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# On the general $K$-interpolation method for the sum and the intersection

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**Abstract**

Let $(A_0, A_1)$ be a compatible couple of normed spaces. We study the interrelation of the general $K$-interpolation spaces of the couple $(A_0 + A_1, A_0 \cap A_1)$ with those of the couples $(A_0, A_1)$, $(A_0 + A_1, A_0)$, $(A_0 + A_1, A_1)$, $(A_0, A_0 \cap A_1)$, and $(A_1, A_0 \cap A_1)$.

**MSC:** 46B70

**Keywords:** sum space; intersection space; $K$-functional; the general $K$-interpolation method; limiting $K$-interpolation methods

## 1 Introduction

Let $(A_0, A_1)$ be a compatible couple of normed spaces, *i.e.* we assume that both $A_0$ and $A_1$ are continuously embedded in a topological vector space $\mathcal{A}$. The sum of $A_0$ and $A_1$, denoted by $A_0 + A_1$, is the set of elements $f \in \mathcal{A}$ that can be represented as $f = f_0 + f_1$ where $f_0 \in A_0$ and $f_1 \in A_1$. The norm on the sum space $A_0 + A_1$ is given by

$$\|f\|_{A_0+A_1} = \inf\left\{\|f_0\|_{A_0} + \|f_1\|_{A_1} : f_0 \in A_0, f_1 \in A_1, f = f_0 + f_1\right\}.$$

The norm on the intersection space $A_0 \cap A_1$ is given by

$$\|f\|_{A_0\cap A_1} = \max\left\{\|f\|_{A_0}, \|f\|_{A_1}\right\}.$$

The Peetre's $K$-functional is defined, for each $f \in A_0 + A_1$ and $t > 0$, by

$$K(t, f; A_0, A_1) = \inf\left\{\|f_0\|_{A_0} + t\|f_1\|_{A_1} : f_0 \in A_0, f_1 \in A_1, f = f_0 + f_1\right\}.$$

Let $\Phi$ be a normed space of Lebesgue measurable functions, defined on $(0, \infty)$, with monotone norm: $|g| \le |h|$ implies $\|g\|_\Phi \le \|h\|_\Phi$. Further assume that

$$t \longmapsto \min\{1, t\} \in \Phi. \tag{1.1}$$

By definition, the general $K$-interpolation space $(A_0, A_1)_\Phi$ is a subspace of $A_0 + A_1$ having the following norm:

$$\|f\|_{(A_0,A_1)_\Phi} = \left\|K(t, f; A_0, A_1)\right\|_\Phi.$$

Here $\Phi$ is often termed the parameter of the $K$-interpolation method. We refer to [1] for a complete treatment of the general $K$-interpolation method.

Set

$$\Gamma = \big([0,1] \times [1,\infty]\big) \setminus \big(\{0,1\} \times [1,\infty)\big).$$

Let $(\theta,p) \in \Gamma$, then the classical scale of $K$-interpolation spaces $(A_0,A_1)_{\theta,q}$ (see [2] or [3]) is obtained when $\Phi$ is taken to be the weighted Lebesgue space $L_q(t^{-\theta})$ defined by the norm

$$\|g\|_\Phi = \begin{cases} (\int_0^\infty t^{-\theta}|g(t)|^p \frac{dt}{t})^{1/p}, & (\theta,p) \in (0,1) \times [1,\infty), \\ \sup_{0<t<\infty} t^{-\theta}|g(t)|, & (\theta,p) \in [0,1] \times \{\infty\}. \end{cases}$$

The following identity was proved by Maligranda [4]:

$$(A_0 + A_1, A_0 \cap A_1)_{\theta,p} = \begin{cases} (A_0,A_1)_{\theta,p} + (A_0,A_1)_{1-\theta,p}, & (\theta,p) \in \Gamma_1, \\ (A_0,A_1)_{\theta,p} \cap (A_0,A_1)_{1-\theta,p}, & (\theta,p) \in \Gamma_2, \end{cases} \tag{1.2}$$

where

$$\Gamma_1 = \big([0,1/2\big) \times [1,\infty]) \setminus \big(\{0\} \times [1,\infty)\big)$$

and

$$\Gamma_2 = \big([1/2,1] \times [1,\infty]\big) \setminus \big(\{1\} \times [1,\infty)\big).$$

Subsequently, Maligranda [5] considered the $K$-interpolation spaces $(A_0,A_1)_{\varrho,p}$, which are obtained when $\Phi$ is given by

$$\|g\|_\Phi = \begin{cases} (\int_0^\infty (\frac{|g(t)|}{\varrho(t)})^p \frac{dt}{t})^{1/p}, & 1 \le p < \infty, \\ \sup_{0<t<\infty} \frac{|g(t)|}{\varrho(t)}, & p = \infty, \end{cases}$$

and extended the identity (1.2) by imposing certain monotonicity conditions on the parameter function $\varrho$. Another related identity, proved by Persson [6], states that

$$(A_0 + A_1, A_0 \cap A_1)_{\varrho,p} = (A_0 + A_1, A_0)_{\varrho,p} \cap (A_0 + A_1, A_1)_{\varrho,p}.$$

Recently, Haase [7] has completely described how the classical $K$-interpolation spaces for the couples $(A_0,A_1)$, $(A_0 + A_1,A_0)$, $(A_0 + A_1,A_1)$, $(A_0,A_0 \cap A_1)$, $(A_1,A_0 \cap A_1)$, and $(A_0 + A_1,A_0 \cap A_1)$ interrelate. The assertions (1.5)-(1.12) in [7], Theorem 1.1, concern the spaces $(A_0 + A_1, A_0 \cap A_1)_{\theta,p}$, and the goal of this paper is to extend these assertions by means of replacing the classical scale $(A_0,A_1)_{\theta,p}$ by the general scale $(A_0,A_1)_\Phi$.

The main ingredient of our proofs will be the estimate in Proposition 2.4 (see below) which relates the $K$-functional of the couple $(A_0 + A_1, A_0 \cap A_1)$ with that of the original couple $(A_0,A_1)$, whereas this estimate has not been used in [7]. Consequently, our arguments of the proofs are different from those in [7].

