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On the general sum-connectivity index and general Randić index of cacti

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Abstract

Let G be a connected graph. The degree of a vertex x of G, denoted by $d_G(x)$, is the number of edges adjacent to x. The general sum-connectivity index is the sum of the weights $(d_G(x) + d_G(y))^\alpha$ for all edges xy of G, where α is a real number. The general Randić index is the sum of weights of $(d_G(x)d_G(y))^\alpha$ for all edges xy of G, where α is a real number. The graph G is a cactus if each block of G is either a cycle or an edge. In this paper, we find sharp lower bounds on the general sum-connectivity index and general Randić index of cacti.

MSC: 05C90

Keywords: cacti; general sum-connectivity index; general Randić index

1 Introduction

Let G be a finite molecular graph of order n and size m with vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex $x \in V(G)$, denoted by $d_G(x)$, is the number of edges adjacent to x. A vertex with degree 1 is called a pendent vertex. The minimum and maximum degrees of G are respectively defined by $\triangle_G = \max\{d_G(x) : x \in V(G)\}$ and $\delta_G = \min\{d_G(x) : x \in V(G)\}$. The set of neighboring vertices of a vertex x is denoted by $N_G(x)$. The graph obtained by deleting a vertex $x \in V(G)$ is denoted by $G - x$. The graph obtained from G by adding an edge xy between two nonadjacent vertices $x, y \in V(G)$ is denoted by $G + xy$.

Historically, the first degree based topological indices are the Zagreb indices, introduced by Gutman and Trinajestić [1]. The first Zagreb index $M_1(G)$ and the second Zagreb index $M_2(G)$ are defined as

$$M_1(G) = \sum_{x \in V(G)} \big(d_G(x)\big)^2 \quad \text{and} \quad M_2(G) = \sum_{xy \in E(G)} d_G(x)d_G(y).$$

The general Randić index (or product-connectivity index) was proposed by Bollobás and Erdős [2] and is defined as follows:

$$R_\alpha(G) = \sum_{xy \in E(G)} \big(d_G(x)d_G(y)\big)^\alpha,$$

where α is a real number. Then $R_{-1/2}(G) = \sum_{xy \in E(G)} (d_G(x)d_G(y))^{-\frac{1}{2}}$ is called the Randić index, which was defined by Randić [3], and R_1 is the second Zagreb index. The Randić index is the most studied, most applied, and most important index among all topologi-

cal indices. Recently, Zhou and Trinajstić [4] modified the concept of Randić index and obtained a new index, called the general sum-connectivity index and defined as follows:

$$\chi_\alpha(G) = \sum_{xy \in E(G)} \left(d_G(x) + d_G(y)\right)^\alpha,$$

where α is a real number. Then $\chi_{-1/2}(G) = \sum_{xy \in E(G)} (d_G(x) + d_G(y))^{-\frac{1}{2}}$ is the sum-connectivity index defined by Zhou and Trinajstić [5], and if $\alpha = 1$, then the general sum-connectivity index becomes the first Zagreb index.

A connected graph G is a cactus if each block of G is either a cycle or an edge. Let $X(n, t)$ be the set of cacti of order n with t cycles. Obviously, $X(n, 0)$ is the set of trees of order n, and $X(n, 1)$ is the set of unicyclic graphs of order n. Denote by $X^0(n, t)$ the n-vertex cactus consisting of t triangles and $n - 2t - 1$ pendent edges such that triangles and pendent edges have exactly one vertex in common.

Lin et al. [6] discuss the sharp lower bounds of the Randić index of cacti with r pendent vertices. Dong and Wu [7] calculated the sharp bounds on the atom-bond connectivity index in the set of cacti with t cycles and also in the set of cacti with r pendent vertices, for all positive integral values of t and r. Li [8] determined the unique cactus with maximum atom-bond connectivity index among cacti with n vertices and t cycles, where $0 \le t \le \lfloor \frac{n-1}{2} \rfloor$, and also among n vertices and r pendent vertices, where $0 \le r \le n - 1$. Ma and Deng [9] calculated the sharp lower bounds of the sum-connectivity index in the set of cacti of order n with t cycles and also in the set of cacti of order $2n$ with perfect matching. Lu et al. [10] gave the sharp lower bound on the Randić index of cacti with t number of cycles.

