & \mathbb{F}_{n-1}^{(k)} & \mathbb{F}_0^{(k)}\end{pmatrix}.$$

That is, $\mathbb{F}_{r^*}^{(k)} = A^{(k)} \circ B^{(k)}$. Then we obtain

$$r_1\left(A^{(k)}\right) = \max_{1\le i\le n}\sqrt{\sum_{j=1}^{n}\left|a_{ij}^{(k)}\right|^2} = \sqrt{n-1}$$

and

$$c_1\left(B^{(k)}\right) = \max_{1\le j\le n} \sqrt{\sum_{i=1}^{n} \left|b_{ij}^{(k)}\right|^2} = \sqrt{\sum_{s=0}^{n-1} \left(\mathbb{F}_s^{(k)}\right)^2}.$$

From Lemma 1 and (4), we have

$$\left\|\mathbb{F}_{r^*}^{(k)}\right\|_2 \le \sqrt{n-1}\,\mathbb{F}_{n-1}^{(k+1)}.$$

Thus we have

$$\frac{|r|^n}{\sqrt{n}}\mathbb{F}_{n-1}^{(k+1)} \le \left\|\mathbb{F}_{r^*}^{(k)}\right\|_2 \le \sqrt{n-1}\,\mathbb{F}_{n-1}^{(k+1)}. \qquad \square$$

4 Conclusion

In this paper we approximated lower and upper bounds of the spectral norms of geometric circulant matrices with the generalized Fibonacci and hyperharmonic Fibonacci numbers. If we take $p = 1$ and $p = 2$ in Theorem 1, we obtain lower and upper bounds of the spectral norms of geometric circulant matrices with the Fibonacci and Pell numbers, respectively. Similarly if we take $p = 1$ and $p = 2$ in Theorem 2, we obtain lower and upper bounds of the spectral norms of geometric circulant matrices with the Lucas and Pell-Lucas numbers, respectively.

In the future it may be possible that one can generalize our results to the Horadam, tribonacci and tribonacci-like sequences.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors contributed to each part of this work equally and read and approved the final version of the manuscript.

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