We will also apply our general results to the limiting $K$-interpolation spaces $(A_0,A_1)_{0,p;K}$ and $(A_0,A_1)_{1,p;K}$ recently introduced by Cobos, Fernández-Cabrera, and Silvestre [8]. Namely, if the parameter spaces $\Phi_0$ and $\Phi_1$ are given by the norms

$$\|g\|_{\Phi_0} = \left(\int_0^1 |g(s)|^p \frac{ds}{s}\right)^{\frac{1}{p}} + \sup_{s>1}|g(s)| \tag{1.3}$$

and

$$\|g\|_{\Phi_1} = \sup_{0<s<1} \frac{|g(s)|}{s} + \left(\int_1^\infty \left(\frac{|g(s)|}{s}\right)^p \frac{ds}{s}\right)^{\frac{1}{p}}, \tag{1.4}$$

where $1 \le p < \infty$, then $(A_0,A_1)_{\Phi_0} = (A_0,A_1)_{0,p;K}$ and $(A_0,A_1)_{\Phi_1} = (A_0,A_1)_{1,p;K}$. Note that, for limiting values $\theta = 0,1$, the space $(A_0,A_1)_{\theta,p}$ is trivial (containing only zero element) when $p$ is finite. The space $(A_0,A_1)_{0,p;K}$ corresponds to the limiting value $\theta = 0$, and the space $(A_0,A_1)_{1,p;K}$ corresponds to the limiting value $\theta = 1$. We will, for convenience, write $(A_0,A_1)_{\{0\},p}$ for $(A_0,A_1)_{0,p;K}$, and $(A_0,A_1)_{\{1\},p}$ for $(A_0,A_1)_{1,p;K}$.

The paper is organised as follows. In Section 2, we establish all necessary background material, whereas Section 3 contains the main results.

## 2 Background material

In the following we will use the notation $A \lesssim B$ for non-negative quantities to mean that $A \le cB$ for some positive constant $c$ which is independent of appropriate parameters involved in $A$ and $B$. If $A \lesssim B$ and $B \lesssim A$, we will write $A \approx B$. Moreover, we will use the symbol $X \hookrightarrow Y$ to show that $X$ is continuously embedded in $Y$.

The elementary but useful properties of the $K$-functional are collected in the following proposition.

**Proposition 2.1** ([3]) *Let $(A_0,A_1)$ be a compatible couple of normed spaces. Then $K(t,f;A_0,A_1)$ is non-decreasing in $t$, and $K(t,f;A_0,A_1)/t$ is non-increasing in $t$. Moreover, we have*

$$K(t,f;A_0,A_1) \le \|f\|_{A_0}, \quad f \in A_0, t > 0; \tag{2.1}$$

$$K(t,f;A_0,A_1) \le t\|f\|_{A_1}, \quad f \in A_1, t > 0; \tag{2.2}$$

$$K(t,f;A_0,A_1) = tK\left(t^{-1},f;A_1,A_0\right), \quad f \in A_0 + A_1, t > 0; \tag{2.3}$$

$$K(t,f+g;A_0,A_1) \le K(t,f;A_0,A_1) + K(t,g;A_0,A_1), \quad f,g \in A_0 + A_1, t > 0. \tag{2.4}$$

In the next three propositions, we describe some formulas which relate the $K$-functional of the couples $(A_0 + A_1,A_1)$, $(A_0,A_0 \cap A_1)$, and $(A_0 + A_1,A_0 \cap A_1)$ with that of the original couple $(A_0,A_1)$.

**Proposition 2.2** *Let $(A_0,A_1)$ be a compatible couple of normed spaces, and let $f \in A_0 + A_1$. Then*

$$K(t,f;A_0 + A_1,A_1) = K(t,f;A_0,A_1), \quad 0 < t < 1.$$

*Proof* In view of (2.3), the proof follows immediately from the following relation:

$$K(t,f;A_0,A_0+A_1)=K(t,f;A_0,A_1),\quad t>1, \tag{2.5}$$

which has been derived in [7], Lemma 2.1. □

For the proof of the next result, we refer to [7], Lemma 2.3.

**Proposition 2.3** *Let* $(A_0,A_1)$ *be a compatible couple of normed spaces, and let* $f\in A_0$. *Then*

$$K(t,f;A_0,A_0\cap A_1)\lesssim K(t,f;A_0,A_1)+t\|f\|_{A_0},\quad 0<t<1.$$

The next result is derived in [4], Theorem 3.

**Proposition 2.4** *Let* $(A_0,A_1)$ *be a compatible couple of normed space, and let* $f\in A_0+A_1$. *Then*

$$K(t,f;A_0+A_1,A_0\cap A_1)\approx K(t,f;A_0,A_1)+tK\bigl(t^{-1},f;A_0,A_1\bigr),\quad 0<t<1.$$

In our proofs, we will make use of the fact that, for a parameter space $\Phi$, both $\|s\chi_{(0,1)}(s)\|_\Phi$ and $\|\chi_{(1,\infty)}\|_\Phi$ are finite. This fact is a simple consequence of (1.1). Moreover, in view of the monotonicity of the norm $\|\cdot\|_\Phi$ and the fact that $K(t,f;A_0,A_1)=\|f\|_{A_0+A_1}$, we have

$$\|f\|_{(A_0,A_1)_\Phi}\approx\bigl\|\chi_{(0,1)}(t)K(t,f;A_0,A_1)\bigr\|_\Phi+\bigl\|\chi_{(1,\infty)}(t)K(t,f;A_0,A_1)\bigr\|_\Phi. \tag{2.6}$$

We will make use of the next result, without explicitly mentioning it, in our proofs.

**Proposition 2.5** *Let* $(A_0,A_1)$ *be a compatible couple of normed spaces, and assume that* $A_1\hookrightarrow A_0$. *Then*

$$\|f\|_{(A_0,A_1)_\Phi}\approx\bigl\|\chi_{(0,1)}(s)K(s,f;A_0,A_1)\bigr\|_\Phi.$$

*Proof* It will suffice to derive

$$\|f\|_{(A_0,A_1)_\Phi}\lesssim\bigl\|\chi_{(0,1)}(s)K(s,f;A_0,A_1)\bigr\|_\Phi, \tag{2.7}$$

as the converse estimate is trivial. Using (2.6) and (2.1), we get

$$\begin{aligned}\|f\|_{(A_0,A_1)_\Phi}&\lesssim\bigl\|\chi_{(0,1)}(s)K(s,f;A_0,A_1)\bigr\|_\Phi+\|f\|_{A_1}\|\chi_{(1,\infty)}\|_\Phi\\&\approx\bigl\|\chi_{(0,1)}(s)K(s,f;A_0,A_1)\bigr\|_\Phi+\|f\|_{A_1},\end{aligned}$$

as our assumption $A_1\hookrightarrow A_0$ implies that $\|f\|_{A_0}\approx\|f\|_{A_0+A_1}$, so

$$\|f\|_{(A_0,A_1)_\Phi}\lesssim\bigl\|\chi_{(0,1)}(s)K(s,f;A_0,A_1)\bigr\|_\Phi+\|f\|_{A_0+A_1}. \tag{2.8}$$