The present paper is motivated by the results of papers [9, 10]. The goal of this paper is to compute the sharp lower bounds for the general sum-connectivity index and the general Randić index of cacti with fixed number of cycles.

2 General sum-connectivity index

In this section, we find sharp lower bound for the general sum-connectivity index of cacti. Let

$$F(n, t) = 2t(n + 1)^\alpha + (n - 2t - 1)n^\alpha + 4^\alpha t,$$

where $n \ge 3$ and $t \ge 0$. First, we give some lemmas that will be used in the main result.

Lemma 2.1 ([4]) *Let $n \ge 5$ be a positive integer, $\alpha < 0$ be a real number, and $G \in X(n, 0)$. Then*

$$\chi_\alpha(G) \ge (n - 1)n^\alpha.$$

Equality holds if and only if $G \cong X^0(n, 0)$.

Lemma 2.2 ([11]) *Let $n \ge 5$ be a positive integer, $-1 \le \alpha < 0$ be a real number, and $G \in X(n, 1)$. Then*

$$\chi_\alpha(G) \ge 2(n + 1)^\alpha + (n - 3)n^\alpha + 4^\alpha.$$

Equality holds if and only if $G \cong X^0(n, 1)$.

Lemma 2.3 *Let x, k be positive integers with $x \geq k \geq 1$, and $\alpha < 0$ be a real number. Define*

$$f(x) = x^\alpha - (x - k)^\alpha.$$

Then $f(x)$ is an increasing function.

Proof It is easily seen that $f'(x) = \alpha(x^{\alpha-1} - (x - k)^{\alpha-1}) > 0$ for $\alpha < 0$. Therefore, $f(x)$ is an increasing function. □

Lemma 2.4 *Let x, k be positive integers with $x \geq k \geq 1$, and $-1 \leq \alpha < 0$ be a real number. Define*

$$f(x) = k(x + 1)^\alpha + (x - k)(x + 2)^\alpha - (x - k)(x + 2 - k)^\alpha.$$

Then $f(x)$ is a decreasing function.

Proof Let $g(x) = k(x+1)^\alpha - k(x+2)^\alpha$ for $x \geq k \geq 1$. We get $g'(x) = \alpha k((x+1)^{\alpha-1} - (x+2)^{\alpha-1}) < 0$. So, $g(x)$ is a decreasing function.

Let $h(x) = x(x+2)^\alpha$ for $x \geq k \geq 1$. Then $h''(x) = \alpha((\alpha+1)x+4)(x+2)^{\alpha-2} < 0$ for $-1 \leq \alpha < 0$. Hence, $h(x) - h(x-k)$ is a decreasing function. Note that $f(x) = g(x) + (h(x) - h(x-k))$. Thus, $f(x)$ is a decreasing function. □

Lemma 2.5 *Let x, k be positive integers with $x \geq k \geq 1$, and $-1 \leq \alpha < 0$ be a real number. Define*

$$f(x) = x(x + 2)^\alpha - (x - 2)x^\alpha.$$

Then $f(x)$ is a decreasing function.

Proof Let $h(x) = x(x + 2)^\alpha$ for $x \geq k \geq 1$. Then $h''(x) = \alpha((\alpha + 1)x + 4)(x + 2)^{\alpha-2} < 0$ for $-1 \leq \alpha < 0$. But $f(x) = h(x) - h(x - 2)$. Thus, $f(x)$ is a decreasing function. □

Theorem 2.1 *Let $-1 \leq \alpha < 0$ be a real number, $n \geq 3$ be a positive integer, and $G \in X(n, t)$. Then*

$$\chi_\alpha(G) \geq F(n, t). \tag{2.1}$$

Equality holds if and only if $G \cong X^0(n, t)$.

Proof We use mathematical induction on n and t. If $t = 0$ or $t = 1$, then Lemmas 2.1 and 2.2 give inequality (2.1). If $n = 5$ and $t = 2$, then there is only one possibility that $X^0(5, 2)$ is a graph with two cycles that have only one common vertex (see Figure 1) and inequality (2.1) holds. If $n \geq 6$ and $t \geq 2$, then we will discuss the following two cases.

Case 1: $G \in X(n, t)$ has at least one pendent vertex.

Let $x \in V(G)$ be a pendent vertex and adjacent with vertex y of degree d, where $2 \leq d \leq n - 1$. The set of neighbors of y in G is $N_G(y) \setminus \{x\} = \{u_1, u_2, u_3, \ldots, u_{d-1}\}$. Without lost of generality, we assume that $u_1, u_2, u_3, \ldots, u_{k-1}$ are pendent vertices and $u_k, u_{k+1}, \ldots, u_{d-1}$

Figure 1 $X^0(5, 2)$.
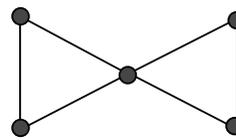
are nonpendent vertices, where $k \geq 1$. If $k = 1$, then $u_1, u_2, u_3, \ldots, u_{d-1}$ are nonpendent vertices. Obviously, we have $t \leq \lfloor \frac{n-k-1}{2} \rfloor$. If H is a graph obtained by deleting vertices $x, u_1, u_2, \ldots, u_{k-1}$ from G, that is, $H = G - x - u_1 - u_2 - \cdots - u_{k-1}$, then $H \in X(n - k, t)$. By inductive assumption and using Lemmas 2.3 and 2.4, we obtain

$$\chi_\alpha(G) = \chi_\alpha(H) + k(d+1)^\alpha + \sum_{i=k}^{d-1} \left(\left(d + d_G(u_i) \right)^\alpha - \left(d - k + d_G(u_i) \right)^\alpha \right)$$

$$\geq F(n-k,t) + k(d+1)^\alpha + \sum_{i=k}^{d-1} \left(\left(d + d_G(u_i) \right)^\alpha - \left(d - k + d_G(u_i) \right)^\alpha \right)$$

$$= F(n,t) + 2t(n-k+1)^\alpha - 2t(n+1)^\alpha + (n-k-2t-1)(n-k)^\alpha - (n-2t-1)n^\alpha$$

$$\quad + k(d+1)^\alpha + \sum_{i=k}^{d-1} \left(\left(d + d_G(u_i) \right)^\alpha - \left(d - k + d_G(u_i) \right)^\alpha \right)$$

$$\geq F(n,t) + 2t(n-k+1)^\alpha - 2t(n+1)^\alpha + (n-k-2t-1)(n-k)^\alpha - (n-2t-1)n^\alpha$$

$$\quad + k(d+1)^\alpha + (d-k)\left((d+2)^\alpha - (d-k+2)^\alpha \right)$$

$$\geq F(n,t) + 2t(n-k+1)^\alpha - 2t(n+1)^\alpha + (n-k-2t-1)(n-k)^\alpha - (n-2t-1)n^\alpha$$

$$\quad + kn^\alpha + (n-1-k)(n+1)^\alpha - (n-1-k)(n+1-k)^\alpha$$

$$= F(n,t) + (n-1-k-2t)\left((n+1)^\alpha - (n-k+1)^\alpha - n^\alpha + (n-k)^\alpha \right)$$

$$\geq F(n,t).$$

The equality holds if and only if $d = n - 1$, $2t = n - k - 1$, and $H \cong X^0(n - k, t)$. Therefore, we have that $\chi_\alpha(G) = F(n, t)$ if and only if $G \cong X^0(n, t)$.

Case 2: $G \in X(n, t)$ has no pendent vertex. Then we consider edges $x_0 x_1, x_0 x_2 \in E(G)$ such that x_0 and x_1 have degree 2 and x_2 has degree d, where $d \geq 3$. Now there arise two subcases.