Since $K(t,f;A_0,A_1)/t$ is non-increasing in $t$, we obtain

$$\left\| \chi_{(0,1)}(s)K(s,f;A_0,A_1)\right\|_\Phi \geq K(1,f;A_0,A_1)\|s\chi_{(0,1)(s)}\|_\Phi,$$

which gives

$$\|f\|_{A_0+A_1} \lesssim \left\| \chi_{(0,1)}(s)K(s,f;A_0,A_1)\right\|_\Phi. \tag{2.9}$$

Now (2.7) follows from (2.8) and (2.9). The proof is complete. □

## 3 Main results

**Theorem 3.1** *Let $(A_0,A_1)$ be a compatible couple of normed spaces. Then, for an arbitrary parameter space $\Phi$, we have with equivalent norms*

$$(A_0 + A_1,A_0)_\Phi \cap (A_0 + A_1,A_1)_\Phi = (A_0 + A_1,A_0 \cap A_1)_\Phi.$$

*Proof* Put $B_0 = (A_0 + A_1,A_0)_\Phi$, $B_1 = (A_0 + A_1,A_1)_\Phi$, and $B = (A_0 + A_1,A_0 \cap A_1)_\Phi$. Let $f \in A_0 + A_1$. Then by Proposition 2.4

$$\|f\|_B \approx \left\| \chi_{(0,1)}(s)K(s,f;A_0,A_1)\right\|_\Phi + \left\| \chi_{(0,1)}(s)sK\left(s^{-1},f;A_0,A_1\right)\right\|_\Phi,$$

next making use of (2.3), we arrive at

$$\|f\|_B \approx \left\| \chi_{(0,1)}(s)K(s,f;A_0,A_1)\right\|_\Phi + \left\| \chi_{(0,1)}(s)K(s,f;A_1,A_0)\right\|_\Phi.$$

Finally, appealing to Proposition 2.2, we get

$$\|f\|_B \approx \|f\|_{B_0} + \|f\|_{B_1},$$

which concludes the proof. □

**Remark 3.2** The result of Theorem 3.1 generalizes the assertion (1.5) in [7], Theorem 1.1.

**Theorem 3.3** *Let $(A_0,A_1)$ be a compatible couple of normed spaces. Then, for an arbitrary parameter space $\Phi$, we have with equivalent norms*

$$(A_0,A_0 \cap A_1)_\Phi + (A_1,A_0 \cap A_1)_\Phi = (A_0 + A_1,A_0 \cap A_1)_\Phi.$$

*Proof* Put $B_0 = (A_0,A_0 \cap A_1)_\Phi$, $B_1 = (A_1,A_0 \cap A_1)_\Phi$ and $B = (A_0 + A_1,A_0 \cap A_1)_\Phi$. Let $f \in B_0 + B_1$, and take an arbitrary decomposition $f = f_0 + f_1$ with $f_0 \in B_0$ and $f_1 \in B_1$. Then by (2.4), we have

$$\begin{aligned}\|f\|_B &\lesssim \left\| \chi_{(0,1)}(s)K(s,f_0;A_0 + A_1,A_0 \cap A_1)\right\|_\Phi \\ &\quad + \left\| \chi_{(0,1)}(s)K(s,f_1;A_0 + A_1,A_0 \cap A_1)\right\|_\Phi,\end{aligned}$$

now applying the simple fact that

$$K(t,f_j;A_0 + A_1,A_0 \cap A_1) \leq K(t,f_j;A_j,A_0 \cap A_1) \quad (j = 0,1), t > 0,$$

we obtain

$$\|f\|_B \lesssim \|f_0\|_{B_0} + \|f_1\|_{B_1},$$

from which the estimate $\|f\|_B \lesssim \|f\|_{B_0+B_1}$ follows as the decomposition $f = f_0 + f_1$ is arbitrary. In order to establish the converse estimate, we take $f \in B$ and note that there exists (by definition of the norm on $A_0 + A_1$) a particular decomposition $f = f_0 + f_1$ with $f_0 \in A_0$ and $f_1 \in A_1$ such that

$$\|f_0\|_{A_0} + \|f_1\|_{A_1} \lesssim \|f\|_{A_0+A_1}. \tag{3.1}$$

By Proposition 2.3,

$$\begin{aligned}\|f_0\|_{B_0} &\lesssim \left\|\chi_{(0,1)}(s)K(s,f_0;A_0,A_1)\right\|_\Phi + \left\|s\chi_{(0,1)}(s)\right\|_\Phi \|f_0\|_{A_0} \\ &\approx \left\|\chi_{(0,1)}(s)K(s,f_0;A_0,A_1)\right\|_\Phi + \|f_0\|_{A_0},\end{aligned}$$

since $f_0 = f - f_1$, we get by (2.4)

$$\|f_0\|_{B_0} \lesssim \left\|\chi_{(0,1)}(s)K(s,f;A_0,A_1)\right\|_\Phi + \left\|\chi_{(0,1)}(s)K(s,f_1;A_0,A_1)\right\|_\Phi + \|f_0\|_{A_0},$$

next we use (2.2) to obtain

$$\begin{aligned}\|f_0\|_{B_0} &\lesssim \left\|\chi_{(0,1)}(s)K(s,f;A_0,A_1)\right\|_\Phi + \left\|s\chi_{(0,1)}(s)\right\|_\Phi \|f_1\|_{A_1} + \|f_0\|_{A_0} \\ &\approx \left\|\chi_{(0,1)}(s)K(s,f;A_0,A_1)\right\|_\Phi + \|f_1\|_{A_1} + \|f_0\|_{A_0}\end{aligned}$$

and, using (3.1), we get

$$\|f_0\|_{B_0} \lesssim \left\|\chi_{(0,1)}(s)K(s,f;A_0,A_1)\right\|_\Phi + \|f\|_{A_0+A_1},$$

in accordance with (2.9), we deduce that

$$\|f_0\|_{B_0} \lesssim \left\|\chi_{(0,1)}(s)K(s,f;A_0,A_1)\right\|_\Phi.$$

Analogously, we can obtain

$$\|f_1\|_{B_1} \lesssim \left\|\chi_{(0,1)}(s)sK\left(s^{-1},f;A_0,A_1\right)\right\|_\Phi.$$

Therefore, combining the previous two estimates, we find that

$$\|f_0\|_{B_0} + \|f_1\|_{B_1} \lesssim \left\|\chi_{(0,1)}(s)K(s,f;A_0,A_1)\right\|_\Phi + \left\|\chi_{(0,1)}(s)sK\left(s^{-1},f;A_0,A_1\right)\right\|_\Phi,$$

from which, in view of Proposition 2.4, it follows that

$$\|f\|_{B_0+B_1} \lesssim \|f\|_B,$$

which completes the proof. □

**Remark 3.4** The result of Theorem 3.3 generalizes the assertion (1.6) in [7], Theorem 1.1.

In order to formulate the further results, we need the following conditions on the parameter spaces $\Phi_0$ and $\Phi_1$:

($C_1$) $\|\chi_{(0,1)}(s)g(s)\|_{\Phi_0} \lesssim \|\chi_{(0,1)}(s)g(s)\|_{\Phi_1}$.
($C_2$) $\|\chi_{(0,1)}(s)g(s)\|_{\Phi_1} \lesssim \|\chi_{(0,1)}(s)g(s)\|_{\Phi_0}$.
($C_3$) $\|\chi_{(1,\infty)}(s)g(s)\|_{\Phi_1} \lesssim \|\chi_{(1,\infty)}(s)g(s)\|_{\Phi_0}$.
($C_4$) $\|\chi_{(1,\infty)}(s)g(s)\|_{\Phi_0} \approx \|\chi_{(0,1)}(s)sg(1/s)\|_{\Phi_1}$.
($C_5$) $\|\chi_{(1,\infty)}(s)g(s)\|_{\Phi_1} \approx \|\chi_{(0,1)}(s)sg(1/s)\|_{\Phi_0}$.

**Remark 3.5** Let $(\theta,p) \in \Gamma$, and assume that $\Phi_0$ and $\Phi_1$ are given by the norms

$$\|g\|_{\Phi_0} = \begin{cases} (\int_0^\infty t^{-\theta}|g(t)|^p \frac{dt}{t})^{1/p}, & (\theta,p) \in (0,1) \times [1,\infty), \\ \sup_{0<t<\infty} t^{-\theta}|g(t)|, & (\theta,p) \in [0,1] \times \{\infty\}, \end{cases} \tag{3.2}$$

and

$$\|g\|_{\Phi_1} = \begin{cases} (\int_0^\infty t^{1-\theta}|g(t)|^p \frac{dt}{t})^{1/p}, & (\theta,p) \in (0,1) \times [1,\infty), \\ \sup_{0<t<\infty} t^{1-\theta}|g(t)|, & (\theta,p) \in [0,1] \times \{\infty\}. \end{cases} \tag{3.3}$$

Then it is easy to see that ($C_1$) and ($C_3$) hold for $(\theta,p) \in \Gamma_1$, and ($C_2$) holds for $(\theta,p) \in \Gamma_2$. The conditions ($C_4$) and ($C_5$) hold trivially for all $(\theta,p) \in \Gamma$.

**Remark 3.6** Let $1 \leq p < \infty$, and assume that $\Phi_0$ and $\Phi_1$ are given by (1.3) and (1.4). Then we note that ($C_1$), ($C_3$), ($C_4$), and ($C_5$) hold.

**Theorem 3.7** *Let $(A_0,A_1)$ be a compatible couple of normed spaces, and assume that the parameter spaces $\Phi_0$ and $\Phi_1$ satisfy* ($C_1$), ($C_3$) *and* ($C_4$). *Then we have with equivalent norms*

$$(A_0,A_1)_{\Phi_0} \cap (A_0,A_1)_{\Phi_1} = (A_0 + A_1, A_0 \cap A_1)_{\Phi_1}.$$

*Proof* Put $B_0 = (A_0,A_1)_{\Phi_0}$, $B_1 = (A_0,A_1)_{\Phi_1}$ and $B = (A_0 + A_1, A_0 \cap A_1)_{\Phi_1}$. Let $f \in A_0 + A_1$. Then

$$\begin{aligned} \|f\|_{B_0} + \|f\|_{B_1} \approx{}& \big\|\chi_{(0,1)}(s)K(s,f;A_0,A_1)\big\|_{\Phi_0} + \big\|\chi_{(1,\infty)}(s)K(s,f;A_0,A_1)\big\|_{\Phi_0} \\ &+ \big\|\chi_{(0,1)}(s)K(s,f;A_0,A_1)\big\|_{\Phi_1} + \big\|\chi_{(1,\infty)}(s)K(s,f;A_0,A_1)\big\|_{\Phi_1}, \end{aligned}$$

which, in view of ($C_1$) and ($C_3$), reduces to

$$\|f\|_{B_0} + \|f\|_{B_1} \approx \big\|\chi_{(0,1)}(s)K(s,f;A_0,A_1)\big\|_{\Phi_1} + \big\|\chi_{(1,\infty)}(s)K(s,f;A_0,A_1)\big\|_{\Phi_0},$$

at this point we use ($C_4$) to obtain

$$\|f\|_{B_0} + \|f\|_{B_1} \approx \big\|\chi_{(0,1)}(s)K(s,f;A_0,A_1)\big\|_{\Phi_1} + \big\|\chi_{(0,1)}(s)K\big(s^{-1},f;A_0,A_1\big)\big\|_{\Phi_1},$$

finally, applying Proposition 2.4, we conclude that

$$\|f\|_{B_0} + \|f\|_{B_1} \approx \|f\|_B.$$

The proof is complete. □

**Remark 3.8** Applying Theorem 3.7 to the parameter spaces $\Phi_0$ and $\Phi_1$ given by (3.2) and (3.3), we get back the result (1.7) in [7], Theorem 1.1, for $(\theta,p) \in \Gamma_1$. Note that the case when $(\theta,p) \in \Gamma_2$ follows from the case when $(\theta,p) \in \Gamma_1$ by replacing $\theta$ by $1-\theta$.

**Corollary 3.9** *Let* $(A_0,A_1)$ *be a compatible couple of normed spaces, and let* $1 \le p < \infty$. *Then we have with equivalent norms*

$$(A_0,A_1)_{\{0\},p} \cap (A_0,A_1)_{\{1\},p} = (A_0 + A_1, A_0 \cap A_1)_{\{1\},p}.$$

*Proof* The proof follows by applying Theorem 3.7 to the parameter spaces $\Phi_0$ and $\Phi_1$ given by (1.3) and (1.4). □

**Theorem 3.10** *Let* $(A_0,A_1)$ *be a compatible couple of normed spaces, and assume that the parameter spaces* $\Phi_0$ *and* $\Phi_1$ *satisfy* $(C_1)$, $(C_3)$, *and* $(C_5)$. *Then we have with equivalent norms*

$$(A_0,A_1)_{\Phi_0} + (A_0,A_1)_{\Phi_1} = (A_0 + A_1, A_0 \cap A_1)_{\Phi_0}.$$