Subcase 2.1: x_1 and x_2 are nonadjacent vertices in G. Then we construct a new graph H by deleting a vertex x_0 and adding an edge between x_1 and x_2, that is, $H = G - x_0 + x_1 x_2 \in X(n - 1, t)$. Then by inductive assumption we have

$$\chi_\alpha(G) = \chi_\alpha(H) + (d+2)^\alpha + 4^\alpha - (d+2)^\alpha$$

$$\geq F(n-1,t) + 4^\alpha$$

$$= F(n,t) + 2tn^\alpha - 2t(n+1)^\alpha + (n-2t-2)(n-1)^\alpha - (n-2t-1)n^\alpha + 4^\alpha$$

$$= F(n,t) + 2t\left(n^\alpha - (n-1)^\alpha \right) + (n-2t-2)\left((n-1)^\alpha - n^\alpha \right) + 4^\alpha - n^\alpha$$

$$> F(n,t).$$

The last inequality holds if $n \geq 7$. If $n = 6$, then $t \leq 2$, and we easily show the inequality.

Subcase 2.2: x_1 and x_2 are adjacent in G. Then we construct a new graph H by deleting two vertices x_0 and x_1, that is, $H = G - x_0 - x_1 \in X(n-2, t-1)$. Let the set of neighbors of x_2 be $N_G(x_2) \setminus \{x_0, x_1\} = \{u_1, u_2, \ldots, u_{d-2}\}$, where all $u_1, u_2, \ldots, u_{d-2}$ are nonpendent vertices. By inductive assumption and using Lemmas 2.3 and 2.5, we obtain

$$
\begin{aligned}
\chi_\alpha(G) &= \chi_\alpha(H) + 4^\alpha + 2(d+2)^\alpha + \sum_{i=1}^{d-2}\left(\left(d + d_G(u_i)\right)^\alpha - \left(d - 2 + d_G(u_i)\right)^\alpha\right) \\
&\geq F(n-2, t-1) + 4^\alpha + 2(d+2)^\alpha + \sum_{i=1}^{d-2}\left(\left(d + d_G(u_i)\right)^\alpha - \left(d - 2 + d_G(u_i)\right)^\alpha\right) \\
&= F(n,t) + 2(t-1)(n-1)^\alpha - 2t(n+1)^\alpha + (n-2t-1)(n-2)^\alpha - (n-2t-1)n^\alpha \\
&\quad + (t-1)4^\alpha - t4^\alpha + 4^\alpha + 2(d+2)^\alpha + \sum_{i=1}^{d-2}\left(\left(d + d_G(u_i)\right)^\alpha - \left(d - 2 + d_G(u_i)\right)^\alpha\right) \\
&= F(n,t) + 2(t-1)(n-1)^\alpha - 2t(n+1)^\alpha + (n-2t-1)(n-2)^\alpha - (n-2t-1)n^\alpha \\
&\quad + (t-1)4^\alpha - t4^\alpha + 4^\alpha + 2(d+2)^\alpha + (d-2)\left((d+2)^\alpha - d^\alpha\right) \\
&= F(n,t) + 2(t-1)(n-1)^\alpha - 2t(n+1)^\alpha + (n-2t-1)(n-2)^\alpha - (n-2t-1)n^\alpha \\
&\quad + d(d+2)^\alpha - (d-2)d^\alpha \\
&= F(n,t) + 2(t-1)(n-1)^\alpha - 2t(n+1)^\alpha + (n-2t-1)(n-2)^\alpha - (n-2t-1)n^\alpha \\
&\quad + (n-1)(n+1)^\alpha - (n-3)(n-1)^\alpha \\
&= F(n,t) + (n-1-2t)\left((n+1)^\alpha - (n-1)^\alpha + (n-2)^\alpha - n^\alpha\right) \\
&\geq F(n,t).
\end{aligned}
$$

The equality holds if and only if $d = n-1$, $2t = n-1$, and $H \cong X^0(n-2, t-1)$. Therefore, we have $\chi_\alpha(G) = F(n,t)$ with equality if and only if $G \cong X^0(n,t)$. □

3 General Randić index
In this section, we find sharp lower bound for the general Randić index of cacti. Let

$$
F_1(n,t) = 2^{\alpha+1}t(n-1) + t4^\alpha + (n-1-2t)(n-1)^\alpha,
$$

where $n \geq 3$ and $t \geq 0$. First, we give some lemmas that will be helpful in the proof of the main result.

Lemma 3.1 ([12]) *Let $n \geq 5$ be a positive integer, $\alpha < 0$ be a real number, and $G \in X(n, 0)$. Then*

$$
R_\alpha(G) \geq (n-1)^{\alpha+1}.
$$

Equality holds if and only if $G \cong X^0(n, 0)$.

Lemma 3.2 ([13]) *Let $n \geq 5$ be a positive integer, $-1 \leq \alpha < 0$ be a real number, and $G \in X(n, 1)$. Then*

$$R_\alpha(G) \geq (n-3)(n-1)^\alpha + 2^\alpha(n-1)^\alpha + 4^\alpha.$$

Equality holds if and only if $G \cong X^0(n, 1)$.

Lemma 3.3 *Let x, d, k be positive integers with $d \geq k \geq 1, x \geq 2$, and $\alpha < 0$ be a real number. Define*

$$f(x) = x^\alpha \left(d^\alpha - (d - k)^\alpha\right).$$

Then $f(x)$ is an increasing function.

Proof It is easily seen that $f'(x) = \alpha x^{\alpha-1}(d^\alpha - (d - k)^\alpha) > 0$ for $\alpha < 0$. Therefore, $f(x)$ is an increasing function. □

Lemma 3.4 *Let x, k be positive integers with $x \geq k \geq 1$, and $-1 \leq \alpha < 0$ be a real number. Define*

$$f(x) = kx^\alpha + 2^\alpha(x - k)x^\alpha - 2^\alpha(x - k)^{\alpha+1}.$$

Then $f(x)$ is a decreasing function.

Proof Let $g(x) = kx^\alpha(1 - 2^\alpha)$ for $x \geq k \geq 1$. We get $g'(x) = \alpha kx^{\alpha-1}(1 - 2^\alpha) < 0$. So, $g(x)$ is a decreasing function.

Let $h(x) = 2^\alpha x^{\alpha+1}$ for $x \geq k \geq 1$. Then $h''(x) = 2^\alpha(\alpha + 1)x^{\alpha-1} < 0$ for $-1 \leq \alpha < 0$. Hence, $h(x) - h(x - k)$ is a decreasing function. Note that $f(x) = g(x) + (h(x) - h(x - k))$. Thus, $f(x)$ is a decreasing function. □

Lemma 3.5 *Let x, k be positive integers with $x \geq k \geq 1$, and $-1 \leq \alpha < 0$ be a real number. Let*

$$f(x) = x^{\alpha+1} - (x - 2)^{\alpha+1}.$$

Then $f(x)$ is a decreasing function.

Proof Let $h(x) = x^{\alpha+1}$ for $x \geq k \geq 1$. Then $h''(x) = \alpha(\alpha + 1)x^{\alpha-1} < 0$ for $-1 \leq \alpha < 0$. But $f(x) = h(x) - h(x - 2)$. Thus $f(x)$ is a decreasing function. □

Theorem 3.1 *Let $-1 \leq \alpha < 0$ be a real number, $n \geq 3$ be a positive integer, and $G \in X(n, t)$. Then*

$$R_\alpha(G) \geq F_1(n, t). \tag{3.1}$$

Equality holds if and only if $G \cong X^0(n, t)$.

Proof We use mathematical induction on n and t. If $t = 0$ or $t = 1$, then inequality (3.1) holds by Lemma 3.1 and Lemma 3.2, respectively. If $n = 5$ and $t = 2$, then there is only one graph $X^0(5, 2)$, and the result for $X^0(5, 2)$ is trivial. Now if $n \geq 6$ and $t \geq 2$, then we will consider the following two cases.

Case 1: $G \in X(n,t)$ has at least one pendent vertex.