*Proof* Put $B_0 = (A_0,A_1)_{\Phi_0}$, $B_1 = (A_0,A_1)_{\Phi_1}$ and $B = (A_0 + A_1, A_0 \cap A_1)_{\Phi_0}$. Let $f \in B_0 + B_1$, and write $f = f_0 + f_1$, where $f_0 \in B_0$ and $f_1 \in B_1$. Now by Proposition 2.4, we have

$$\|f\|_B \approx \big\| \chi_{(0,1)}(s)K(s,f;A_0,A_1)\big\|_{\Phi_0} + \big\| \chi_{(0,1)}(s)sK\big(s^{-1},f;A_0,A_1\big)\big\|_{\Phi_0},$$

using $(C_5)$ gives

$$\|f\|_B \approx \big\| \chi_{(0,1)}(s)K(s,f;A_0,A_1)\big\|_{\Phi_0} + \big\| \chi_{(1,\infty)}(s)K(s,f;A_0,A_1)\big\|_{\Phi_1},$$

since $f = f_0 + f_1$, so by (2.4), we have

$$\begin{aligned}\|f\|_B \lesssim{}& \big\| \chi_{(0,1)}(s)K(s,f_0;A_0,A_1)\big\|_{\Phi_0} + \big\| \chi_{(0,1)}(s)K(s,f_1;A_0,A_1)\big\|_{\Phi_0} \\ &+ \big\| \chi_{(1,\infty)}(s)K(s,f_0;A_0,A_1)\big\|_{\Phi_1} + \big\| \chi_{(1,\infty)}(s)K(s,f_1;A_0,A_1)\big\|_{\Phi_1},\end{aligned}$$

by $(C_1)$ and $(C_3)$, we arrive at

$$\begin{aligned}\|f\|_B \lesssim{}& \big\| \chi_{(0,1)}(s)K(s,f_0;A_0,A_1)\big\|_{\Phi_1} + \big\| \chi_{(0,1)}(s)K(s,f_1;A_0,A_1)\big\|_{\Phi_0} \\ &+ \big\| \chi_{(1,\infty)}(s)K(s,f_0;A_0,A_1)\big\|_{\Phi_1} + \big\| \chi_{(1,\infty)}(s)K(s,f_1;A_0,A_1)\big\|_{\Phi_0},\end{aligned}$$

which gives

$$\|f\|_B \lesssim \|f_0\|_{B_0} + \|f_1\|_{B_1},$$

from which the estimate $\|f\|_B \lesssim \|f\|_{B_0+B_1}$ follows. To derive the other estimate, take $f \in B$, and choose a particular decomposition $f = f_0 + f_1$, with $f_0 \in A_0$ and $f_1 \in A_1$, satisfying (3.1). Then

$$\begin{aligned}\|f_0\|_{B_0} &\approx \big\|\chi_{(0,1)}(s)K(s,f_0;A_0,A_1)\big\|_{\Phi_0} + \big\|\chi_{(1,\infty)}(s)K(s,f_0;A_0,A_1)\big\|_{\Phi_0}\\ &\lesssim \big\|\chi_{(0,1)}(s)K(s,f_0;A_0,A_1)\big\|_{\Phi_0} + \big\|\chi_{(1,\infty)}\|_{\Phi_0}\|f_0\big\|_{A_0}\\ &\approx \big\|\chi_{(0,1)}(s)K(s,f_0;A_0,A_1)\big\|_{\Phi_0} + \|f_0\|_{A_0},\end{aligned}$$

where we have used (2.1). Next proceeding in the same way as in the proof of Theorem 3.3, we obtain

$$\|f_0\|_{B_0} \lesssim \big\|\chi_{(0,1)}(s)K(s,f;A_0,A_1)\big\|_{\Phi_0}. \tag{3.4}$$

Also, we can show that

$$\|f_1\|_{B_1} \lesssim \big\|\chi_{(1,\infty)}(s)K(s,f;A_0,A_1)\big\|_{\Phi_1},$$

which, in view of ($C_5$), becomes

$$\|f_1\|_{B_1} \lesssim \big\|\chi_{(0,1)}(s)sK\big(s^{-1},f;A_0,A_1\big)\big\|_{\Phi_0},$$

which, combined with (3.4), yields

$$\|f_0\|_{B_0} + \|f_1\|_{B_1} \lesssim \big\|\chi_{(0,1)}(s)K(s,f;A_0,A_1)\big\|_{\Phi_0} + \big\|\chi_{(0,1)}(s)sK\big(s^{-1},f;A_0,A_1\big)\big\|_{\Phi_0},$$

which, in view of Proposition 2.4, gives

$$\|f_0\|_{B_0} + \|f_1\|_{B_1} \lesssim \|f\|_B,$$

from which the desired estimate $\|f\|_{B_0+B_1} \lesssim \|f\|_B$ follows. The proof of the theorem is finished. □

**Remark 3.11** Theorem 3.10, applied to the parameter spaces $\Phi_0$ and $\Phi_1$ given by (3.2) and (3.3), gives back (1.8) in [7], Theorem 1.1.

**Corollary 3.12** *Let* $(A_0,A_1)$ *be a compatible couple of normed spaces, and let* $1 \le p < \infty$. *Then we have with equivalent norms*

$$(A_0,A_1)_{\{0\},p} + (A_0,A_1)_{\{1\},p} = (A_0 + A_1, A_0 \cap A_1)_{\{0\},p}.$$

*Proof* Apply Theorem 3.10 to the parameter spaces $\Phi_0$ and $\Phi_1$ given by (1.3) and (1.4). □

**Theorem 3.13** *Let* $(A_0,A_1)$ *be a compatible couple of normed spaces, and assume that the parameter spaces* $\Phi_0$ *and* $\Phi_1$ *satisfy* ($C_1$). *Then we have with equivalent norms*

$$(A_0, A_0 \cap A_1)_{\Phi_0} \cap (A_0 + A_1, A_0 \cap A_1)_{\Phi_1} = (A_0, A_0 \cap A_1)_{\Phi_1}.$$

*Proof* Denote $B_0 = (A_0, A_0 \cap A_1)_{\Phi_0}$, $B_1 = (A_0 + A_1, A_0 \cap A_1)_{\Phi_1}$, and $B = (A_0, A_0 \cap A_1)_{\Phi_1}$. Let $f \in A_0$. The estimate $\|f\|_{B_0} + \|f\|_{B_1} \lesssim \|f\|_B$ follows thanks to the condition $(C_1)$ and the following simple inequality:

$$K(t,f;A_0 + A_1, A_0 \cap A_1) \le K(t,f;A_0, A_0 \cap A_1), \quad t > 0. \tag{3.5}$$

To derive the converse estimate, we apply Proposition 2.3 to obtain

$$\|f\|_B \lesssim \left\| \chi_{(0,1)}(s) K(s,f;A_0,A_1) \right\|_{\Phi_1} + \|f\|_{A_0}. \tag{3.6}$$