Let x be a vertex that is adjacent only with vertex y. Then y has degree d with $2 \leq d \leq n-1$. The set of neighbors of y in G is $N_G(y) \setminus \{x\} = \{u_1, u_2, u_3, \ldots, u_{d-1}\}$. Without lost of generality, we assume that $d_G(u_j) \geq 2$ for $k \leq j \leq d-1$, where $k \geq 1$, and the remaining neighboring vertices of y are pendent vertices. If $k = 1$, then G has no pendent vertex. Obviously, we have $t \leq \lfloor \frac{n-k-1}{2} \rfloor$. If H is a new graph obtained by deleting vertices $x, u_1, u_2, \ldots, u_{k-1}$, that is, $H = G - x - u_1 - u_2 - \cdots - u_{k-1}$, then $H \in X(n-k, t)$. By inductive assumption and using Lemmas 3.3 and 3.4, we obtain

$$
\begin{aligned}
R_\alpha(G) &= R_\alpha(H) + kd^\alpha + \sum_{j=k}^{d-1}\left(\left(dd_G(u_j)\right)^\alpha - (d-k)^\alpha d_G(u_j)^\alpha\right) \\
&\geq F_1(n-k, t) + kd^\alpha + \sum_{j=k}^{d-1}\left(\left(dd_G(u_j)\right)^\alpha - (d-k)^\alpha d_G(u_j)^\alpha\right) \\
&= F_1(n,t) + 2^{\alpha+1}t(n-k+1)^\alpha - 2^{\alpha+1}t(n-1)^\alpha + (n-k-2t-1)(n-k-1)^\alpha \\
&\quad - (n-2t-1)(n-1)^\alpha + kd^\alpha + \sum_{j=k}^{d-1}\left(\left(dd_G(u_j)\right)^\alpha - (d-k)^\alpha d_G(u_j)^\alpha\right) \\
&\geq F_1(n,t) + 2^{\alpha+1}t(n-k+1)^\alpha - 2^{\alpha+1}t(n-1)^\alpha + (n-k-2t-1)(n-k-1)^\alpha \\
&\quad - (n-2t-1)(n-1)^\alpha + kd^\alpha + (d-k)\left((2d)^\alpha - 2^\alpha(d-k)^\alpha\right) \\
&\geq F_1(n,t) + 2^{\alpha+1}t(n-k+1)^\alpha - 2^{\alpha+1}t(n-1)^\alpha + (n-k-2t-1)(n-k-1)^\alpha \\
&\quad - (n-2t-1)(n-1)^\alpha + k(n-1)^\alpha + 2^\alpha(n-1-k)(n-1)^\alpha - 2^\alpha(n+1-k)^{\alpha+1} \\
&= F_1(n,t) + (n-1-k-2t)\left(1-2^\alpha\right)\left((n-k-1)^\alpha - (n-1)^\alpha\right) \\
&\geq F_1(n,t).
\end{aligned}
$$

The equality holds if and only if $H \cong X^0(n-k, t)$, $d = n-1$, and $2t = n-k-1$. Therefore, we have that $R_\alpha(G) = F_1(n,t)$ if and only if $G \cong X^0(n,t)$.

Case 2: If G has no pendent vertex, then we consider $x_0 x_1, x_0 x_2$ edges of a cycle such that x_0 and x_1 are vertices of degree two and $d_G(x_2) = d$, where $d \geq 3$. Next, we discuss this in the following two subcases.

Subcase 2.1: If x_1 and x_2 are nonadjacent vertices in G, then H is a new graph obtained by deleting x_0 and adding an edge $x_1 x_2$, that is, $H = G - x_0 + x_1 x_2 \in X(n-1, t)$. Therefore, we get

$$
\begin{aligned}
R_\alpha(G) &= R_\alpha(H) + (2d)^\alpha + 4^\alpha - (2d)^\alpha \\
&\geq F_1(n-1, t) + 4^\alpha \\
&= F_1(n,t) + 2^{\alpha+1}t(n-2)^\alpha - 2^{\alpha+1}t(n-1)^\alpha + (n-2t-2)(n-2)^\alpha \\
&\quad - (n-2t-1)(n-1)^\alpha + 4^\alpha \\
&= F_1(n,t) + 2^{\alpha+1}t\left((n-2)^\alpha - (n-1)^\alpha\right) + (n-2t-2)\left((n-1)^\alpha - n^\alpha\right) \\
&\quad + 4^\alpha - (n-1)^\alpha \\
&> F_1(n,t).
\end{aligned}
$$

The last inequality holds if $n \geq 7$. If $n = 6$, then $t \leq 2$, and we easily show the inequality.