Next, since $K(t,f;A_0,A_1)/t$ is non-increasing in $t$, observe that

$$\left\| \chi_{(0,1)}(s) K(s,f;A_0, A_0 \cap A_1) \right\|_{\Phi_0} \ge K(1,f;A_0, A_0 \cap A_1) \left\| s\chi_{(0,1)}(s) \right\|_{\Phi_0},$$

noting $K(1,f;A_0,A_0 \cap A_1) = \|f\|_{A_0}$, we have

$$\|f\|_{A_0} \lesssim \|f\|_{B_0}. \tag{3.7}$$

By Proposition 2.4, we also have

$$\left\| \chi_{(0,1)}(s) K(s,f;A_0,A_1) \right\|_{\Phi_1} \lesssim \|f\|_{B_1}. \tag{3.8}$$

Finally, combining (3.6), (3.7), and (3.8), we obtain $\|f\|_B \lesssim \|f\|_{B_0} + \|f\|_{B_1}$. The proof is finished. □

**Remark 3.14** By applying Theorem 3.13 to the parameter spaces $\Phi_0$ and $\Phi_1$ given by (3.2) and (3.3), we get back (1.9) in [7], Theorem 1.1, for $(\theta, p) \in \Gamma_1$.

**Corollary 3.15** *Let* $(A_0, A_1)$ *be a compatible couple of normed spaces, and let* $1 \le p < \infty$. *Then we have with equivalent norms*

$$(A_0, A_0 \cap A_1)_{\{0\},p} \cap (A_0 + A_1, A_0 \cap A_1)_{\{1\},p} = (A_0, A_0 \cap A_1)_{\{1\},p}.$$

*Proof* Apply Theorem 3.13 to the parameter spaces $\Phi_0$ and $\Phi_1$ given by (1.3) and (1.4). □

**Theorem 3.16** *Let* $(A_0, A_1)$ *be a compatible couple of normed spaces, and assume that the parameter spaces* $\Phi_0$ *and* $\Phi_1$ *satisfy* $(C_2)$. *Then we have with equivalent norms*

$$(A_0, A_0 \cap A_1)_{\Phi_0} \cap (A_0 + A_1, A_0 \cap A_1)_{\Phi_1} = (A_0, A_0 \cap A_1)_{\Phi_0}.$$

*Proof* It will suffice to establish that $(A_0, A_0 \cap A_1)_{\Phi_0} \hookrightarrow (A_0 + A_1, A_0 \cap A_1)_{\Phi_1}$. Let $f \in (A_0, A_0 \cap A_1)_{\Phi_0}$, then by (3.5) we have

$$\left\| \chi_{(0,1)}(s) K(s,f;A_0 + A_1, A_0 \cap A_1) \right\|_{\Phi_1} \le \left\| \chi_{(0,1)}(s) K(s,f;A_0, A_0 \cap A_1) \right\|_{\Phi_1},$$

consequently, in view of condition $(C_2)$, we obtain

$$\left\| \chi_{(0,1)}(s) K(s,f;A_0 + A_1, A_0 \cap A_1) \right\|_{\Phi_1} \lesssim \left\| \chi_{(0,1)}(s) K(s,f;A_0, A_0 \cap A_1) \right\|_{\Phi_0},$$

which concludes the proof. □

**Remark 3.17** For $(\theta, p) \in \Gamma_2$, the result (1.9) in [7], Theorem 1.1, follows from Theorem 3.16, applied to the parameter spaces $\Phi_0$ and $\Phi_1$ given by (3.2) and (3.3).

**Corollary 3.18** *Let $(A_0, A_1)$ be a compatible couple of normed spaces, and let $1 \leq p < \infty$. Then we have with equivalent norms*

$$(A_0, A_0 \cap A_1)_{\{1\},p} \cap (A_0 + A_1, A_0 \cap A_1)_{\{0\},p} = (A_0, A_0 \cap A_1)_{\{1\},p}.$$

*Proof* Apply Theorem 3.16 to the parameter spaces $\Phi_0$ and $\Phi_1$ given by the norms

$$\|g\|_{\Phi_0} = \sup_{0<s<1} \frac{|g(s)|}{s} + \left(\int_1^\infty \left(\frac{|g(s)|}{s}\right)^p \frac{ds}{s}\right)^{\frac{1}{p}} \tag{3.9}$$

and

$$\|g\|_{\Phi_1} = \left(\int_0^1 |g(s)|^p \frac{ds}{s}\right)^{\frac{1}{p}} + \sup_{s>1} |g(s)|. \tag{3.10}$$

□

**Theorem 3.19** *Let $(A_0, A_1)$ be a compatible couple of normed spaces, and assume that the parameter spaces $\Phi_0$ and $\Phi_1$ satisfy $(C_2)$. Then we have with equivalent norms*

$$(A_0 + A_1, A_1)_{\Phi_0} + (A_0 + A_1, A_0 \cap A_1)_{\Phi_1} = (A_0 + A_1, A_1)_{\Phi_1}.$$

*Proof* Put $B_0 = (A_0 + A_1, A_1)_{\Phi_0}$, $B_1 = (A_0 + A_1, A_0 \cap A_1)_{\Phi_1}$, and $B = (A_0 + A_1, A_1)_{\Phi_1}$. Let $f \in B_0 + B_1$, and take an arbitrary decomposition $f = f_0 + f_1$ with $f_0 \in B_0$ and $f_1 \in B_1$. Then by (2.4)

$$\|f\|_B \lesssim \left\|\chi_{(0,1)}(s) K(s, f_0; A_0 + A_1, A_1)\right\|_{\Phi_1} + \left\|\chi_{(0,1)}(s) K(s, f_1; A_0 + A_1, A_1)\right\|_{\Phi_1},$$

using condition $(C_2)$ and the fact that

$$K(t, f_1; A_0 + A_1, A_1) \leq K(t, f_1; A_0 + A_1, A_0 \cap A_1), \quad t > 0,$$

we obtain

$$\|f\|_B \lesssim \|f_0\|_{B_0} + \|f_1\|_{B_1},$$

whence, since $f = f_0 + f_1$ is an arbitrary decomposition, we get $\|f\|_B \lesssim \|f\|_{B_0+B_1}$. For the converse estimate, let $f \in B$, and choose a particular decomposition $f = f_0 + f_1$, with $f_0 \in A_0$ and $f_1 \in A_1$, satisfying (3.1). By Proposition 2.4,

$$\|f_0\|_{B_1} \approx \left\|\chi_{(0,1)}(s) K(s, f_0; A_0, A_1)\right\|_{\Phi_1} + \left\|\chi_{(0,1)}(s) s K\left(s^{-1}, f_0; A_0, A_1\right)\right\|_{\Phi_1},$$

using (2.1), we obtain

$$\|f_0\|_{B_1} \lesssim \left\|\chi_{(0,1)}(s) K(s, f_0; A_0, A_1)\right\|_{\Phi_1} + \|f_0\|_{A_0},$$

which, since $f_0 = f - f_1$, gives

$$\|f_0\|_{B_1} \lesssim \|\chi_{(0,1)}(s)K(s,f;A_0,A_1)\|_{\Phi_1} + \|\chi_{(0,1)}(s)K(s,f_1;A_0,A_1)\|_{\Phi_1} + \|f_0\|_{A_0},$$

now using (2.2), it follows that

$$\|f_0\|_{B_1} \lesssim \|\chi_{(0,1)}(s)K(s,f;A_0,A_1)\|_{\Phi_1} + \|f_1\|_{A_1} + \|f_0\|_{A_0}. \tag{3.11}$$

Using (2.2) also gives

$$\begin{aligned}\|f_1\|_{B_0} &\approx \|\chi_{(0,1)}(s)K(s,f_1;A_0+A_1,A_1)\|_{\Phi_0}\\ &\lesssim \|f_1\|_{A_1}\|\chi_{(0,1)}(s)\|_{\Phi_0}\\ &\approx \|f_1\|_{A_1},\end{aligned}$$

which, together with (3.11), leads to

$$\|f_0\|_{B_1} + \|f_1\|_{B_0} \lesssim \|\chi_{(0,1)}(s)K(s,f;A_0,A_1)\|_{\Phi_1} + \|f_0\|_{A_0} + \|f_1\|_{A_1},$$

whence, in view of (3.1), it follows that

$$\|f_0\|_{B_1} + \|f_1\|_{B_0} \lesssim \|\chi_{(0,1)}(s)K(s,f;A_0,A_1)\|_{\Phi_1} + \|f\|_{A_0+A_1},$$

according to 2.9, we arrive at

$$\|f_0\|_{B_1} + \|f_1\|_{B_0} \lesssim \|\chi_{(0,1)}(s)K(s,f;A_0,A_1)\|_{\Phi_1},$$

appealing to Proposition 2.2 yields

$$\|f_0\|_{B_1} + \|f_1\|_{B_0} \lesssim \|f\|_B$$

from which the desired estimate $\|f\|_{B_0+B_1} \lesssim \|f\|_B$ follows. The proof is complete. □

**Remark 3.20** We recover (1.10) in [7], Theorem 1.1, for $(\theta,p) \in \Gamma_2$, by an application of Theorem 3.19 to the parameter spaces $\Phi_0$ and $\Phi_1$ given by (3.2) and (3.3).

**Corollary 3.21** *Let $(A_0,A_1)$ be a compatible couple of normed spaces, and let $1 \le p < \infty$. Then we have with equivalent norms*

$$(A_0 + A_1, A_1)_{\{1\},p} + (A_0 + A_1, A_0 \cap A_1)_{\{0\},p} = (A_0 + A_1, A_1)_{\{0\},p}.$$

*Proof* Apply Theorem 3.19 to the parameter spaces $\Phi_0$ and $\Phi_1$ given by (3.9) and (3.10). □

**Theorem 3.22** *Let $(A_0,A_1)$ be a compatible couple of normed spaces, and assume that the parameter spaces $\Phi_0$ and $\Phi_1$ satisfy* ($C_1$). *Then we have with equivalent norms*

$$(A_0 + A_1, A_1)_{\Phi_0} + (A_0 + A_1, A_0 \cap A_1)_{\Phi_1} = (A_0 + A_1, A_1)_{\Phi_0}.$$

*Proof* It suffices to show that

$$(A_0 + A_1, A_0 \cap A_1)_{\Phi_1} \hookrightarrow (A_0 + A_1, A_1)_{\Phi_0}.$$

Let $f \in (A_0 + A_1, A_0 \cap A_1)_{\Phi_1}$. Then, using condition ($C_1$) and the elementary fact that

$$K(t,f;A_0 + A_1, A_1) \leq K(t,f;A_0 + A_1, A_0 \cap A_1), \quad t > 0,$$

we have

$$\begin{aligned} \|\chi_{(0,1)}(s)K(s,f;A_0 + A_1, A_1)\|_{\Phi_0} &\lesssim \|\chi_{(0,1)}(s)K(s,f;A_0 + A_1, A_1)\|_{\Phi_1} \\ &\leq \|\chi_{(0,1)}(s)K(s,f;A_0 + A_1, A_0 \cap A_1)\|_{\Phi_1}, \end{aligned}$$

which finishes the proof. □

**Remark 3.23** Theorem 3.22, applied to the parameter spaces $\Phi_0$ and $\Phi_1$ given by (3.2) and (3.3), gives back (1.10) in [7], Theorem 1.1, for $(\theta, p) \in \Gamma_1$.

**Corollary 3.24** *Let* $(A_0, A_1)$ *be a compatible couple of normed spaces, and let* $1 \leq p < \infty$. *Then we have with equivalent norms*

$$(A_0 + A_1, A_1)_{\{0\},p} + (A_0 + A_1, A_0 \cap A_1)_{\{1\},p} = (A_0 + A_1, A_1)_{\{0\},p}.$$

*Proof* Apply Theorem 3.22 to the parameter spaces $\Phi_0$ and $\Phi_1$ given by (1.3) and (1.4). □

**Theorem 3.25** *Let* $(A_0, A_1)$ *be a compatible couple of normed spaces, and assume that the parameter spaces* $\Phi_0$ *and* $\Phi_1$ *satisfy* ($C_1$). *Then we have with equivalent norms*

$$(A_0, A_0 \cap A_1)_{\Phi_0} + (A_0 + A_1, A_0 \cap A_1)_{\Phi_1} = (A_0, A_1)_{\Psi},$$

*where*

$$\|g\|_{\Psi} = \|\chi_{(0,1)}(s)g(s)\|_{\Phi_0} + \|\chi_{(0,1)}(s)sg(1/s)\|_{\Phi_1}.$$

*Proof* Set $B_0 = (A_0, A_0 \cap A_1)_{\Phi_0}$ and $B_1 = (A_0 + A_1, A_0 \cap A_1)_{\Phi_1}$. Let $f \in B_0 + B_1$, and write $f = f_0 + f_1$ with $f_0 \in B_0$ and $f_1 \in B_1$. Making use of (2.4), we have

$$\|f\|_{(A_0,A_1)_{\Psi}} \lesssim I_1 + I_2, \tag{3.12}$$

where

$$I_1 = \|\chi_{(0,1)}(s)K(s,f_0;A_0,A_1)\|_{\Phi_0} + \|\chi_{(0,1)}(s)K(s,f_1;A_0,A_1)\|_{\Phi_0}$$

and

$$I_2 = \|\chi_{(0,1)}(s)sK(s^{-1},f_0;A_0,A_1)\|_{\Phi_1} + \|\chi_{(0,1)}(s)sK(s^{-1},f_1;A_0,A_1)\|_{\Phi_1}.$$