Subcase 2.2: If x_1 and x_2 are connected by an edge, then $H = G - x_0 - x_1 \in X(n-2, t-1)$. Let the set of neighbors of x_2 be $N_G(x_2) \setminus \{x_0, x_1\} = \{u_1, u_2, \ldots, u_{d-2}\}$, where u_j for $1 \leq j \leq d-2$ are nonpendent vertices. By inductive assumption and using Lemmas 3.3 and 3.5, we obtain

$$R_\alpha(G) = R_\alpha(H) + 4^\alpha + 2^{\alpha+1}d^\alpha + \sum_{j=1}^{d-2}\left(\left(dd_G(u_j)\right)^\alpha - (d-2)^\alpha d_G(u_j)^\alpha\right)$$

$$\geq F_1(n-2, t-1) + 4^\alpha + 2^{\alpha+1}d^\alpha + \sum_{j=1}^{d-2}\left(\left(dd_G(u_j)\right)^\alpha - (d-2)^\alpha d_G(u_j)^\alpha\right)$$

$$= F_1(n,t) + 2^{\alpha+1}(t-1)(n-3)^\alpha - 2^{\alpha+1}t(n-1)^\alpha + (n-2t-1)(n-3)^\alpha$$
$$\quad - (n-2t-1)(n-1)^\alpha + (t-1)4^\alpha - t4^\alpha + 4^\alpha + 2^{\alpha+1}d^\alpha$$
$$\quad + \sum_{j=1}^{d-2}\left(\left(dd_G(u_j)\right)^\alpha - (d-2)^\alpha + d_G(u_j)^\alpha\right)$$

$$= F_1(n,t)2^{\alpha+1}(t-1)(n-3)^\alpha - 2^{\alpha+1}t(n-1)^\alpha + (n-2t-1)(n-3)^\alpha$$
$$\quad - (n-2t-1)(n-1)^\alpha + (t-1)4^\alpha - t4^\alpha + 4^\alpha + 2^{\alpha+1}d^\alpha$$
$$\quad + 2^\alpha(d-2)\left(d^\alpha - (d-2)^\alpha\right)$$

$$= F_1(n,t) + 2^{\alpha+1}(t-1)(n-3)^\alpha - 2^{\alpha+1}t(n-1)^\alpha + (n-2t-1)(n-3)^\alpha$$
$$\quad - (n-2t-1)(n-1)^\alpha + 2^\alpha d^{\alpha+1} - 2^\alpha(d-2)^{\alpha+1}$$

$$= F_1(n,t) + 2^{\alpha+1}(t-1)(n-3)^\alpha - 2^{\alpha+1}t(n-1)^\alpha + (n-2t-1)(n-3)^\alpha$$
$$\quad - (n-2t-1)(n-1)^\alpha + 2^\alpha(n-1)^\alpha - 2^\alpha(n-3)^\alpha$$

$$= F_1(n,t) + (n-1-2t)\left(1-2^\alpha\right)\left((n-3)^\alpha - (n-1)^\alpha\right)$$

$$\geq F_1(n,t).$$

The equality holds if and only if $H \cong X^0(n-2, t-1)$, $d = n-1$, and $2t = n-1$. Therefore, we have that $R_\alpha(G) = F_1(n,t)$ if and only if $G \cong X^0(n,t)$. $\square$

4 Conclusion

In this paper, we determined the sharp lower bounds for the general sum-connectivity index and the general Randić index of cacti with fixed number of cycles for $-1 \leq \alpha < 0$. The general sum-connectivity index and general Randić index of cacti for other values of α remains an open problem. Moreover, some topological indices and polynomials are still unknown for cacti with fixed number of cycles and fixed number of pendent vertices.

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