The condition ($C_1$), along with the following simple inequality:

$$K(t,f_0;A_0,A_1) \le K(t,f_0;A_0,A_0\cap A_1), \quad t>0,$$

implies that

$$I_1 \lesssim \left\| \chi_{(0,1)}(s)K(s,f_0;A_0,A_0\cap A_1)\right\|_{\Phi_0} + \left\| \chi_{(0,1)}(s)K(s,f_1;A_0,A_1)\right\|_{\Phi_1}. \tag{3.13}$$

Next we observe that $f_0 \in A_0$ as $B_0 \subset A_0$. Therefore, we can apply (2.1) to arrive at

$$I_2 \lesssim \|f_0\|_{A_0} + \left\| \chi_{(0,1)}(s)sK\left(s^{-1},f_1;A_0,A_1\right)\right\|_{\Phi_1}. \tag{3.14}$$

The proof of the estimate

$$\|f_0\|_{A_0} \lesssim \left\| \chi_{(0,1)}(s)K(s,f_0;A_0,A_0\cap A_1)\right\|_{\Phi_0} \tag{3.15}$$

is the same as that of (3.7). Finally, inserting estimates (3.13) and (3.14) in (3.12) and then using (3.15) and Proposition 2.4, we get

$$\|f\|_{(A_0,A_1)_\Psi} \lesssim \|f_0\|_{B_0} + \|f_1\|_{B_1},$$

which gives the estimate $\|f\|_{(A_0,A_1)_\Psi} \lesssim \|f\|_{B_0+B_1}$. In order to prove the other estimate, we take $f \in (A_0,A_1)_\Psi$, and select a particular decomposition $f = f_0 + f_1$, with $f_0 \in A_0$ and $f_1 \in A_1$, satisfying condition (3.1). Then proceeding in the same way as in the proof of Theorem 3.3, we obtain

$$\|f_0\|_{B_0} \lesssim \left\| \chi_{(0,1)}(s)K(s,f;A_0,A_1)\right\|_{\Phi_0}.$$

Also, we have

$$\|f_1\|_{B_1} \lesssim \left\| \chi_{(0,1)}(s)sK\left(s^{-1},f;A_0,A_1\right)\right\|_{\Phi_1}.$$

Therefore, these estimates, along with the definition of $\Psi$, imply that

$$\|f_0\|_{B_0} + \|f_1\|_{B_1} \lesssim \|f\|_{(A_0,A_1)_\Psi},$$

whence we get $\|f\|_{B_0+B_1} \lesssim \|f\|_{(A_0,A_1)_\Psi}$. The proof is finished. □

**Remark 3.26** Take $\Phi_0$ and $\Phi_1$ to be given by (3.2) and (3.3), then we see that $\Psi = \Phi_0$. Thus, we recover the result (1.11) in [7], Theorem 1.1, for $(\theta,p) \in \Gamma_1$. Since the case when $(\theta,p) \in \Gamma_2$ follows from the case when $(\theta,p) \in \Gamma_1$, Theorem 3.25 provides a generalization of the assertion (1.11) in [7], Theorem 1.1.

**Corollary 3.27** *Let $(A_0,A_1)$ be a compatible couple of normed spaces, and let $1 \le p < \infty$. Then we have with equivalent norms*

$$(A_0,A_0\cap A_1)_{\{0\},p} + (A_0+A_1,A_0\cap A_1)_{\{1\},p} = (A_0,A_1)_{\{0\},p}.$$

*Proof* Apply Theorem 3.25 to the parameter spaces $\Phi_0$ and $\Phi_1$ given by (1.3) and (1.4). □

**Theorem 3.28** *Let* $(A_0, A_1)$ *be a compatible couple of normed spaces, and assume that the parameter spaces* $\Phi_0$ *and* $\Phi_1$ *satisfy* $(C_2)$. *Then we have with equivalent norms*

$$(A_0 + A_1, A_0)_{\Phi_0} \cap (A_0 + A_1, A_0 \cap A_1)_{\Phi_1} = (A_0, A_1)_{\Psi},$$

*where*

$$\|g\|_{\Psi} = \left\| \chi_{(0,1)}(s)g(s) \right\|_{\Phi_1} + \left\| \chi_{(0,1)}(s)sg(1/s) \right\|_{\Phi_0}.$$

*Proof* Set $B_0 = (A_0 + A_1, A_0)_{\Phi_0}$ and $B_1 = (A_0 + A_1, A_0 \cap A_1)_{\Phi_1}$. Let $f \in A_0 + A_1$. Applying Proposition 2.2 to the compatible couple $(A_1, A_0)$, we get

$$\|f\|_{B_0} \approx \left\| \chi_{(0,1)}(s)K(s,f;A_1,A_0) \right\|_{\Phi_0},$$

using (2.3), we have

$$\|f\|_{B_0} \approx \left\| \chi_{(0,1)}(s)sK\left(s^{-1},f;A_0,A_1\right) \right\|_{\Phi_0}. \tag{3.16}$$

By Proposition 2.4,

$$\|f\|_{B_1} \approx \left\| \chi_{(0,1)}(s)K(s,f;A_0,A_1) \right\|_{\Phi_1} + \left\| \chi_{(0,1)}(s)sK\left(s^{-1},f;A_0,A_1\right) \right\|_{\Phi_1},$$

combining this with (3.16) and making use of $(C_2)$, we arrive at

$$\|f\|_{B_0} + \|f\|_{B_1} \approx \|f\|_{(A_0,A_1)_{\Psi}},$$

which completes the proof. □

**Remark 3.29** Theorem 3.28 generalizes the result (1.12) in [7], Theorem 1.1.

**Corollary 3.30** *Let* $(A_0, A_1)$ *be a compatible couple of normed spaces, and let* $1 \leq p < \infty$. *Then we have with equivalent norms*

$$(A_0 + A_1, A_0)_{\{1\},p} + (A_0 + A_1, A_0 \cap A_1)_{\{0\},p} = (A_0, A_1)_{\{0\},p}.$$

*Proof* Apply Theorem 3.28 to the parameter spaces $\Phi_0$ and $\Phi_1$ given by (3.9) and (3.10). □

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors have equally contributed toward the article. All authors read and approved the final manuscript.

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